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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

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VOL. XXVIII.

NOVEMBER, 1896, TO NOVEMBER, 1897.

LONDON:
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1897. a.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.

THE LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT BY THE TREASURER FOR THE YEAR 1895–6 (NOV. 14TH, 1895, TO NOV. 12TH, 1896).

Cr. *GENERAL CASH ACCOUNT OF THE SOCIETY.* **Dr.**

				£.	s.	d.
Balance from 1894–95 :—						
General Fund	£113	12	1			
Life Composition Fund ...	42	0	0			
				155	12	1
Interest on Capital—						
Lieut.-Col. Campbell's Fund	14	17	8			
Life Composition Fund ...	45	13	7			
Invested Surplus Fund ...	17	6	0			
				77	17	3
Interest on Lord Rayleigh's Fund				45	11	1
9 Entrance Fees				9	9	0
6 Life Compositions				63	0	0
124 Annual Subscriptions—						
2 for 1892–93	2	2	0			
22 for 1893–94	23	2	0			
41 for 1894–95	43	1	0			
54 for 1895–96	56	14	0			
5 for 1896–97 (in advance)	5	5	0			
				130	4	0
Sale of *Proceedings*				99	7	10
				£581	1	3

				£.	s.	d.
Life Composition Fund—						
Purchase of £76. 1*s*. 10*d*. New South Wales $3\frac{1}{2}$ per Cent. Inscribed Stock of 1918...				84	0	0
Printing *Proceedings*, &c.				301	1	4
Purchase of Journals, &c., for Library				12	11	1
Postage and Sundries				8	18	2
Rent				19	9	0
Attendance				7	7	0
Subscription to Mittag-Leffler Testimonial ...				5	0	0
Cost of address to Lord Kelvin on his Jubilee				5	5	0
Balance at Bank—						
General Fund	£116	9	8			
Life Composition Fund	21	0	0			
				137	9	8
				£581	1	3

Audited and found correct,
(Signed) THOMAS R. TERRY.

Dec. 3, 1896.

TRUST FUND ACCOUNTS.

I. Lord Rayleigh's Fund.

	£.	s.	d.		£.	s.	d.
By Interest on Capital	45	11	1	Transferred to the General Account of the Society	45	11	1

II. De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Interest on Capital, from previous account ...	8	18	5	Cost of Medal for period 1893–96	9	7	6
Interest during the year	2	18	0	Balance at Bank	2	8	11
	£11	16	5		£11	16	5

STATEMENT OF CAPITAL, November, 1896.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
1. Lieut.-Col. Campbell's Fund... ...	499	7	5 ...	... 385	0	0	Guaranteed 4 per Cent. London & North Western Railway Consolidated Stock.
2. Life Composition Fund, from 1894–5	1298	10	0 ...	1388	2	10	New South Wales $3\frac{1}{2}$ per Cent. Inscribed Stock of 1918.
Amount received during 1895–6	63	0	0 ...	21	0	0	Cash in Bank awaiting investment.
3. Invested Surplus Fund	600	0	0 ...	... 596	6	6	India 3 per Cent. Stock.
Trust Funds.							
1. Lord Rayleigh's Fund	1000	0	0 ...	... 870	0	0	Guaranteed 5 per Cent. Great Indian Peninsula Railway Capital Stock.
2. De Morgan Medal Fund	102	5	9 ...	... 60	0	0	Great Western Railway 5 per Cent. Preference Stock.

Audited and found correct,

(Signed) THOMAS R. TERRY.

Dec. 3, 1896.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXVIII.

THIRTY-THIRD SESSION, 1896–97

(since the Formation of the Society, January 16th, 1865).

November 12*th,* 1896.

THE THIRD ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

Present, twenty-four members and one visitor.

The following members were admitted into the Society: Miss Hardcastle, Mr. Gallop, and Mr. Western.

The Treasurer gave an abstract of his report. Its reception was moved by Mr. Kempe, and seconded by Mr. S. Roberts. The former gentleman thought that more strenuous efforts might be made by present members to increase the number of members. The reception of the report was carried unanimously.

The Rev. T. R. Terry having consented, at the request of the President, to again act as Auditor, he was appointed to that office.

The senior Secretary stated that the present number of members was 226, of whom 114 were life compounders. He also mentioned that the following additional exchanges had been made during the Session:—with the Physical Society, of London, with the *Monatshefte für Mathematik und Physik*, Vienna, and with the American Mathematical Society.

The following communications had been made or received :—

On the Stability and Instability of certain Fluid Motions (iii.), and on the Propagation of Waves upon the Plane Surface separating Two Portions of Fluid of different Vorticities : Lord Rayleigh.

Note on Matrices : Mr. J. Brill.

Determination of the Volumes of certain Species of Tetrahedra without employment of the Method of Limits : Prof. M. J. M. Hill.

Some Algebraical Theorems connected with the Theory of Partitions : Prof. Forsyth.

Certain general Series : Mr. F. H. Jackson.

An Extension of Sylvester's Constructive Theory of Partitions : Major MacMahon.

Note on the Representation of a Conic by a Linear Equation : Mr. J. Griffiths.

On the Representation of a Number as a Sum of Squares : Prof. Mathews.

Researches in the Calculus of Variations—Pt. vii., Limiting conditions in Multiple Integrals ; Pt. viii., Reduction of the Problem of the Discrimination of Maxima and Minima Values in Double Integrals with Variable Limits to a New Problem in Single Integrals : Mr. E. P. Culverwell.

A Note on certain Forms of the Equation of Normals to Conic Sections : Mr. J. L. S. Hatton.

On the Evaluation of a certain Dialytic Determinant : Mr. W. W. Taylor.

Criterion of 2 as a 16-ic Residue, with remarks upon some of Mersenne's Numbers : Lt.-Col. Cunningham.

Note on the Convergency of Series : Dr. R. Bryant.

On the Distribution of Electricity induced on an Infinite Plane Disc with a Circular Hole in it : Mr. H. M. Macdonald.

On the Double Foci of a Bicircular Quartic, and the Nodal Focal Curves of a Cyclide : Dr. R. Lachlan.

Note on a Ternary Cubic : Prof. Tanner.

On Boltzmann's Law of the Equality of Mean Kinetic Energy for each Degree of Freedom : Mr. S. H. Burbury.

Examples illustrating Lord Rayleigh's Theory of the Stability or Instability of certain Fluid Motions : Mr. Love.

Geodesics on Quadrics, not of Revolution : Prof. Forsyth.

The Continuity of Pressure in Vortex Motion, and an Ellipsoidal Vortex : Mr. R. Hargreaves.

On the Enumeration of Groups of Totitives : Prof. Tanner.

The Motion of the Top, and the Catenary on the Paraboloid and Cone : Prof. Greenhill.

On the Isomorphism of a Group with itself : Prof. W. Burnside.

The Division of the Lemniscate : Prof. Mathews.

On some general Formulæ for the Potentials of Ellipsoids, Shells, and Discs : Dr. Hobson.

The Compensation for Difference of Capital in Gambling *à outrance:* A Contribution to the Theory of Duration of Play : Major MacMahon.

On the Stability of a Frictionless Liquid—Theory of Critical Planes : Mr. Basset.

The Bitangents of a Plane Quartic Curve and the Straight Lines of a Cubic Surface: Mr. Baker.
On the Application of the Principal Function to the Solution of Delaunay's Canonical System of Equations: Prof. E. W. Brown.
Waves in Canals: Mr. H. M. Macdonald.
On the a, b, c Form of the Binary Quintic: Mr. J. Hammond.
Construction for the Four Normals to a Central Conic drawn through a given Point: Prof. Mathews.
On a Two-fold Generalization of Stieltjes' Theorem: Dr. H. Taber.
The Construction of Nasik Squares: Rev. A. H. Frost.

The same journals had been subscribed for and the same exchanges of *Proceedings*, in addition to those named above, made as in the preceding Session. No presents had been made to the Society's Albums.

The President stated the grounds of the award by the Council of the Society of the De Morgan Medal to Mr. Samuel Roberts. After receiving the medal, Mr. Roberts cordially thanked the Council and the members generally for the honour they had conferred upon him, and, in a brief review of his long connexion with the Society, said that it had been most useful as well as agreeable to him.

Prof. Tanner and Mr. S. O. Roberts having been appointed Scrutators, the ballot was taken, with the result that the following gentlemen, nominated by the Council, were elected to serve as the Council for the ensuing Session:—Prof. Elliott, F.R.S., President; Dr. Hobson, F.R.S., Mr. M. Jenkins, and Major MacMahon, R.A., F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Mr. R. Tucker, and Mr. A. E. H. Love, F.R.S., Hon. Secs. Other Members of the Council:—Lt.-Col. Cunningham, R.E., Mr. H. T. Gerrans, Dr. Glaisher, F.R.S., Prof. Greenhill, F.R.S., Prof. M. J. M. Hill, F.R.S., Prof. W. H. H. Hudson, Mr. A. B. Kempe, F.R.S., Mr. F. S. Macaulay, and Mr. D. B. Mair.

Prof. Elliott, having taken the Chair, called upon Major MacMahon to read his Valedictory Address, which was entitled "Combinatory Analysis: a Review of the present State of Knowledge." Upon the motion of the President, seconded by Prof. Greenhill, Major MacMahon consented to the publication of his address in the Society's *Proceedings*.

Mr. S. Roberts gave a brief abstract of an Essay by Herr E. Lasker, "On the Geometrical Calculus."

The following papers, owing to the lateness of the hour, were communicated by their titles only:—

Symbolic Logic: Mr. H. MacColl.

On a General Integral with some Physical Applications: Mr. G. J. Hurst.

On Ratio: Prof. M. J. M. Hill.

On the Geometrical Construction of Models of Cubic Surfaces: Mr. W. H. Blythe.

Theory of Vortex Rings: Mr. H. S. Carslaw.

Differentiation of Spherical Harmonics: Mr. E. G. Gallop.

(i.) On the application of Jacobi's Dynamical Method to the General Problem of Three Bodies, and (ii.) On certain Properties of the Mean Motions and the Secular Accelerations of the Principal Arguments used in the Lunar Theory: Prof. E. W. Brown.

Note on the Symmetric Group: Prof. W. Burnside.

Note on the Capacity of a Conductor in the Form of Two Intersecting Spheres: Mr. W. D. Niven.

The following presents were received for the Library:—

"Mathematical Questions and Solutions from the 'Educational Times,'" edited by W. J. C. Miller, Vol. LXV., 8vo; London, 1896.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 3; 1896.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 9; Leipzig, 1896.

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. I.; October, 1896.

"Proceedings of the Royal Society," Nos. 361, 362; 1896.

"Monatshefte für Mathematik und Physik," Vol. VII., Hefte 10, 11, 12; Wien, 1896.

"Proceedings of the Physical Society," Vol. XIV., Pts. 10, 11; 1896.

"Bulletin of the American Mathematical Society," Vol. III., No. 1, October, 1896; New York.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki 7, Dai 6.

"Nieuw Archief voor Wiskunde," 2e Reeks, Deel III., St. 1; Amsterdam, 1896.

"Bulletin des Sciences Mathématiques," Tome XX., Sept., 1896; Paris.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. V., Fasc. 6, 7, 8; Roma, 1896.

"Acta Mathematica," XX., 2; Stockholm, 1896.

"Transactions of the Cambridge Philosophical Society," Vol. XVI., Pt. 1; 1896.

"Educational Times," November, 1896.

"Annali di Matematica," Serie 2, Tome XXIV., Fasc. 4; Milano, 1896.

"Journal für die reine und angewandte Mathematik," Bd. CXVII., Heft 1; Berlin, 1896.

"Indian Engineering," Vol. XX., Nos. 12–16, Sept. 19th to Oct. 17th, 1896.

"Philosophical Transactions of the Royal Society," A. Vol. CLXXXVI., Pts. 1, 2; B. Vol. CLXXXVI., Pts. 1, 2; 1896.

Combinatory Analysis: A Review of the Present State of Knowledge.[*] By Major P. A. MacMahon, R.A., F.R.S. Read November 12th, 1896.

Since my predecessor vacated the chair which for two years past I have had the honour to fill, important events have occurred in the world of mathematical science. Death has claimed Helmholtz and Cayley, men in whose work little similarity can be detected, but who were alike in their ardent devotion to exact science. Eloquent tributes to their brilliant achievements have been written by such competent pens that I will restrict myself, previous to entering upon the theme of my address, to some comments upon that particular work of Cayley with which I had for some years a special acquaintance, in consequence of his exceeding kindness to me at the time, some fourteen years ago, when I first placed my foot upon the thorny path of mathematical research.

Then the theory of perpetuant invariants of binary quantics initiated by Sylvester had made some progress, to which Cayley and Hammond had also contributed, but its advancement had been arrested by difficulties connected with the syzygies (*i.e.*, linear relations) connecting forms of the same weight. A new impulse had, however, just been given by the discovery of a transformation of a general character of the functions of differences of the roots of a binary quantic into functions of the power-sums of degree exceeding unity. This development, as Sylvester remarked, immediately converted the algebra into arithmetic and made it possible to consider the question without any reference to algebraic quantity. It is to Cayley we owe the ingenious algorithm (*A. M. J.*, Vol. VII.) by which this possibility became a reality. It gave the requisite facility in dealing with combinations of forms represented in the notation of partitions. The great advance thus made will be apparent when it is stated that it became comparatively easy to deal with forms of as high a weight as forty or fifty and to assign the syzygies. Combined with the algorithm were the converse processes of capitation and decapitation which had been used by myself for some time. They were discovered independently and named by Cayley.

It was not recognised at the time that the algorithm was an arithmetical exhibition of a definite process of differential operation. It was created by Cayley synthetically by continued application of combinatory analysis. It is essentially constructive in its nature and singularly well adapted to its immediate purpose, for the multiplication of *two* forms. Its weakness becomes apparent when more than two forms have to be multiplied together, the final coefficient sought being only arrived at after a long process in which there are many chances of error. It is here that differential operation is successful with comparatively small risk of error. The particular term of the result of multiplication which it may be desirable to isolate is formed directly by means of a differential operation, the form of which depends upon the form of the term in question.

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[*] Address delivered on retiring from the office of President.

Further, the differential operation is capable of much more than the constructive algorithm, as the operation can be stopped at any point, so as to derive from any identity identities of lower degree and weight. It can be equally well applied to forms whose partition representations contain zero and negative parts: whereas the extension of the algorithm to these cases is not obvious.

The process of "decapitation" of a form or identity, the truth of which was derived by Cayley from algebraical reasoning, is a similar process of differential operation. Viewed in this way, decapitation, to continue the figure of speech, emerges as a particular case of dismemberment, a general process which is difficult to reach without reference to the infinitesimal calculus.

This work enabled mathematicians to grapple with the syzygies, so long a stumbling block and fruitful source of error. The simplest perpetuant of degree 6 was thus found to have the weight 31. I may remark that at one time Cayley was of opinion that no perpetuant of this or of any higher degree existed, hazarding the conjecture that all non-unitary forms were expressible in terms of the perpetuants of the first five degrees. At this period the generating functions of perpetuants of degree inferior to 6 had been established. That corresponding to the degree 6 was the next desideratum. A conjecture was made as to the form of the generating function for the degree n, which has been finally confirmed by Stroh (*Math. Ann.*, t. XXXVI.).

The foregoing important work of Cayley led him to desire a purely algebraic proof of Gordan's theorem concerning the finality of the covariants of quantics of finite order. At the time alluded to the only proof in existence was the symbolic one of Gordan. This has been finally accomplished by Hilbert. The symbolism employed is essentially that initiated by Cayley, who had little fancy for the method, which, dropping from his hands, was carried on with great results by the mathematicians of Germany.

The algebraic characters of the simplest ground forms of quantics of finite order were also sought, and with partial success. Some mathematicians in similar cases are content to enumerate the forms of different orders and classes; and have no desire to see the actual algebraic structures. In no case did Cayley's interest seem to cease with solving the question of enumeration. He desired, up to a certain point, to see the things themselves, and, moreover, considered it very important to attain this end. His analytical skill was a great power in this respect in all departments of pure mathematics. Writing a clear hand, he shone in numerical accuracy and combined great inventive powers in mathematical shorthand with untiring patience. No mathematician of Cayley's time possessed comparable powers over long expressions and tiresome calculations. He was a great intuitionist, but his reputation in this respect may have suffered from his extraordinary skill as a formalist. Much of his work might, and should, have been relegated to others. This, without doubt, would have been so had there existed a school of pure mathematics at Cambridge, similar to those in many German Universities. These remarks arise from my recollections of the vast amount of purely numerical labour, that might have been handed over to a calculator, undertaken by Cayley about the year 1885 when engaged in calculating ground forms and syzygies of binary quantics. But, while thus occupied with minutiæ, he never for a moment lost the power of ascending to an elevated position to survey the theory as a whole and consider its connexion with others. Only a man of his ability could retainthis faculty unimpaired while devoting time, extending over weeks and months, to details. With a lesser man it would have become a vice. In truth, it has been well said that attention to details is frequently the refuge of a man who is destitute of, or who has lost, the power of discussing broad and general questions. That Cayley never neglected the work of others may be seen from what he once said to a friend of mine when discussing the Royal Society Subject-Index—that he did not see its use, as, of course, one *always* knew what had been done in one's own department of science. Few memoirs escaped his vigilant eye. In the theory of invariants, however, in the later years of his life, he worked largely on his own initiative, although well acquainted with contemporary work on the Continent and in the United States of America. With the exception of his exposition and explanation of Stroh's method, his last contribution to the theory related to the literal forms of the symmetric functions of differences which represent covariants. He sought, by a particular ordering of the literal terms, to be able to recognise in the structure of the leading and ending terms the reducibility or non-reducibility of the forms. In this he was only partially successful. The subject bristled with difficulties and exceptional cases. Much ingenious work was suppressed as not giving sufficient promise of clearing up the whole question.

This was a matter of keen regret to Cayley. Several letters in my possession show that it was in his mind to again attack the problem at some convenient time. Alas! that the time never came.

As your retiring President has the opportunity of making a few remarks upon some special topic, I will now pass on to that subject which may possibly be of some interest to those here to-night. It is the combinatory analysis, and my selection is influenced by the belief that this subject is not at present upon a proper footing. Its importance is not fully recognised, because much that properly belongs to it appears under other headings in all recent attempts to organize and arrange the various departments of mathematical science. I hope to show that this is unnecessary, as the associations and connexions sought to be carried out and emphasized can be suitably secured by cross-references. In order to understand what is properly included under the heading, let us first turn to the *Index du répertoire bibliographique des sciences mathématiques*, published by La Commission permanente du Répertoire, Paris, 1893. We find—

Classe J 1. Analyse combinatoire, with the sub-division

a. Groupes où l'on tient compte de l'ordre :

  α. Permutations et arrangements de toutes sortes, nombre et loi de formation ;

  β. Structure, séquences, &c.;

  γ. Manière de ramener les permutations les unes aux autres par permutation circulaire de groupes d'éléments, permutations semblables, &c.

b. Groupes où l'ordre est indifférent :

  α. Combinaisons simples et avec répétition ;

  β. Combinaisons régulières, complètes.

c. Fonctions qui se présentent dans l'analyse combinatoire, théorie des chemins.

d. Applications :

  α. A la recherche des termes généraux des séries ;

β. Des tables;

γ. Des coefficients de certains développements.

Examining this description, we may remark that—

a treats of permutations, and includes under γ the first elements of the theory of substitutions.

b treats of combinations in the sense usually understood in text-books.

c of functions which arise from considerations affecting a and b, with the curious addition "théorie des chemins."

d of applications to algebra and to the formation of tables.

The subject-matter of the combinatory analysis is really far more extensive than would appear from the above. The theory of permutations considers the distributions of objects, all different or not all different, into different parcels so that one and only one object is in each parcel. The number of objects may exceed the number of parcels, so that a selection of a number of parcels has to be made, or, on the other hand, the number of parcels may exceed the number of objects, and then a selection of parcels has to be made.

The theory of combinations, apparently included under b, considers the distribution of objects into similar or indifferent parcels, one object in each parcel, when in general the number of objects exceeds the number of parcels, and accordingly a selection of objects is necessary.

There are particular cases of the general theory of distribution of which a general problem may be stated as follows:—

n objects may be specified by the notation

$$(pqr \dots)$$

$$p+q+r+\dots=n,$$

meaning that p objects are of one kind, q of a second, r of a third, and so forth.

Similarly m parcels may be specified by

$$(p_1 q_1 r_1 \dots)$$

$$p_1+q_1+r_1+\dots=m$$

where $m \overset{>}{\underset{<}{=}} n$ and we have to enumerate the distributions of the objects amongst the parcels both when permutations of the objects in any one parcel are and are not permissible. The discussion of this problem under various imposed conditions, and the determination of the properties of the various distributions, appears to me the true *rôle* of the "Analyse combinatoire."

The theory of the partitions of numbers is at once seen to be properly included. The partitions of a number n into m parts are given by the distributions of n similar objects into m similar parcels, it being allowed to place more than one object in a parcel. We have, in fact, to distribute objects specified by (n) into parcels specified by (m). More generally we may distribute objects specified by $(pqr \ldots)$ into parcels specified by (m), and thus come to the consideration of the partitions of multipartite numbers.

Observe that the French classification distinctly excludes "partitions" from "Classe J" and the "Analyse combinatoire," and we find the subject placed in "Classe I 10," which deals with the theory of numbers and the higher arithmetic. A cross-reference to (F 8 c α) includes the applications of the elliptic functions to the partition of numbers.

The combinatory analysis in my opinion holds the ground between the theory of numbers and algebra, and is the proper passage between the realms of discontinuous and continuous quantity. It would appear advisable to restrict the theory of numbers to the enormous tract of country given up to the cultivation of the theories of congruences and of homogeneous forms, and to consider the theory of partitions an important part of combinatory analysis.

To show that the theory of partitions is intimately connected with other branches of combinatory analysis, it is merely necessary to remark that there is a one-to-one correspondence between the partitions of unipartite numbers and combinations of another nature of bipartite numbers. The essential character of a partition is that the order of the parts is unessential. The combinations of bipartite numbers above referred to are such that the order of parts is essential; the parcels instead of being similar are different. There is no special feature about partitions of numbers justifying a departure from allied theories universally considered to be in the domain of combinatory analysis.

The greatest method in combinatory analysis is without doubt that of symmetric functions, and this is not, I think, suitably provided for in the *Répertoire.* We find A 3 "Théorie des équations," and under *b* "Relations entre les coefficients et les racines. Fonctions symétriques des racines d'une équation; fonctions symétriques des différences des racines; équations différentielles de ces fonctions," with a cross-reference B 3 *c*, which is "Définition, formation et propriétés du discriminant." Further on we find under B 3 "Elimination; *b*. Fonctions symétriques des racines communes à deux equations."

First observe that the theory of symmetric functions is regarded as a part of the theory of equations. I think that this idea can be shown to be without foundation. The thing itself is confounded with one of its applications. It is frequently convenient to regard the quantities which are placed in symmetrical relation as being the roots of an equation, and there are certain important applications to the theory of equations, but there is no essential connexion between the two theories. There is a beautiful theory of symmetric algebra quite apart from any theory of equations which has applications, at least as important, in several other branches of mathematics. Moreover, symmetric algebra deals uniformly and easily with many sets of quantities. It is convenient to refer the quantities to a number of simultaneous equations of which they are the common roots; as a matter of convenience, not of necessity. An important application is to the theory of elimination, but this theory nowise includes that of symmetric functions of systems of quantities. The latter theory has other uses, particularly in connexion with various problems of combinatory analysis which have nothing to say to elimination. Again, in a recent scheme, the theory of symmetric functions was placed as a part of the theory of invariants—an extraordinary instance of confused classification. The general problem of combinatory analysis, enunciated above, involves the theory of distributions on a line, the line being regarded as fixed and not reversible end for end. The enumeration of distributions of the latter kind is less simple, and has not yet been completely worked out. We may also consider distributions on a fixed circle and on a fixed regular polygon—two problems with which but small progress has been made. We have also to consider the reversible circle and regular polygon—problems so far scarcely attacked.

As far as the general theory is concerned, the problems of enumeration are completely solved by identities between symmetric functions. The numbers sought are obtained as results of symmetric function multiplication. This was fully set forth in my paper, "Symmetric Functions and the Theory of Distributions." The mere association of symmetric algebra with theories of distribution leads to theorems of reciprocity and expressibility in the algebra, which, when given full generality, are of considerable complexity, and which have never been discovered in any other manner. *A posteriori* probably these theorems may be established by the method of differential operation. Up to a certain point this additional proof has already been given. The method of demonstration, however, is not so instantaneous and convincing as that which rests upon purely arithmetical reasoning. In this connexion an interesting question has arisen which has resulted in a real advance being made in the fundamental notions of combinatory analysis. It has been usual in symmetric algebra to restrict investigation to functions

$$\Sigma \alpha^p \beta^q \gamma^r \ldots$$

in which $p, q, r, \ldots$ are positive non-zero integers. Distribution theorems concerning the placing of objects in parcels are then applicable: in particular the important case in which the number of objects is equal to the number of parcels, and one object finds a place in each parcel. How are we to make the correspondence when the integers may be negative or zero as well as positive?

As I have shown elsewhere, this may be accomplished by a generalized method of distribution, the objects being comprised in an upper and in a lower group or set. The number of objects is taken to be :—

The number in upper group *minus* the number in lower group.

We thus arrive at the notion of objects enumerated by zero or any negative integer. We apply the same notion to the enumeration of parcels, and are thus able, under rules and restrictions which have been fully set forth, to arrive at a complete theory of the symmetric functions

$$\Sigma \alpha^p \beta^q \gamma^r \ldots$$

$p, q, r, \ldots$ positive, zero, or negative, connecting product identities with definite distributions in the enlarged sense.*

This method of extending the combinatory analysis to zero and negative quantity has only been partially examined, and presents an attractive field for future research. The algebra of symmetric functions, and the method of operators, being both to the fullest extent applicable, the subject is full of hope.

It will be seen that, while, on the one hand, the symmetric algebra contains implicitly the complete solution of the main problems of combinatory analysis, on the other hand, the combinatory analysis explains and greatly enlarges the theory of symmetric functions. Moreover, through the medium of symmetric functions, it enables knowledge to be advanced with regard to the algebra of differential operators. Nothing in analysis is more striking, or more astonishing, than the analogies between differential operation and pure quantity, and these so far have only been discovered, to their furthest extent, by considerations of combinatory analysis.

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* See *Amer. Jour. of Math.*, 1888–90.

The course taken usually by the stream of discovery may be described as follows:—(i.) By purely arithmetical reasoning the enumeration of certain modes of distribution is shown to be representable by a symmetric function identity; (ii.) a corresponding identity connecting differential operators is formed by reason of the analogy between the algebras of pure quantity and differential operations; (iii.) the operator identity is then used as an operation upon the symmetric function identity according to established laws, with the result of obtaining a relation between the cofficients.

This relation constitutes a theorem in algebra, and also by interpretation arithmetically a theorem in combinatory analysis. The origination of this valuable procedure is due to James Hammond, M.A.

The *general* problem, above enunciated, of distributions on a line has only been seriously attacked by the employment of symmetric algebra. Moreover, it does not appear likely that any other method can be equally effective.

On coming to attack special problems of distribution we find that special methods are available. The method of symmetric functions does not in general succeed in giving analytical expressions for the numbers under consideration. To reach this end the particular problem necessitates a particular method.

The general problem, as it affects positive integers, is concerned, as above remarked, with distributions on a line. If zero and negative integers be also included, the problem is in regard to distributions on two parallel lines.

The divisions of the subject that have hitherto principally attracted the attention of mathematicians are—

(i.) The line distributions of objects of given specification amongst different parcels, the number of objects being equal to the number of parcels.

(ii.) The line distributions of similar objects amongst an equal or lesser number of similar parcels.

The former involves the theory of permutations; the latter, the theory of partitions of unipartite numbers. The theory of permutations is so intimately connected with the theory of groups of substitutions that the requirements of the latter theory have, to a great extent, guided researches in the former. It is probably due to this cause that many investigations have been restricted to the particular case in which the objects are all different. Results in regard to the unrestricted problem have thus frequently been overlooked. As an example, may be quoted a generalization of a well-known theorem that I have recently given.

It is to find the number of permutations of n letters

$$x_1 x_2 x_3 \dots x_n$$

which possess the property that no letter is in the place originally occupied. The solution is easily obtained by the finite series

$$n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \dots \pm \frac{1}{n!} \right\}.$$

The more general problem considers the letters in the product

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

where we have to enumerate the permutations such that no letter x_s is in a position originally occupied by x_s. Here also the solution in the form of a finite series with an external factor

$$\frac{(\Sigma\xi)!}{\xi_1!\, \xi_2! \dots \xi_n!}$$

is easy to arrive at. The important points to observe are that the more general problem admits of solution by a generating function, and that this is not so in the case of the particular problem.

This arises from an algebraic theorem of some importance.

It has been shown that the coefficient of

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$$

in the product

$$(a_1 x_1 + a_2 x_2 + \dots + a_n x_n)^{\xi_1} (b_1 x_1 + b_2 x_2 + \dots + b_n x_n)^{\xi_2} \dots$$
$$\dots (n_1 x_1 + n_2 x_2 + \dots + n_n x_n)^{\xi_n}$$

is equal to the coefficient of the same term in the reciprocal of the expression

$$|\, (1 - a_1 x_1)(1 - b_2 x_2) \dots (1 - n_n x_n) \,|,$$

which is in a symbolic form, the meaning being that after multiplication a literal product $a_1 c_3 d_4 f_6 \dots$ is to be placed in determinant brackets; so,

$$|\, a_1 c_3 a_4 f_6 \dots \,|,$$

and denotes a coaxial minor of the determinant

$$\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ . & . & \dots & . \\ . & . & \dots & . \\ n_1 & n_2 & \dots & n_n \end{vmatrix}.$$

Applying this theorem to the case in hand, there is, first of all, a redundant, or crude, generating function

$$(x_2+x_3+\ldots+x_n)^{\xi_1}(x_1+x_3+\ldots+x_n)^{\xi_2}\ldots(x_1+x_2+\ldots+x_{n-1})^{\xi_n},$$

and by consideration of the determinant

$$\begin{vmatrix} 0 & 1 & 1 & 1 & \ldots \\ 1 & 0 & 1 & 1 & \\ 1 & 1 & 0 & 1 & \ldots \\ . & . & . & . & \\ . & . & . & . & \\ . & . & . & . & \end{vmatrix},$$

of order n, we readily arrive at the real generating function

$$\frac{1}{1-\Sigma x_1x_2-2\Sigma x_1x_2x_3-\ldots-(n-1)\,x_1x_2\ldots x_n};$$

an elegant result that remains concealed until the problem is given suitable enlargement.

Many theorems in regard to the permutations of a sequence of different letters require generalization for the case when the letters are not all different. The above theorem is an important instrument for effecting such an extension.

In regard to the restricted problem, much interesting work has been accomplished by M. Désiré André in his memoirs upon "Le Triangle des Séquences." He employs difference equations, a method of frequent use in investigations of like character.

Amongst permutation problems may be mentioned one that appears to be well fortified against attack.

In a restricted form, it seems to have been first dealt with by H. A. Rothe (A.D. 1800), who defines "conjugate permutations" (*verwandte Permutationen*) as follows:—

"Two permutations of the numbers 1, 2, 3, ... n are called conjugate when each number and the number of the place which it occupies in the one permutation are interchanged in the case of the other permutation."

He then states that, if U_n denotes the number of self-conjugate permutations of the first n integers,

$$U_n = U_{n-1}+(n-1)\,U_{n-2},$$

with $$U_2 = 2U_1 = 2.$$

We may consider the similar problem in regard to permutations of the product

$$x_1^{\xi_1}x_2^{\xi_2}\ldots x_n^{\xi_n}$$

with the object of obtaining the real, as distinct from the crude, generating function.

It is easy to show that for $n = 3$ the real generating function is

$$(1-x_1-x_2-x_3)^{-\frac{1}{2}}\,(1-x_1-x_2-x_3+4x_1x_2x_3)^{-\frac{1}{2}}.$$

For $n = 4$ we first obtain the redundant product

$$(x_1+x_2+\alpha x_3+\beta x_4)^{\xi_1}\,(x_1+x_2+x_3+\gamma x_4)^{\xi_2}$$
$$\times\left(\frac{x_1}{\alpha}+x_2+x_3+x_4\right)^{\xi_3}\left(\frac{x_1}{\beta}+\frac{x_2}{\gamma}+x_3+x_4\right)^{\xi_4},$$

the enumeration being given by the coefficient of $x_1^{\xi_1}x_2^{\xi_2}x_3^{\xi_3}x_4^{\xi_4}$ after terms containing α, β, or γ have been struck out. We can, as usual, substitute for the product a fraction which is free from the exponents ξ. This fraction is the reciprocal of the expression

$$1-x_1-x_2-x_3-x_4-\begin{vmatrix} 1 & 1 & \alpha \\ 1 & 1 & 1 \\ \frac{1}{\alpha} & 1 & 1 \end{vmatrix} x_1x_2x_3-\begin{vmatrix} 1 & 1 & \beta \\ 1 & 1 & \gamma \\ \frac{1}{\beta} & \frac{1}{\gamma} & 1 \end{vmatrix} x_1x_2x_4$$

$$-\begin{vmatrix} 1 & \alpha & \beta \\ \frac{1}{\alpha} & 1 & 1 \\ \frac{1}{\beta} & 1 & 1 \end{vmatrix} x_1x_3x_4-\begin{vmatrix} 1 & 1 & \gamma \\ 1 & 1 & 1 \\ \frac{1}{\gamma} & 1 & 1 \end{vmatrix} x_2x_3x_4+\begin{vmatrix} 1 & 1 & \alpha & \beta \\ 1 & 1 & 1 & \gamma \\ \frac{1}{\alpha} & 1 & 1 & 1 \\ \frac{1}{\beta} & \frac{1}{\gamma} & 1 & 1 \end{vmatrix} x_1x_2x_3x_4.$$

The isolation from this fraction of those terms in the expanded form which are free from α, β, and γ does not appear to be easy, and, in fact, has not yet been carried out. The problem for greater values of n has not yet been attempted. The fractional redundant form can always be obtained: the true, or non-redundant, forms are the desiderata.

Much remains to be done in special researches in that part of the theory of permutations where geometrical methods are available, and have been employed by several investigators.

Let us turn now to the second branch of the subject—the theory of the partitions of numbers—which has given birth to many researches.

As generally understood, the theory refers to the distributions of similar objects in similar parcels. The enlarged view of partition which is concerned with the distributions of objects, not all similar, into similar parcels has been only considered in a few notes; it is virgin soil laden with germinative seeds which only await the application of the husbandman.

There is again another nature of partitionment excluded usually from the domain of "partitions." It is that in which the parcels, instead of being similar, are all different. Such distributions, whether of unipartite or multipartite numbers, have been termed "compositions." This latter theory has only been recently developed. Compared with the theory of partitions, it is quite simple, though by no means lacking in interest, as the results are elegant, and, moreover, shed light upon the more difficult theory with which it is in somewhat intimate relationship. The numerous bonds which link "partitions" to other parts of combinatory analysis lead to the conviction that this department, rather than the theory of numbers, is its proper home. I do not overlook the fact that special problems in "partitions" are closely connected with the theory of congruences and the theory of homogeneous forms—witness the problem of the expressibility of a number as a sum of squares, and many others which could be cited—but these *are* special problems, and surely it is to the general problems that we must look when seeking to assign places in the mathematical scheme.

Since the time of Euler, the most important advances in the theory of partitions of unipartite numbers are due to Sylvester and Cayley, who appear to have been led to their investigations by the requirements of the theory of invariants—an edifice which was, in the spring and summer of 1855, rapidly rising from its foundations. It had been found that the enumeration of certain invariant forms depended upon generating functions which were algebraic fractions of the same nature as those constructed by Euler for the solution of questions in the partitions of numbers. It therefore became the duty of the mathematicians mentioned to seek theorems concerning the expansion of algebraic fractions which should not only facilitate computation of the coefficients of the expansion, but should also throw light upon the structure of those coefficients. In order to appreciate the considerable success met with, particularly by Sylvester, it is proper to glance at the previous history of the theory. Euler, in 1750, had transformed the problem of partition so as to make it dependent upon the expansion of an algebraic fraction; and he was followed by Paoli and other Italian mathematicians, without, however, much advance being made. Lacroix, 1819, and Legendre, 1830, touched upon the subject, and Jacobi, in 1846, arrived at results, of a particular character, connected with the series met with in the study of the elliptic functions. Interest in this country was aroused in the first instance by De Morgan, who, in a letter to Henry Warburton, about the year 1846, suggested that combinatory analysis stood in great need of development, and alluded to the theory of partitions. Warburton then took the matter up, and, to some extent, under the guidance of De Morgan, prosecuted researches with the aid of the theory of finite differences. He obtained accurate expressions for the enumeration of partitions in the simplest cases, as may be seen in the *Transactions of the Cambridge Philosophical Society* for the year 1849. The importance of this work lies in the fact that it drew attention to the subject in England, and inspired a valuable memoir by Sir John Herschel in the *Phil. Trans. Roy. Soc.* of 1850.

Herschel introduced the idea and notation of the "circulating function," which he refers to as his own invention thirty years previously.

He writes
$$\tfrac{1}{3}\Sigma\rho^x = s_x,$$

where ρ is an s^{th} root of unity, and consequently s_x is unity when $x \equiv 0 \bmod s$, and zero in other cases. His circulating function then consists of numerical multiples of the quantities s_x, x varying.

His formula for three-part partitions is written

$$\overset{3}{\Pi}(x) = \tfrac{1}{12}\{x^2 - 6_{x-1} - 4.6_{x-2} + 3.6_{x-3} - 4.6_{x-4} - 6_{x-5}\},$$

and he further shows how, in general, to obtain similar expressions.

He did not obtain a general formula, and his investigation, from the standpoint of later researches, is of a laborious character. It was, however, of a pioneering nature, and, to some extent, anticipated, as we shall presently see, the work of Cayley.

In 1855 the subject was attacked simultaneously by Cayley and Sylvester. The former, in his memoir of May, 1855, added the notion of prime circulators to that of the circulating functions of Herschel, and showed how by their use to construct a formula for any particular case of partitionment; but gave no general formula. On hearing of Sylvester's great results, which included general formulæ, he was able to apply Sylvester's method to his own theory of prime circulators (occupying a position similar to that of Sylvester's waves), and thus to obtain a formula which led more directly to the actual expression of the number of partitions.

This was accomplished in the memoir of 1857, and examples, illustrative of the method, given.

This work of Cayley, involving improvements and extensions of the method of prime circulators, due to the discoveries of Sylvester, was, at the time it was published, the best accessible exposition of these researches.

Denoting by $P(a, b, c, \ldots)\,q$ the number of ways in which the number q can be partitioned into the elements $a, b, c, \ldots$, each element being repeatable an indefinite number of times, we have the general form of the function $P(a, b, c, \ldots)\,q$ expressed in Cayley's notation:—

$$P(a,b,c,\ldots)\,q = Aq^{k-1}+Bq^{k-2}+\ldots+Lq+M+\Sigma q^r(A_0, A_1, \ldots A_{l-1})\operatorname{per} l_q;$$

in which k denotes the number of elements; l is any divisor of one or more of these elements (unity excluded); and the summation, in the case of each such divisor, extends from $r = 0$ to $r = k'-1$, where k' is the number of elements $a, b, c, \ldots$ of which l is a divisor. The investigations show how the coefficients A of the circulators are obtained, and they also determine the coefficients $A, B, \ldots L, M$ of the non-circulating part.

When the coefficients have been computed, the number which enumerates the partitions of q, or—the same thing—which gives the coefficient of x^q in the ascending expansion of the fraction

$$(1-x^a)^{-1}(1-x^b)^{-1}(1-x^c)^{-1}\ldots,$$

is obtained as follows:—The non-circulating part is directly calculated, and then each term $q^r(A_0, A_1, \ldots A_{l-1})\operatorname{per} l_q$ gives an added number $q^r A_\lambda$, where $q \equiv \lambda \bmod l$.

Turning now to Sylvester, we find that in the *Quarterly Journal of Mathematics*, 1855, he had given an outline of his researches, and the general formulæ, reserving fuller explanations to a later period. The beauty of this work must strike every student of the works of the masters of pure mathematics. It is incomparably the finest contribution that has ever been made to combinatory analysis. What may be regarded as a full account of the theorem was not given by Sylvester till twenty-seven years later, in Vol. v. of the *American Journal of Mathematics.*

Adopting Cauchy's use of the word "residue" to denote the coefficient of $\frac{1}{x}$ in an ascending expansion of a function of x, he establishes the following lemma:—

"The constant term in any proper algebraic fraction developed in ascending powers of its variable is the same as the residue, with its sign changed, of the sum of the fractions obtained by substituting in the given fraction, in lieu of the variable, its exponential multiplied in succession by each of its values (zero excepted if there be such), which makes the given fraction infinite."

Writing $$F(x)=\Sigma\Sigma\frac{c_{\lambda,\mu}}{(a_\mu-x)^\lambda}+\Sigma\frac{\gamma_\lambda}{x^\lambda},$$

he shows that the residue of $\Sigma F(a_\nu e^x)$, the summation being for the infinity roots a_ν of $F(x)$, is equal to the constant term in $-F(x)$.

From this it follows at once that the coefficient of x^n in a rational function $f(x)$ is equal to the residue of

$$\Sigma\left\{r^{-n}e^{nx}f(re^{-x})\right\};$$

r denoting each finite-infinity root of $f(x)$ taken in turn, provided only that $x^{-n}f(x)$ is a proper algebraic fraction.

He introduces the important notion and notation of denumeration. The coefficient of x^n in the expansion of the fraction

$$\frac{1}{(1-x^a)(1-x^b)(1-x^c)\dots(1-x^l)}$$

he terms the denumerant of n *quâ* the elements or components $a, b, c, \dots l$, or the denumerant to the equation

$$ax+by+\dots+lt=n,$$

and denotes it by $$\frac{n,}{a,\,b,\,c,\,\dots\,l}.$$

A denumerant with a single component, such as $\frac{n,}{a}$, he calls an elementary denumerant. Its value obviously is 1 or 0, according as a is, or is not, a divisor of n. He further regards $\frac{n,}{a}$ as an analytical function, defining it as the mean of the a values of ρ^n, where ρ is any root of the equation $\rho^a-1=0$. The advantage of this definition is that $\frac{n,}{a}$ preserves a meaning for negative values of n, and, as he shows subsequently, enables a more perfect expression of his results to be made.

Proceeding with the development of his theorem in the particular application to partitions, he arranges the infinity roots into groups, all those which are primitive roots of unity of the same order appearing in one group. To each group corresponds a sum of fractions which constitute a period or wave of the expansion. If r denote a primitive i^{th} root of unity, he constructs the i^{th} wave

$$\begin{aligned}W_i &= \text{residue of } \Sigma\, r_i^{-n}e^{nx}F(r_ie^{-x})\\ &= \text{residue of } \Sigma\, r_i^{n}e^{nx}F(r_i^{-1}e^{-x}).\end{aligned}$$

A great step now is to write

$$n = \nu - \tfrac{1}{2}(a+b+\ldots+l),$$

leading to $$W_i = \text{residue of } \Sigma \frac{r_i^{\nu} e^{\nu x}}{\Pi\left(r_i^{\frac{1}{2}a} e^{\frac{1}{2}ax} - r_i^{-\frac{1}{2}a} e^{-\frac{1}{2}ax}\right)},$$

the product symbol having reference to the indices $a, b, \ldots$.

The importance of this transformation lies in the fact that W_i, regarded as a function of ν, is either an even, or an uneven, function, and thus the number of terms to be calculated is reduced by one-half.

He ascertains the number of terms contained in the several waves, and gives the general form of the wave W_i, which consists of a set of products of polynomial functions of ν each multiplied by a sum of exponential quantities, consisting of pairs of the form

$$c\Sigma\left(r_i^{\nu+\delta} \pm r_i^{\nu-\delta}\right),$$

where δ is half the totient of i.

The general form with undetermined coefficients having been written down, he shows that $F(\pm\nu)$ is zero for all values of ν from 0 to $\frac{1}{2}(a+b+\ldots+l)-1$ if $a+b+\ldots+l$ be even, and from $\frac{1}{2}$ to $\frac{1}{2}(a+b+\ldots+l)-1$ if $a+b+\ldots+l$ be uneven, and that this fact alone is sufficient to give exactly the number of homogeneous equations required to determine the coefficients.

The quantity $\Sigma\left(r_i^{\nu+\delta} \pm r_i^{\nu-\delta}\right)$ is expressed in terms of elementary denumerants by calculating the sums of the powers of the roots of the equation giving the primitive roots r^i.

He gives the particular result

$$\frac{n}{1,2,3,} = \frac{\nu^2}{12} - \frac{7}{72} - \frac{1}{8}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right) + \frac{1}{9}\left(2\frac{\nu}{3,} - \frac{\nu+1}{3,} - \frac{\nu-1}{3,}\right),$$

where $$\nu = n+3,$$

which, being compared, agrees with Cayley's formula:—

$$P(1,2,3)\,q$$
$$= \tfrac{1}{72}\left\{6q^2+36q+47+9\,(1,-1)\text{ per }2_q+8\,(2,-1,-1,)\text{ per }3_q\right\}.$$

It may be stated that Sylvester completely solved the problem of enumerating the solutions of the indeterminate equation

$$ax+by+cz+\ldots+lw = n,$$

all the symbols denoting positive integers.

In the *Philosophical Magazine* for 1858, Vol. XVI., he gives a surprising generalization of the problem of partition and its solution; and shows how to represent algebraically the value of

$$\Sigma x^{\alpha} y^{\beta} z^{\gamma} \ldots,$$

where the sign of summation extends over all the simultaneous solutions of the equation

$$ax+by+cz+\ldots = n.$$

Observe that $\alpha=\beta=\gamma=\ldots=0$ gives the particular case which enumerates the solutions. The theorem is most remarkable; the more so, it appears to me, from the circumstance that it seems to have been overlooked by subsequent investigators. In Sylvester's own words:—

"This is a considerable advance upon the conception (itself before my discovery entirely unrecognised) of the explicit representability of the mere number of the solving systems $x, y, z, \ldots$ by general algebraical formulæ. By this new theorem we pass, as it were, from the shadow to the substance."

What a wonderful extension of the theory of the symmetric functions of the roots of an equation!

The theory requires further elucidation and development, and it is to be hoped that workers in our science will, now after a period of forty years, give it some attention.

It should be noted that in 1861 (*Quarterly Journal of Mathematics*, Vol. IV.) Samuel Roberts, the De Morgan Medallist of this year, gave an additional proof of Sylvester's main theorem of partitions.

In 1874 (*British Association Report*) J. W. L. Glaisher attacked the problem of the partitions of numbers in quite a novel manner. His method appears to be only applicable to the expansion of the special fractions

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots,$$

and not to those of the more general form

$$(1-x^a)^{-1}(1-x^b)^{-1}(1-x^c)^{-1}\ldots.$$

He starts with the notion of Arbogast's rule of literal derivation known as the "rule of the last and last but one," and obtains difference equations which lead easily to the simpler results, and shows the general method of treatment. Solutions of an interesting class of questions which arise in connexion with the derivations are also obtained.

Leaving the analytical theory of partitions, we will now pass to the diagrammatic method of research, brought forward by Sylvester in his "Constructive Theory of Partitions." As compared with the former theory, it is for the most part arithmetical and intuitive, and not algebraical.

In quite ancient times it was usual to depict a composite number by a rectangle of dots, and to thence establish the commutative nature of arithmetical multiplication,

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

ex. gr., $3\times 2 = 2\times 3 = 6.$

This is, in fact, what Klein would call a point-lattice. It is the representation of a number as a sum of equal parts. Sylvester, in like manner, exhibits a number as a sum of unequal parts. Thus the partition (322) of 7 is depicted by the incomplete point-lattice

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \end{array}$$

According to Sylvester, this method was first adopted by Ferrers, who also reached the generalization of commutative arithmetical multiplication by a reading by columns instead of rows.

It is of great importance to bear in mind the circumstance that the lattice is in evidence, because we are thus able to associate the theory of partitions with other branches of mathematics in which representation by lattice is a principal instrument of research.

Sylvester in his excellent paper derives many results of value in the theory of special algebraic series. He also inspired some good work of the same nature by Dr. F. Franklin and Mr. Durfee.

The dissection of a symmetrical graph into a square and two symmetrically placed appendages was imagined by Durfee, and led in Sylvester's hands to results of great consequence.

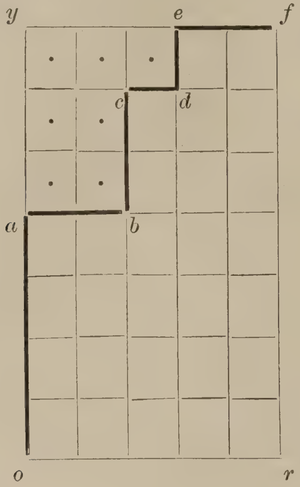
A fine portion of the paper is that which deals with the graphical conversion of continued products into series.

He thus deals with the elliptic transcendant $\Theta_1 x$, and also with Euler's continued product

$$(1-x)(1-x^2)(1-x^3)\ \ldots\ \text{ad inf.},$$

and Jacobi's series for its cube.

In a paper shortly to be published in the *Philosophical Transactions of the Royal Society*, I have shown, *inter alia*, that the graphical method, upon which Sylvester's paper is founded, involves an intimate bond of connexion between unipartite and bipartite numbers, in such wise that there is correspondence between the partitions of unipartite numbers and the compositions (*i.e.*, partitions in which the *order* of the parts is taken into account) of bipartite numbers. To see this it is merely necessary to form a complete lattice about a Sylvester-graph. *Ex. gr.*,



Here we have a Sylvester-graph of the partition (322) of the unipartite number 7. Taking 0 as origin, and forming the thick boundary line *oabcdef*, we write down the coordinates of *a* with respect to 0, of *c* with respect to *a*, of *e* with respect to *c*, and of *f* with respect to *e*, and thus reach the composition ($\overline{04}$, $\overline{22}$, $\overline{11}$, $\overline{20}$) of the bipartite number $\overline{57}$. This notion is developed in the paper (*loc. cit.*). It shows, I believe, that Sylvester's graph should be considered from the point of view of the incomplete lattice, and that it is not expedient to dissociate partitions from other branches of combinatory analysis.

Sylvester at the commencement of his paper writes: "In the new method of partitions it is essential to consider a partition as a *definite thing*." Pondering over this led me some years since to consider the partition (or, as I term it, the "separation") of a partition. Indeed this is absolutely necessary in the modern theory of symmetric functions, which as a consequence of this notion has made enormous strides. At the outset the theory of the separations of a partition involves the whole theory of the partitions of multipartite numbers. This is inevitable, since the notation

$$(lmn\ldots)$$

may serve either to denote the partition of $l+m+n+\ldots$ things all of the same kind, into groups containing l, m, n, ... things respectively, or to represent $l+m+n+\ldots$ things, l of one kind, m of a second, n of a third kind, &c. ...

The enumeration of the separations of the partition $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$, $\pi_1, \pi_2, \pi_3, \ldots$ indicating repetitions of parts, is identical with the enumeration of the partitions of the multipartite numbers

$$(\overline{\pi_1\,\pi_2\,\pi_3\,\ldots}).$$

This at present is the great question for consideration by workers at combinatory analysis.

In general, after Ferrers and Sylvester, there is no graphical method available; but in particular much may be possible. A certain class of partitions may be depicted upon an incomplete lattice in three dimensions, and theorems, analogous to those of Sylvester, evolved.

This solidification of the plane theory I have brought under consideration in the memoir alluded to, now passing through the press.

I have under view those partitions only which are graphically regularized. Such is the partition

$$(\overline{a_1a_2a_3\ldots},\ \ \overline{b_1b_2b_3\ldots},\ \ \overline{c_1c_2c_3\ldots},\ \ \overline{\ldots\ldots})$$

of the multipartite number

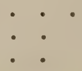
$$(\overline{a_1+b_1+c_1+\ldots},\ \ \overline{a_2+b_2+c_2+\ldots},\ \ \overline{a_3+b_3+c_3+\ldots},\ \ \overline{\ldots\ldots}),$$

when the sets of numbers

$$\begin{array}{ll}
a_1a_2a_3\ \ldots, & a_1b_1c_1\ \ldots, \\
b_1b_2b_3\ \ldots, & a_2b_2c_2\ \ldots, \\
c_1\,c_2\,c_3\ \ldots, & a_3b_3c_3\ \ldots, \\
\ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots
\end{array}$$

are all in descending order of numerical magnitude.

The Sylvester-graphs of $(a_1a_2a_3\ldots)$, $(b_1b_2b_3\ldots)$, $(c_1c_2c_3\ldots)$, ... can then be placed in successive layers so as to form a graph or incomplete lattice in three dimensions. In general each graph can be read in six ways, and we have the notions of symmetrical and semi-symmetrical graphs. The six graphs will of course be of the same total content, but will not generally appertain to the same multipartite number. Even the Sylvester two-dimensional graphs admit of six interpretations when viewed from the three-dimensional standpoint. *Ex. gr.*, the graph



represents the partitions

(322)
(331) of the unipartite number 7,

$(\overline{322})$
$(\overline{111}\ \ \overline{111}\ \ \overline{100})$ of the tripartite number $\overline{322}$,

$(\overline{331})$
$(\overline{111}\ \ \overline{110}\ \ \overline{110})$ of the tripartite number $\overline{331}$.

On attempting to enumerate the graphs of given content great difficulties present themselves. It is, indeed, not difficult to write down a crude or redundant generating function when the three dimensions are restricted in any desired manner.

The *desideratum* is the reduced generating function. The simplest case is that in which one dimension is limited to contain at most two nodes, a second dimension is limited to at most s nodes, and the third dimension is unrestricted.

Forsyth has succeeded in this case in reducing the redundant generating function. He finds the form

$$\frac{1}{1-x.1-x^2.1-x^3\ldots 1-x^s}\times\frac{1}{1-x^2.1-x^3\ldots 1-x^s.1-x^{s+1}},$$

which is in agreement with the solution of the general problem, which I predicted at an early stage of the investigation. This result and the similar ones that may be obtained show that the partitions under investigation may be represented *in plano* by two sorts of nodes —say, by black and red nodes. In other cases, more than two sorts of nodes are requisite. The problem is, so to speak, flattened out into two dimensions.

Much yet remains to be done, both in the way of rigorous demonstration of the fundamental theorems, and in applications to the theory of algebraic series.

The possibility of constructing an analytical theory of multipartite denumeration after Sylvester has only been considered by Sylvester himself. Euler had formulated the problem and so transformed it as to make it depend upon the expansion of algebraic fractions involving more than one variable, but neither he nor any other had made any progress with the solution of the question. The bipartite case which is concerned with the fraction

$$\left\{\Pi\left(1-x^{\alpha}y^{\beta}\right)\right\}^{-1}$$

was known to Euler and his contemporaries and successors as "the Problem of the Virgins," and the process of obtaining the solutions of the allied equations

$$ax+by+cz+\ldots+lw=m,$$
$$a'x+b'y+c'z+\ldots+l'w=m',$$

as "the Rule of the Virgins," or "the Rule of Ceres." Sylvester, fresh from his conquest over simple denumeration, applied himself to the question, and in the *Philosophical Magazine*, for 1858, Vol. XVI., stated he had made the discovery "that the problem of compound partition in its utmost generality is capable of a complete solution—in a word, that this problem may in all cases be made to depend on that of simple partition." He further stated that the theorem by which this is effected had been already confided to Cayley, and would "be shortly committed to the *Transactions* of one of our learned societies"—and, remarking that he will confine himself to a disclosure of the general character of the theorem without going into any details, gives a theorem in the following general statement:—

"Any given system of simultaneous simple equations to be solved in positive integers being proposed, the determination of the number of solutions of which they admit may in all cases be made to depend upon the like determination for one or more systems of equations of a certain fixed standard form. When a system of r equations between n variables of the aforesaid standard form is given, the determination of the number of solutions in positive integers of which it admits may be made to depend on the like determination for $\binom{n}{r-1}$ single independent equations derived from those of the given systems by the ordinary process of elimination, with a slight modification; the final result being obtained by taking the sum of certain numerical multiples (some positive, others negative) of the numbers corresponding to those independent determinations. This process admits of being applied in a variety of modes, the resulting sum of course remaining unaltered in value whichever mode is employed, only appearing for each such mode made up of a different set of component parts."

Further, he gives as a fundamental theorem of compound partition:—

"If there be r simultaneous simple equations between n variables (in which the coefficients are all positive or negative integers) forming a definite system (*i.e.*, one in which no variable can become indefinitely great in the positive direction without one or more of the others becoming negative), and if the r coefficients belonging to each of the same variables are exempt from a factor common to them all, and if not more than $r-1$ of the variables can be eliminated simultaneously between the r equations, then the determination of the number of positive integer solutions of the given system may be made to depend on like determinations for each of n derived independent systems in each of which the number of variables and equations is one less than in the original system. This reduction in general can be effected in a great, but limited, variety of modes. When only two equations are concerned the number of modes is always two—neither more nor less. So that, in fact, we are still navigating in the narrows, and have not fairly entered upon the wide ocean of the theory of compound partition until we have passed the case of double partition. When the given system, supposed definite, is one of three equations between four variables, the number of modes of reduction is twelve or sixteen, according to that type out of two (to one or the other of which it must of necessity belong) under which the system falls. The theory of types applicable to any system of simultaneous simple equations with rational coefficients, here faintly shadowed forth, constitutes, I apprehend, a new and important branch in the theory of inequalities."

It will scarcely be credited that Sylvester has never, in any of the scientific journals, said another word on this subject of compound partition, and all that the scientific world knows concerning the great theorems which he considered himself to have established is contained in the above short quotations from the only paper on the subject which his pen has ever written. Two more quotations from this paper are interesting:

"Thus the virgins who appeared to Euler, but with their forms muffled and their faces veiled, have not disdained to reveal themselves to me under their natural aspect."

"In the first instance I discovered the theorem above given by a method of induction, aided by an effort of imagination, and confirmed by numerous trials; but I have since obtained a very simple, although somewhat subtle, general proof of it. Mr. Cayley, on his part, and independently, has also laid the foundation of a most ingenious and instructive method of demonstration entirely distinct from my own. I reason upon the equations, Mr. Cayley upon the Eulerian generating function; but it was by operations performed upon this function that I was myself originally led to a perception of the transcendental analogies out of which I was enabled to evolve the law."

Observe that the theorem is alluded to as "above given." Obviously it was not given at all in a real sense, but only shadowed forth in its general nature. Some deplorable mischance seems to have diverted Sylvester's attention from the subject so that it was allowed to drop completely, and the promised theorems were never even enunciated. In June and July of the following year, 1859, Sylvester delivered a course of seven lectures on the "Partitions of Numbers," at King's College, London. Printed outlines of these lectures were distributed amongst his hearers and a few others who were not present; they were never published, and copies are now extremely rare. In these lectures he appears to have traversed the whole ground of compound denumeration; he employed arithmetical, algebraical, or geometrical methods as seemed to be convenient. A perusal of the "Outlines," a privilege for which I am indebted to my friend Mr. R. F. Scott, M.A., of St. John's College, Cambridge, convinces me that they are of vastly greater importance than anything that has been published on the subject of combinatory analysis. They are full of inspiration. The geometrical notions of point and ray clusters and of "aspects" are singularly beautiful. The publication, even at this late period, of these "Outlines" would rescue a great work from oblivion, and give an impulse to the whole theory, which, at the present time, it sorely needs.

One other clue may be found in the paper of Cayley (*Collected Works*, No. 255) originally published in *Phil. Mag.* (1858). This is the work alluded to by Sylvester. It is Cayley's single contribution to the theory. In it is given his own proof of a particular case of a theorem that had been communicated to him by Sylvester. The special problem of double partitions considered is the determination of the coefficient of $x^m y^\mu$ in the development of the fraction

$$\frac{1}{(1-x^a y^\alpha)(1-x^b y^\beta)(1-x^c \gamma^\gamma)\ldots},$$

under the conditions that $\frac{a}{\alpha}, \frac{b}{\beta}, \frac{c}{\gamma}, \ldots$ are unequal fractions, each in its least terms.

It appears that Sylvester reasoned from the indeterminate equations

$$ax+by+cz+\ldots=m,$$
$$\alpha x+\beta y+\gamma z+\ldots=n,$$

and, eliminating each of the r variables in succession, he wrote down

r equations of the form

$$(ab-a\beta)\,y+(ac-a\gamma)\,z+\ldots=am-a\mu.$$

He then appears to have discovered that under the above-mentioned conditions the sought denumerant might be expressed as a linear function of the denumerants of each of these r equations. Cayley establishes this by an examination of the generating function, but gives no hint either as to the method of investigation or as to the nature of the solution when the above conditions are not satisfied, and deplores the circumstance that Sylvester had only published an outline of his researches.

So the matter of compound denumeration remains for the present. The rediscovery of the theorems, which were without doubt in Sylvester's possession, is at present the desideratum in combinatory analysis. Before leaving this part of the subject I venture to suggest a method of research in multipartite denumeration which appears to give promise of leading to a general solution of the question.

The fraction $$\frac{1}{1-x^{a_1}y^{b_1}}$$

is equal to the constant term in the development of the fraction

$$\frac{1}{(1-px^{a_1})\left(1-\frac{1}{p}y^{b_1}\right)}$$

regarded as a function of p.

This is shown by the identity

$$\frac{1-x^{a_1}y^{b_1}}{(1-px^{a_1})\left(1-\frac{1}{p}y^{b_1}\right)}=1+\frac{px^{a_1}}{1-px^{a_1}}+\frac{\frac{1}{p}y^{b_1}}{1-\frac{1}{p}y^{b_1}}.$$

Hence the fraction

$$\frac{1}{(1-x^{a_1}y^{b_1})(1-x^{a_2}y^{b_2})(1-x^{a_3}y^{b_3})\ldots}$$

is the constant term in the development of the fraction

$$\frac{1}{(1-p_1x^{a_1})(1-p_2x^{a_2})(1-p_3x^{a_3})\ldots}\times\frac{1}{(1-p_1^{-1}y^{b_1})(1-p_2^{-1}y^{b_2})(1-p_3^{-1}y^{b_3})\ldots}$$

regarded as a function of $p_1, p_2, p_3, \ldots$.

Say this is $F_1(x)\times F_2(y)$.

By Sylvester's theorem we can find an expression for the coefficient of x^my^n in $F_1(x)\,F_2(y)$, for this is coefficient of x^m in $F_1(x)\times$ coefficient of y^n in $F_2(y)$, and will be of the form

$$\phi_1(p_1, p_2, p_3, \ldots)\times\phi_2(p_1, p_2, p_3, \ldots).$$

The constant term in $\phi_1\phi_2$ regarded as a function of the quantities

p_3 will then obviously be the coefficient of $x^m y^n$ in the fraction

$$(1-x^{a_1}y^{b_1})^{-1}(1-x^{a_2}y^{b_3})^{-1}(1-x^{a_3}y^{b_3})^{-1}\ldots.$$

In general, there is a similar theory applicable to multipartite denumeration, arising from the circumstance that the fraction

$$\frac{1}{1-x^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_s^{a_s}}$$

is the constant term in the development of the faction

$$\frac{1}{(1-p_1x_1^{a_1})(1-p_2p_1^{-1}x_2^{a_2})(1-p_3p_2^{-1}x_3^{a_3})\ldots(1-p_{s-1}p_{s-2}^{-1}x_{s-1}^{a_{s-1}})(1-p_{s-1}^{-1}x^{a_s})}$$

regarded as a function of $p_1, p_2, \ldots p_{s-1}$.

The principle of adding to an unfactorizable expression certain terms, always identifiable, which make the augumented expression resoluble into factors is a very important one which has been applied in other parts of combinatory analysis.

I have not yet found leisure to examine this theory except in the simplest cases. The results so far obtained are quite satisfactory, and appear likely to lead to a general theory.

Allusion should be made to the method of "Trees" which has been applied by Cayley and others to various combinatory problems; in particular to the theory of chemical compounds.

The study of combinatory analysis on a circle and on a regular polygon is likely to lead to striking results in the proper theory of numbers. The number which gives the enumeration under assigned conditions is of necessity an integer. If an analytical expression for this number can be obtained, a theorem in congruencies is almost certainly reached. Theorems which are generalizations of some of the simplest theorems of divisibility in arithmetic have in this manner been recently established. An example is the congruence

$$\Sigma\phi(d)\,p^{n/d} \equiv 0 \bmod n,$$

where p and n are any positive integers, d any divisor of n, and $\phi(d)$ the totient of d.

The subject of combinatory analysis has for forty years been much neglected; may I urge that both the beauty of the methods of research available and the value of the results achieved indicate that it is well worthy the attention of mathematicians?

In conclusion, I will quote the words of the greatest living English master of pure mathematics—Sylvester—words, which, I believe, are as applicable to the combinatory analysis as a whole as to that particular branch—the theory of partitions—in connexion with which they were spoken·

"Partitions constitute the sphere in which analysis lives, moves, and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle, and universally permeating, element of algebraical thought and expression."

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* Now being published in the *Proc. L. M. S.*, 1897.

Outlines of Seven Lectures on the Partitions of Numbers. Delivered by J. J. SYLVESTER, F.R.S., at King's College, London, during the year 1859.

PREFACE.

These outlines appertain to lectures delivered by Prof. Sylvester at King's College, London, during the year 1859. The outline of each lecture was printed shortly before its delivery and handed to those in attendance, and a few copies also were privately circulated. They are now published for the first time. The Professor's attention was called away shortly afterwards to another department of mathematics, with the result that his researches on compound partitions were never published. As the lectures constitute the only serious attempt that has ever been made to deal with the subject, and as copies of the outlines are very scarce, Prof. Sylvester has yielded to the suggestion made to him in regard thereto by the Council of the London Mathematical Society, so far as to assent to their publication in the *Proceedings*, with all their imperfections on their heads. The present state of his health and the long lapse of time combine to render any revision upon the part of the Professor impossible. He desires it to be known that he cannot vouch for the correctness of all that appears in the notes, and that they were prepared in a hand-to-mouth manner during the process of investigation between the lectures, and that it is only on the opinion of the Council urgently expressed to him that the work should not entirely perish that he has consented at this late hour to the publication.

The Council desires to acknowledge the assistance it has derived from Prof. H. W. Lloyd Tanner, of University College, Cardiff, who kindly placed his annotated copy of the outlines at its disposal, and also to Mr. R. F. Scott, of St. John's College, Cambridge, who presented a copy to the Society.

FIRST LECTURE.*

INTRODUCTORY REMARKS.

Resolution of an integer into parts.

Resolution of an integer into parts limited in number.

Resolution of a number into parts limited in magnitude.

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* Delivered at King's College, London, on the 6th June, 1859.

Euler's law of reciprocity, viz.,

As many ways as an integer n can be resolved into parts not exceeding m in number, so many ways can it be resolved into parts not exceeding m in amount.

Ferrers' Proof.—Example.—$n=5,\ m=3$,

$$\left.\begin{array}{llll} 111 & 11 & 11 & 1\\ 11 & 11 & 1 & 1\\ & 1 & 1 & 1\\ & & 1 & 1\\ & & & 1\end{array}\right\}\begin{array}{llll} \multicolumn{4}{c}{\text{may be read as}}\\ 3, 2; & 2, 2, 1; & 2, 1, 1, 1; & 1, 1, 1, 1, 1;\\ \text{or } 2, 2, 1; & 3, 2; & 4, 1; & 5.\end{array}$$

Cayley's application of this law to the calculation of groups of symmetric functions.

Example.—To find Σx^5, Σx^4y, Σx^3y^2, Σx^2y^2z, where x, y, z are roots of $x^3-p_1x^2+p_2x-p_3=0$, $p_1\cdot p_1\cdot p_1\cdot p_1\cdot p_1$, $p_2\cdot p_1\cdot p_1\cdot p_1$, $p_2\cdot p_2\cdot p_1$, $p_3\cdot p_2$ will be linear functions of the quantities to be found.

Euler, Waring, Paoli, De Morgan, Warburton, Herschel, Kirkman, Ferrers, Cayley, in connexion with question of resolution.

The resolution of a number into parts is the problem of ascertaining the different modes of composing n with the elements

$$1, 2, 3, \ldots \text{ up to } n.$$

General problem of *simple* partition is to find in how many ways a given number n can be composed of given elements a, b, c, ... k.

General problem of *binary* partition is to find in how many ways the couple m, m' can be composed of the couples a, a', b, b', c, c', k, k'.

Statement of problem under form of equations.

Denumeration and denumerant defined.

Denumerant of $U=0$ same as that of $kU=0$.

Denumerant of $U=0$, $V=0$ same as that of

$$kU+lV=0, \quad k'U+l'V=0.$$

Coefficient groups and constant group defined.

How the resolution of an equation or system of equations with any *real* coefficients may be made to depend on the inverse problem of the centre of gravity of a system of points.

Example.—A system of two equations.

Origin, coefficient points, primary defined.

Total of coefficient points is called a cluster.

Coefficient points may be denoted by the variables to which they belong.

Weighted cluster.

Weight of primary assumed to be positive unity.

If primary and cluster balance about the origin, the weights at the several points of cluster will satisfy the given system of equations.

Linear cluster; plane cluster; solid cluster.

The cluster origin and primary may be considered apart from the axes used in the construction.

Ray cluster; axis of cluster defined.

Derivative of an equation-system. An equation-system really consists of the universe of its derivatives.

How this universe is contained in the geometrical representation of the system.

A principal derivative of a binary system is the equation resulting from the elimination of any *one* of its variables.

A principal derivative of a ternary system is the equation resulting from the elimination of any *two* of its variables.

Universe or Plexus of Principal Derivatives.

How to construct geometrically the principal derivatives by aid of the cluster, primary, and origin.

(1) For binary system.

(2) For ternary system.

We can thus perform the process of elimination geometrically.

If more than the regular number of variables can be eliminated simultaneously out of the system, this will be evidenced in the plane cluster by three or more points lying in a line, and in the solid cluster by four or more points lying in a plane.*

An equation is said to be homonymous when the coefficients of the variables are all positive or all negative.

It may be congruous or incongruous.

Example.— $2x+3y+4z=10$ congruous,

$2x+3y+4z=-10$ incongruous.

An *omni-positive* solution of an equation or system means a solution in which the variables are all positive.

An *omni-negative* solution is one in which the variables are all negative.

A *homonymous* solution is one which is either omni-positive or omni-negative.

An equation or equation-system may be *definite* or *indefinite.*

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* The general polyhedron *in solido* analogous to the polygon *in plano* is a polyhedron with triangular faces exclusively.

Indefinite when homonymous solutions can be found wherein the variables may be made indefinitely great.

Definite when the variables cannot be made indefinitely great in any homonymous solution.

The equations $ax-by=m$ and $ax+by-cz=m$, where $a, b, c, \dots m$ are any real positive quantities whatever, are indefinite.

The character as to definite or indefinite depends only on the coefficients, and not on the constant term.

A single equation to be definite must be homonymous.

A system of equations to be definite must admit of a homonymous derivative.

If it admit of one, it must admit of an infinite number of such.

Definiteness and indefiniteness of systems depend only on the relative values of coefficients, and not on the constant terms.

Hence, the relative position of origin and cluster must suffice geometrically to determine this character.

Definition of boundary of a plane or solid cluster of points.

Lemma.—The centre of gravity of any weighted cluster is contained inside the boundary, and may be made to lie at any point within it by a due adjustment of the relative magnitudes of the weights at the several points.

Theorem.—If the origin lies within the cluster, the system is indefinite; if outside, definite.

In Fig. 1 the centre of gravity of the cluster may be brought to the position g or g' as near as we please to O on either side of it in a line with PO, and, the sum of the weights

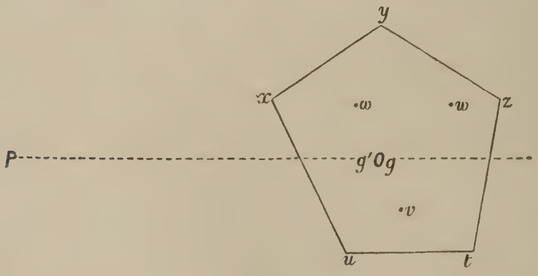

FIG. 1.

$$x+y+z+t+u+v+w+\omega$$

being $\frac{PO}{gO}$ or $-\frac{PO}{g'O}$, may be made indefinitely great either on the positive or negative side of zero.

In Fig. 2, if origin is at O, Σx will lie between $\frac{PO}{gO}$ and $\frac{PO}{g'O}$; if origin is at O', Σx will lie between $-\frac{PO'}{gO'}$ and $\frac{PO'}{g'O'}$;* if at O'', the system cannot by any system of weights, all positive or all negative, be made to balance the weight at P about the origin.

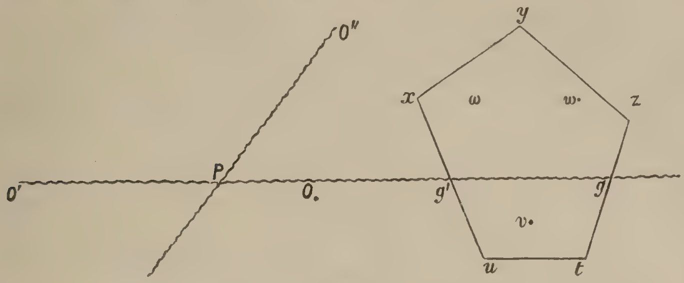

FIG. 2.

The same method is applicable to points *in solido.*

Indefinite systems in general admit of homonymous solutions of both kinds.

The only case of exception is when the origin is in the contour of cluster.

Definite systems admit only of solutions of one kind.

Three Species of Definite Systems.

The system is *positive* or *negative* when the axis cuts the cluster according as primary or cluster lie on opposite or same side of origin.

It is *neuter* when the axis does not cut the cluster. (*Ex. gr.*, origin O''.)

An analytical determination of the genus and species of a system may be deduced from the preceding construction.

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* Hence it may easily be shown that the greatest and least values of Σx in any definite systems of equations

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots = m,$$
$$a_1'x_1 + a_2'x_2 + a_3'x_3 + \ldots = m',$$
$$a_1''x_1 + a_2''x_2 + a_3''x_3 + \ldots = m'',$$

will be the greatest and least values of ρ deduced successively from all the equations that can be formed after the type of the following :—

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ a_1' & a_2' & a_3' & 1 \\ a_1'' & a_2'' & a_3'' & 1 \\ m & m' & m'' & \rho \end{vmatrix} = 0,$$

and in like manner we may derive from the geometrical method a simple rule for determining algebraically the maxima and minima values of each separate variable.

Binary System.

In the indefinite case, if we draw lines from origin to every point in cluster, each such ray divides the cluster into two parts.

In the definite case there are two extreme rays leaving all the points in the cluster on the same side.

Hence, if a system is indefinite, the universe or plexus of principal derivatives will contain no homonymous equations.

If it be definite, it will contain two homonymous equations.

Again, as regards species—

If the system is positive definite, the two homonymous derivatives will be both congruous. If the system is negative, they will be both incongruous.

If the system be neuter, the homonymous will be one congruous, the other incongruous.

Ternary System.

If the system is indefinite, all the planes through the origin and any two points of the cluster divide the cluster into two parts.

If it be definite, the bounding planes of the pyramid formed by joining the origin with each point of the cluster will leave the other points of cluster all on one side.

Hence, when the system is indefinite, the plexus of principal derivatives will contain no homonymous equations; when it is definite, there will be some homonymous derivatives, and the number cannot be less than three or greater than the number of variables. For, if we project the cluster from the origin on a plane cutting the rays all on the same side of origin, the number of sides in contour of this projection will be the number of planes in the pyramid, and n points in a plane cannot form a figure bounded by less than three nor more than n sides.*

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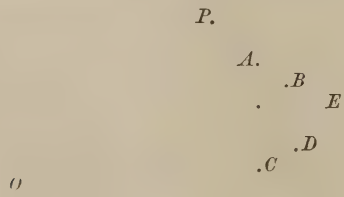
* Thus we see in like manner that the number of homonymous principal derivatives to a definite quaternary system of n variables is some number intermediate to 4 and q (where q is the number of faces in a triangular polyhedron with n summits), *i.e.*, $2n-4$.

This would be difficult to prove by a direct analytical process.

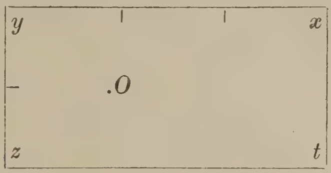
N.B.—In any neuter binary system of which O is origin, P the primary and $ABCDE$ the



The different species of definite will be distinguishable by the homonymous principal derivatives being all congruous, all incongruous, or partly congruous, partly incongruous.

Examples of *indefinite* two-equation systems.—

Let the cluster of coefficient points be at the four angles of a parallelogram x, y, z, t, the origin O being at the distance of one unit from yz, yx, zt, and two units from xt.



The system of equations will be

$$\begin{aligned} 2x-y-z+2t &= n, \\ x+y-z-\ t &= m. \end{aligned}$$

The universe of principal derivatives will be

$$\begin{aligned} 3y-\ z-4t &= 2m-\ n, \\ 3x-2z+\ t &= \ m+\ n, \\ -x+2y-3t &= \ m-\ n, \\ 4x+\ y-3z &= \ n+2m, \end{aligned}$$

all of which are heteronymous or indefinite, showing that the system is indefinite.

If y, z, x, t were a square, y, t as well as x, z would be brought into line with O; equations would become

$$\begin{aligned} x-y-z+t &= m, \\ x+y-z-t &= n, \end{aligned}$$

and, on account of these two syzygies, there would be only two principal derivatives, viz.:

$$\begin{aligned} 2y-2t &= n-m, \\ 2x-2z &= n+m. \end{aligned}$$

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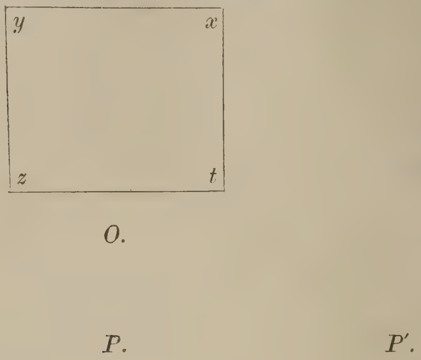
cluster. All the triangles OPA, OPB, &c., following the same order of rotation will represent determinants of the same sign.

This cannot be the case for definite positive or negative, or for indefinite systems.

Hence the neuter case may be recognised by the determinants obtained by conjugating *in situ* each coefficient group in succession with the constant group, never changing sign.

Examples of Definite Systems.[*]



Positive Case.—Take x, y, z, t at the angles of a square, two units each way (breadth and depth).

Let the origin O be at an equal distance from z and t, and from x and y and the primary P in a line with xt. The system referred to OP and OQ at right angles to OP as axes of moment gives rise to the equations

$$x-y-z+t=0,$$

$$(1+c)\,x+(1+c)\,y+cz+ct=m,$$

c being the distance of O from zt, and m its distance from P. The extreme rays being Oz and Ot, the two homonymous principal deri-

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[*] In order that a system may be definite the points of the cluster, whether in line, plane, or solid, must be all *in front* to an eye at the origin. In the last two cases accordingly, a line or a plane may be drawn through the origin, leaving the cluster entirely on one side. Now, as a line in a plane will cut three out of any four quadrants in the plane made by two intersecting lines, and a plane *in solido* will cut seven out of any eight octants made by three intersecting planes, it follows that a binary system *may* be definite when of the four possible combinations of signs affecting the terms of the several coefficient groups, *i.e.*, $\begin{smallmatrix}+&+&-&-\\+&-&+&-\end{smallmatrix}$, three are found among the several groups, and so a ternary system may remain definite even when out of the 8 possible combinations of signs

$$\begin{array}{cccccccccc}+&+&+&+&&&-&-&-&-\\+&+&-&-&&&+&+&-&-\\+&-&+&-&&&+&-&+&-\end{array}$$

all but one are found among the several coefficient groups.

We may then safely infer that, in general, all but one of the possible combinations of sign may occur in the coefficient groups of any system without the system necessarily ceasing to be definite. But, if all possible combinations occur, the system will be necessarily indefinite.

vatives will be the two resultants in respect to z and t, *i.e.*,

$$(1+2c)\,x+y+2ct=m,$$
$$x+(1+2c)\,y+2cz=m,$$

both of which are congruous.

Negative Case.—Figure the same as the preceding, but position of P *reversed* (*i.e.*, passed through origin to an equal distance from it on the other side). The equations will be as above, with the exception of m becoming negative, so that the two homonyms will be incongruous.

Neuter Case.—Same figure as above, but the primary P moved horizontally through d to P' lying to the right of zO produced. This condition implies that $\frac{m}{d}<c$ or $m<cd$.

The two equations now become

$$x-y-z+t=d,$$
$$(1+c)\,x+(1+c)\,y+cz+ct=m,$$

and the two homonyms are

$$(1+2c)\,x+y+2ct=m+cd,$$
$$x+(1+2c)\,y+2cz=-(cd-m),$$

of which the first is congruous, the second incongruous, thereby indicating that the system is neuter.

The determinants

$$\begin{vmatrix} 1 & d \\ 1+c & m \end{vmatrix} \qquad \begin{vmatrix} -1 & d \\ 1+c & m \end{vmatrix} \qquad \begin{vmatrix} -1 & d \\ c & m \end{vmatrix} \qquad \begin{vmatrix} 1 & d \\ c & m \end{vmatrix}$$

i.e. $\quad m-cd-d, \qquad -m-d-cd, \qquad -m-cd, \qquad m-cd,$

being all negative, would also have served to prove the system to be neuter.

Scholium.—If our equation or equation-systems be now supposed to be integer equations, we see that the denumerant will be in all cases zero, if the system be negative or neuter. If it be indefinite, the denumerant in general will be infinite (according to the known theory of numbers), but it may be zero, viz., in the case where the coefficients of any equation or derived equation of the given system have a common factor which is not a factor of the constant term.

The plexus of principal derivatives affords an absolute criterion for determining whether the denumerant of a given indefinite system of equations is infinite or zero.

SECOND LECTURE.*

Definition of denumerant recalled,

$$\text{ᗡ}U, \quad \text{ᗡ}\,(U, V), \quad \text{ᗡ}\,(U, V, W),$$

used as implicit symbols of denumeration.

$$\text{ᗡ}U \text{ in its explicit form } \frac{n\,;}{a,\ b,\ c, \ldots\ l\,;}$$

$$\text{ᗡ}\,(U, V) \quad \text{,,} \quad \text{,,} \quad \frac{n,\ n'\,;}{a,\ a'\,;\ b,\ b'\,;\ c,\ c'\,;\ \ldots\,;\ l,\ l'\,;}$$

&c. &c.

Numeratives and denominatives defined.

Herschel's symbol r_n explained.

Its value as a linear function of n^{th} powers of the r^{th} roots of unity.

$$r_n \text{ in the new theory will be replaced by } \frac{n\,;}{r,}$$

Observation.— $\frac{n\,;}{r\,;} + \frac{n-1\,;}{r\,;} + \ldots \frac{n-(r-1)\,;}{r\,;} = 1.$

More generally,

$$\frac{n\,;}{r'r\,;} + \frac{n-r\,;}{r'r\,;} + \frac{n-2r\,;}{r'r\,;} + \frac{n-(r'-1)\,r\,;}{r'r\,;} = \frac{n\,;}{r\,;}$$

for, if $\frac{n}{r}$ is fractional, so is $\frac{n}{r}-1$, $\frac{n}{r}-2$, ...,

and *a fortiori*,

$$\frac{1}{r'}\cdot\frac{n}{r}, \quad \frac{1}{r'}\left(\frac{n}{r}-1\right), \quad \frac{1}{r'}\left(\frac{n}{r}-2\right), \quad \ldots \quad \frac{1}{r'}\left(\frac{n}{r}-r'-1\right),$$

and, if $\frac{n}{r}$ is integer, one and only one of the above quantities will be an integer.

What $E\left(\frac{n}{r}\right)$ is commonly used to denote; Herschel's notation $\frac{n}{r}$.

$$E\left(\frac{n}{r}\right) = \frac{(n-r)\,;}{1,\ r\,;}$$

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[*] Delivered at King's College, London, on the 9th June, 1859.

Examples.— $x+2y=7, \quad E\left(\frac{7}{2}\right)=3,$

$$x+3y=8, \quad E\left(\frac{8}{3}\right)=2,$$

$$x+5y=15, \quad E\left(\frac{15}{5}\right)=3.$$

N.B.—In the partition theory, zero always counts as a *positive* integer.*

Of course the residue of n to modulus r is

$$n-r\times\frac{(n-r);}{1,\, r;};$$

but it may also be expressed as a binary denumerant.

Simplest class of indeterminate equations

$$x_1+x_2+\ldots+x_r-n=0,$$

$$\frac{n}{1}=1, \quad \frac{n}{1,\,1}=n+1, \quad \frac{n}{1,\,1,\,1}=\frac{(n+1)(n+2)}{2}, \text{ \&c.}$$

Generally $\text{Ȼ}U$ in above equation is coefficient of t^n in $\dfrac{1}{(1-t)^r}$.

The denumerant of the equation

$$ax_1+ax_2+\ldots+ax_r-n=0,$$

$$\frac{n;}{a;}\times\left\{\frac{n+a}{a}\cdot\frac{n+2a}{2a}\ldots\frac{n+(r-1)\,a}{(r-1)\,a}\right\}.$$

Provisional Method of Simple Denumeration.

Any simple denumerant may be expressed in terms of denumerants of the class last treated of.

Example 1.— $x+2y=n;$

x must be of the form 2ξ or $2\xi+1$, two suppositions mutually exclusive.

Hence the denumerant of the given equation is the sum of those of the two equations,

$$2\xi+2y=n \quad \text{and} \quad 2\xi+2y=n-1.$$

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* Consequently, the equation $ax+by+cz\ldots=0$ has the denumerant *unity*, and is not neuter; there being in fact no neuter cases for *simple* partition.

Hence $$\frac{n\,;}{1,\,2\,;}=\frac{n\,;}{2,\,2\,;}+\frac{n-1\,;}{2,\,2\,;}$$

$$=\frac{n\,;}{2\,;}\times\frac{n+2}{2}+\frac{n-1\,;}{2\,;}\times\frac{n+1}{2}.$$

But $$\frac{n\,;}{2\,;}+\frac{n-1\,;}{2\,;}=1.$$

Hence $$\frac{n\,;}{1,\,2\,;}=\frac{2n+3}{4}+\left\{\frac{n\,;}{2\,;}-\frac{(n-1)\,;}{2\,;}\right\}\frac{1}{4}.$$

Observe that $$\left\{\frac{n\,;}{2\,;}-\frac{(n-1)\,;}{2\,;}\right\}\frac{1}{4}=(-)^n\frac{1}{4}.$$

Example 2.— $$x+3y=n\,;$$

x must be of form 3ξ, or $3\xi+1$, or $3\xi+2$.

Hence $$\frac{n\,;}{1,\,3\,;}=\frac{n\,;}{3,\,3\,;}+\frac{(n-1)\,;}{3,\,3\,;}+\frac{(n-2)\,;}{3,\,3\,;}$$

$$=\frac{n\,;}{3\,;}\cdot\frac{n+3}{3}+\frac{(n-1)\,;}{3\,;}\cdot\frac{n+2}{3}+\frac{n-2\,;}{3\,;}\cdot\frac{n+1\,;}{3\,;}$$

$$=\tfrac{1}{3}\left[(n+2)+\left\{\frac{n\,;}{3\,;}-\frac{(n-2)\,;}{3\,;}\right\}\right].$$

Example 3.—To find the denumerant of

$$x+2y+4z=n.$$

4 is the least common multiple of 1, 2, 4.

x is either 4ξ, $4\xi+1$, $4\xi+2$, or $4\xi+3$,

$2y$ is either 4η, or $4\eta+2$,

$4z$ is $4z$.

Thus there are eight cases, each giving rise to an equation of the form

$$4\xi+4\eta+4z+c=n.$$

I combine together those in which the constant on the left side is either the same, or leaves the same residue when divided by 4.

Thus, we obtain

$$\begin{aligned}\frac{n;}{1,\,2,\,4;} = {} & \frac{n;}{4,\,4,\,4;} + \frac{n-4;}{4,\,4,\,4;} \\ & + \frac{n-1;}{4,\,4,\,4;} + \frac{n-5;}{4,\,4,\,4;} \\ & + 2\,\frac{n-2;}{4,\,4,\,4;} + 2\,\frac{n-3;}{4,\,4,\,4;}\end{aligned}$$

and observing that

$$\frac{n-4;}{4;} = \frac{n;}{4;} \quad \frac{n-5;}{4;} = \frac{n-1;}{4;}$$

we obtain

$$\begin{aligned}\frac{n;}{1,\,2,\,4;} = {} & \frac{n;}{4;}\left\{\frac{(n+4)(n+8)}{4.8} + \frac{n\,(n+4)}{4.8}\right\} \\ & + \frac{n-1;}{4;} \cdot \left\{\frac{(n+3)(n+7)}{4.8} + \frac{(n-1)(n+3)}{4.8}\right\} \\ & + \frac{n-2;}{4;} \cdot \frac{(n+2)(n+6)}{4.4} \\ & + \frac{n-3;}{4;} \cdot \frac{(n+1)(n+5)}{4.4}\end{aligned}$$

$$\begin{aligned}= \frac{1}{16}\left\{\frac{n;}{4;}(n^2+8n+16) + \frac{n-1;}{4;}(n^2+6n+9) + \frac{n-2;}{4;}(n^2+8n+12)\right. \\ \left. + \frac{n-3;}{4;}(n^2+6n+5)\right\}.\end{aligned}$$

By aid of the identities,

$$\frac{n}{4} + \frac{n-1;}{4;} + \frac{n-2;}{4;} + \frac{n-3;}{4;} = 1,$$

$$\frac{n;}{4;} + \frac{n-2;}{4;} = \frac{n;}{2;} \quad \frac{n-1;}{4;} + \frac{n-3;}{4;} = \frac{n-1;}{2;}$$

the above equation becomes

$$\frac{n;}{1,\,2,\,4;} = F + \left(\frac{n;}{2;} - \frac{n-1;}{2;}\right) G + \left(\frac{n;}{4;} + \frac{n-1;}{4;} - \frac{n-2;}{4;} - \frac{n-3;}{4;}\right) H,$$

where $\quad F = \frac{1}{16}\left(n^2 + 7n + \frac{21}{2}\right); \quad G = \frac{2n+7}{32}, \quad H = \frac{1}{8}.$

F will be the mean value of the transcendental function $\frac{n;}{1,\,2,\,4;}$.

The mean value of any simple denumerant by virtue of the theorem discovered by the lecturer is always expressible directly as an algebraical function of n, and of the quantities $a_1, a_2, \ldots a_r$ left perfectly indefinite.

Observe that in the multipliers of G and H the sums of the coefficients are all zero.

General direct method of expressing every simple denumerant under a simple form is furnished by theorem above referred to.

The method above given substantially consists in making the denumeration of $a_1x_1+a_2x_2 \ldots +a_rx_r = n$ depend on finding all the solutions of the congruence

$$a_1u_1+a_2u_2+a_3u_3 \ldots +a_ru_r-u_{r+1} = 0 \text{ to modulus } K,$$

K being the least common multiple of $a_1, a_2, \ldots a_r$, and $u_1, u_2, \ldots u_r$ being all limited to be positive integers less than K, but u_{r+1} being left indefinite.

Thus the numbering of the solutions in positive integers of an equation can be brought to depend upon finding the solutions themselves of a congruence in positive integers.

Euler's Method of Generating Fractions.

The denumerant of $ax+by+cz+\ldots+lt = n$

is the coefficient of t^n in the expansion of

$$\frac{1}{(1-t^a)(1-t^b)\ldots(1-t^l)}$$

expanded in ascending powers of t.

Proof that the product of the series generated by $\frac{1}{1-t^a}\cdot\frac{1}{1-t^b}$, &c., gives $\frac{n;}{a, b, c, \ldots l;}$ as the coefficient of t^n.

Note that when $n=0$ the coefficient of t^n is 1.

Thus we see that the denumerant of $x_1+x_2+\ldots+x_r = n$ is the coefficient of t^n in $\frac{1}{(1-t)^r}$ as already found.

Necessity of attending to the *order* of terms in the denominators of generating fractions; $\frac{1}{p-q}$ and $\frac{1}{-q+p}$ distinguished; $\frac{1}{p\sim q}$ may be used to signify one or the other of the two previous forms, the choice being left subject to ulterior determination.

Euler's generating fraction continues to hold good even when any of the coefficients become negative, the expansion becoming *indefinite.*

Example.—The denumerant of $x-y=n$ is generated by the product of $\frac{1}{1-t}$ by that of $\frac{1}{1-t^{-1}}$, that is to say, of the series

$$1+t \quad +t^2 \quad +t^3 \quad + \dots \textit{ ad inf.},$$

by the series

$$1+t^{-1}+t^{-2}+t^{-3}+ \dots \textit{ ad inf.}$$

This product will consist of an ascending and descending branch, and the coefficients of every term in each branch will be infinite, showing that the denumerant of $x-y=n$ is infinite for all integer values of n whether positive or negative.

The cognate forms to a generating fraction defined.

Their number, if there are r factors in the denominator, is 2^r.

In above example $\frac{1}{(1-t)(1-t^{-1})}$ generates a double indefinite development, but the cognate form $\frac{1}{(1-t)(-t^{-1}+1)}$ will generate a series in which the indices of t ascend from 1 to ∞, and the coefficients for any finite value of an index remain finite.

So in general for $\frac{1}{(1-t^a)(1-t^{-b})}$; the coefficient of t^n in a cognate form to this is the coefficient of t^{n-b} in $\frac{1}{(1-t^a)(1-t^b)}$ with the sign changed. So again the coefficient of t^n in a cognate form to

$$\frac{1}{(1-x^a) \dots (1-x^b)(1-x^{-c})(1-x^{-d})}$$

will be the coefficient of t^{n-c-d} in

$$\frac{1}{(1-x^a) \dots (1-x^b)(1-x^c)(1-x^d)},$$

and so on.

Every generating fraction to a single equation contains two cognate forms (of which itself may be one), which admit of development in series with *finite* coefficients. One of these will be purely an ascending, the other purely a descending, series.

The coefficient of t^n in the ascending development I call the connumerant of the equation.

The connumerant is always finite; it may be positive or negative; when the coefficients are all positive the connumerant and denumerant are identical.

The meaning of the symbol $\dfrac{n\,;}{a_1,\, a_2,\, \ldots\, a_r\,;}$ extended and modified.

Rule for transforming a connumerant with some or all of its denominatives negative into one with all its denominatives positive.

Why connumerants are necessary.

When the numerative is a negative quantity the connumerant by virtue of the definition is always zero.

The denumerant of a binary system of equations

$$ax+by+cz+\ldots = m,$$

$$a'x+b'y+c'z+\ldots = m',$$

is the coefficient of $t^m t^{m'}$ in

$$\frac{1}{(1-t^a.t'^{a'})(1-t^b.t'^{b'})(1-t^a.t'^{c'})\ldots}.$$

Unnecessariness of the limitation imposed by Euler upon the signs of the coefficients.

How to exhibit geometrically, the limiting ratios to the values of the indices of t and t' which can appear in the development of the Eulerian fraction containing t and t'.

Hence we see that the series generated by such an Eulerian may consist of a single branch, or of two branches, or of three branches.

So the Eulerian of a definite ternary system developed may have any number of branches from one to seven inclusive.

Definition.—A determinate series is one in which none of the coefficients of terms at a finite distance from the origin become infinite in value. A determinate generating function is one which generates a determinate series.

Then $\dfrac{1}{(1-t)^2}$ is determinate, but $\dfrac{1}{(1-t)(t-1)}$ indeterminate.

So, again, $\dfrac{1}{(1-t)(u-1)(ut-1)}$ is determinate, but

$$\frac{1}{(1-t)(u-1)(t-u)} \text{ indeterminate.}$$

Denumerative function distinguished from denumerant.

Reversal of a point in a cluster defined.

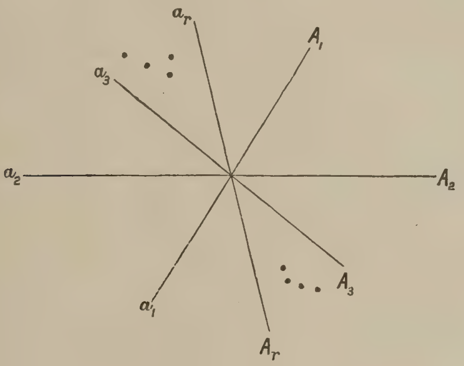
If we change any factor of an Eulerian of the second order

$$\frac{1}{1-t^a.t'^{a'}} \text{ into } \frac{1}{-t^a.t'^{a'}+1},$$

the equation-system by the denumeration of which the coefficient of $t^m t'^m$ may be calculated undergoes a change in its constant terms as well as in its coefficients; but it is only the change in the latter which can influence the character of the system as to being definite or indefinite, and consequently the character of the coefficients of the developed Eulerian as to being finite or infinite.

Hence, it is easy to show geometrically that $2r$ out of the 2^r cognate forms to an Eulerian fraction of the 2nd order will give rise to series with finite coefficients.

For if there be r variables the total number of cognate forms will correspond to the 2^r



clusters consisting of A_1 or a_1 (its reverse), combined with A_2 or a_2 its reverse, with A_3 or a_3 its reverse, and so on.

Now of all these clusters the only ones which do not enclose the origin are the pairs

$$\left.\begin{array}{l} A_1A_2A_3 \ldots\ldots A_r \\ a_1A_r \ldots\ldots A_3A_2 \end{array}\right\},$$

$$\left.\begin{array}{l} A_2A_3 \ldots\ldots A_ra_1 \\ a_2a_1A_r \ldots\ldots A_3 \end{array}\right\},$$

and so on, there being as many pairs of clusters outside the origin as there are points $A_1, A_2, \ldots A_r$.

In like manner an Eulerian fraction of the 3rd order and with r factors in its denominator will admit of as many cognate pairs of forms generating series with finite coefficients as there are combinations of r elements, 2 and 2 together, *i.e.*, $\frac{r\,.\,r-1}{2}$ pairs, and so on, for any order whatever.

Hence it would not be possible without further specification to extend the definition of connumerants (if it were wished to do so) from simple equations to equation-systems.

Happily the necessity for the consideration of such does not arise, as it will be shown that denumerants of all orders may be expressed in terms of *simple* connumerants.

By the connumerant to

$$-ax-by-cz+dt+eu, \text{\&c.} = K,$$

I shall understand the expression

$$\frac{K;}{-a,\ -b,\ -c,\ d,\ e,\ \ldots;}$$

This connumerant will be the same save as to sign (which is or is not to be changed, according as the number of negative coefficients $-a$, $-b$, $-c$ is odd or even) as the denumerant of

$$a(x+1)+b(y+1)+c(z+1)+dt+eu, \text{\&c.} = K.$$

THIRD LECTURE.*

Reduction.

Reduction explained.

Reduction in partitions analogous to elimination in equations.

A prime group defined. Examples.

Syzygy of variables; predicable also (elliptically) of groups.

In a plane cluster, syzygy is evinced by two or more points being in a line with the origin.

In a solid cluster, by three or more points being in the same plane with the origin.

Analytical condition of two groups in a binary system being in syzygy is that the determinant formed by their coefficients vanishes.

Analytical condition of three variables in a ternary system being in syzygy is same as above; and so in general.

If $ab'-a'b=0$,

$$\frac{a}{b}=\frac{a'}{b'};$$

and, if a, b is a prime group, and also a', b', either

$$\begin{matrix} a=a' \\ b=b' \end{matrix} \quad \text{or} \quad \begin{matrix} a=-a' \\ b=-b'. \end{matrix}$$

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* Delivered at King's College, London, on June 16th, 1859.

On the latter supposition, the system would be indefinite (for the origin would either lie *on* the contour of the cluster or *within* it).

Hence two non-identical prime groups cannot be in syzygy.

The same will be true of three non-identical prime groups in a ternary system.

If, in a definite binary system, each of a certain set of groups is a prime group, and no two of the groups the same, the system will be asyzygetic so far as this set of groups or their variables is concerned.

Importance of the case of equal, *i.e.*, identical, coefficients or coefficient groups.

The symmetric functions of the roots of indeterminate equations may be expressed as denumerants to equations or equation-systems with equal coefficients or coefficient groups.

Scheme: its definition as collective name for cluster and primary.

Scheme: linear, plane, or solid.

Centre: Axis: Balancing plane of scheme: denumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given centre or axis; of a solid scheme in respect to a centre, axis, or plane.

Connumerant of a linear scheme in respect to a given centre; of a

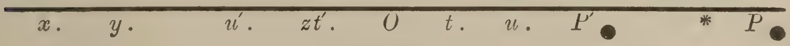

FIG. 1.

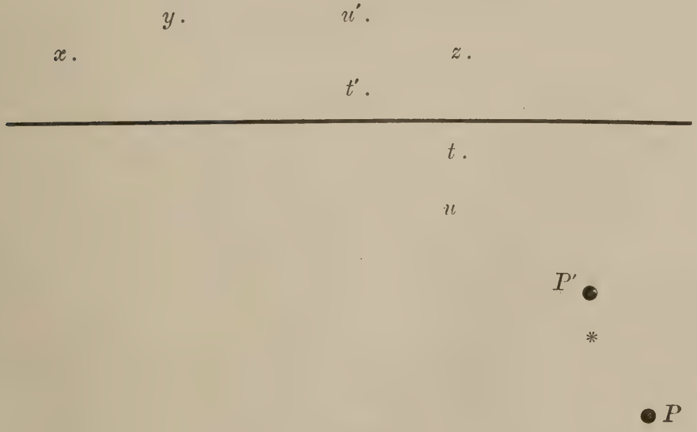

FIG. 2.

plane scheme in respect to a given axis*; of a solid scheme in respect to a given plane.

Notice the algebraical sign of the *connumerant*, which is positive or negative, according as an even number (including *zero* as one) or an odd number to transpositions of cluster-points is *transposed.*

Definitions of rays and planes of cluster recalled and applied to schemes.

The term beam substituted for ray of the primary.

The theory of the reduction of binary systems of equations may be geometrically stated.

Network. In line, plane, or solid.

Nodes and nodal lines.

Prime *point* or prime *ray* in network corresponds to prime groups of coefficients.

Prime *couples* or prime *planes* in network correspond to prime double-groups of coefficients, meaning a pair of groups whose minor determinants form a prime group.

Anticipatory statement, viz.—

The denumerant of a plane scheme in respect to a given centre is the sum of its connumerants with respect to each in succession of the rays which lie on either side (chosen at will) of the beam, provided that all the rays on the side so chosen are prime rays, and the points to which they are drawn are no two of them coincident. If these conditions are satisfied on both sides of the beam, each of the segments of the ray-cluster into which it is divided by the beam will give a distinct solution, and the two sums of connumerants appertaining respectively to the rays in either cluster will be equal to one another.

Observe that, if the system be neuter, all the rays will be on one side of the beam, and there will be but one solution.

The denumerant of a solid scheme in respect to a given centre (corresponding to a ternary system of equations), which satisfies analogous conditions to the preceding, will be shown later on in the course to be expressible in very similar terms (cluster-planes being substituted for cluster-rays), with this remarkable difference, however, that, in lieu of a single dichotomous division of the planes, there will be a considerable number of such, each of which will or may furnish a distinct pair of solutions.

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* If the motion of P should carry it to P' on the opposite side of the axis, the transformed centre and primary will be brought to lie on one side of the axis, and consequently the cluster must have contrary signs to the primary, in order to balance about the axis; and, as there will thus be no omni-positive solution, the connumerant in that case will become zero. If P is sufficiently remote, this cannot take place.

The formation of these dichotomies involves the consideration of the doctrine of *normal orders*, or orders of *perspective sequence*—a branch of the doctrine of free geometry to which allusion was made in the opening address.

The problem of partitions stated as a problem in plane or solid network.

A system of equations in $x, y, z, \ldots u$ may be denoted by $S(x, y, z, \ldots u)$, or, when more convenient, by S alone, with implied reference to $x, y, z, \ldots u$.

Resultants of systems. $R_x S$, where S is binary, defined. $R_{x,y} S$, where S is ternary, defined.

$R_x . S$ is the equation which expresses that the coefficient cluster and primary of S balance about the axis Ox. This will remain good for a ternary system, so that $R_x . S$ will then denote a specific binary system, that which corresponds to *projection* of cluster and primary on a plane through the origin perpendicular to O_x. $R_{x,y} S$ will denote that the centre of gravity of the cluster and primary of S is in the plane xy.

Interpretation of $\mathfrak{C} R_x S$ when S is binary. Interpretation of the same when S is ternary.

$\mathfrak{C} R_x S$ in the latter case is perfectly definite just as much as in the former, although the modes of expressing $R_x S$ are infinitely varied.

If S' is what S becomes when we write in S, $fx+g$, or more generally ϕx, in place of x we may denote S' symbolically by $\frac{\phi x}{x} S$.

Note that $R_x S = R_x \left(\frac{\phi x}{x} S \right)$

$$\left(\frac{\phi x}{x} \cdot \frac{\phi y}{y} \cdot \frac{\phi z}{z} \ldots S \right) \text{ explained.}$$

Order of operative symbols $\frac{\phi x}{x}$, $\frac{\phi y}{y}$, &c. is indifferent. The denumerant of any principal derivative $R_x . S$, if homogeneous, will furnish a superior limit to the denumerant of S; for all the solutions of F must be solutions of $R_x . S$.

Hence, to the denumerant of a definite binary system we can always, by *simple* denumeration, obtain two superior limits; to the denumerant of a ternary system, some number of superior limits, between 3 and n inclusive, such number depending upon the morphological character of the system (as will hereafter be explained).

Examples of superior limits to binary denumerants.

Examples of superior limits to ternary denumerants :—

1. By means of principal *simple* derivatives.
2. By means of principal derivative *binary systems.*

Hereafter we shall find that when the coefficient groups are all prime groups, and none of them alike, these two limits are the respective first terms of two distinct finite series of connumerants, each of which expresses the value of the denumerant of the given binary systems.

Lemma.—If the x group in any system is a prime group, any omni-positive integer solution of $R_x . S$ is in general an omni-positive integer solution either of S or of $\frac{-x}{x} S$.

Proof in case of binary system.

Proof in case of ternary or ultra-ternary system.

How an exception arises when the solution of $R_x . S$, substituted in S, makes $x = 0$.

Were it not for this exception, the equation following would always subsist for any variable x corresponding to a prime group, viz.,

$$\alpha S + \alpha \left(\frac{-x}{x} . S \right) = \alpha R S.$$

The number of omni-positive solutions of system $S(x, y, z, \ldots v)$, subject to the condition $x > k$, is the denumerant of

$$S \left\{ (x+k), y, z, \ldots v \right\}.$$

Thus, if $x > 0$, for x we must substitute $1+x$. Hence the true equation which connects the denumerants referred to *without* exception is

$$\alpha S + \alpha \frac{-1-x}{x} (S) = \alpha R_x . S;$$

or, if we please, $$= \alpha R_x \frac{-1-x}{x} S.$$

In future I shall denote $-x-1$ by $\bar{x}$,

$-y-1$ by $\bar{y}$,

and so on.

We may therefore write

$$\text{Ɑ} S = \text{Ɑ} R_x \overset{\bar{x}}{\underset{x}{=}} S - \text{Ɑ} \left(\overset{\bar{x}}{\underset{x}{=}} . S \right).$$

Now let the y group be also a prime group ; we shall have

$$D \overset{\bar{x}}{\underset{r}{=}} S = S \text{Ɑ} R_y \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} S \right) - \text{Ɑ} \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} S \right);$$

therefore $$\text{Ɑ} S = \text{Ɑ} R_x \overset{\bar{x}}{\underset{x}{=}} S - \text{Ɑ} R_y \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} S \right) + \text{Ɑ} \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} S \right);$$

and so, if the z group be a prime group,

$$\begin{aligned} \text{Ɑ} S = \text{Ɑ} R_x \overset{\bar{x}}{\underset{x}{=}} S - \text{Ɑ} R_y \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} S \right) + \text{Ɑ} \left(R_z \frac{\bar{x}}{x} \; \frac{\bar{y}}{y} \; \frac{\bar{z}}{z} \; S \right) \\ - \text{Ɑ} \left(\frac{\bar{x}}{x} \; \frac{\bar{y}}{y} \; \frac{\bar{z}}{z} \; S \right), \end{aligned}$$

and so on to any extent.

Extensions of this Equation to Systems of Equations of a Higher Order than the first indicated.

This I call the process of eduction. The question above indicated is always *true*, amounting in fact to the assertion of identity as regards the *solutions* themselves (not merely their number) of the systems on one side of the equation and those on the other. But, although true, it will be *nugatory* if any of the systems become indefinite, for then in general their denumerants will be infinite in magnitude.

The above equation applies to systems of any order. Its application will be first studied in respect to binary systems.

By continuing the process of eduction through a sufficient number of steps, we shall find that the equation $R_t \left(\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} \ldots \overset{\bar{z}}{\underset{z}{=}} \; \overset{\bar{t}}{\underset{t}{=}} \right) S$ will become at length *incongruous.* Its denumerant will then vanish.

When this is the case, *a fortiori*, the denumerant of the system $\overset{\bar{x}}{\underset{x}{=}} \; \overset{\bar{y}}{\underset{y}{=}} \ldots \overset{\bar{z}}{\underset{z}{=}}$ will vanish. And thus the series is brought to a close, and the denumerant of S expressed entirely in terms of *simple* denumerants.

FOURTH LECTURE.*

THEORY OF EDUCTION (*continued*).

Process of eduction exemplified. Suppose the system $S(x, y, z, t)$; then

$$\text{Ɑ} S = \text{Ɑ} R_x S - \text{Ɑ} \frac{\bar{x}}{x} R_y S + \text{Ɑ} \frac{\bar{y}}{y} \cdot \frac{\bar{x}}{x} R_z S - \text{Ɑ} \frac{\bar{z}}{z} \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{x}}{x} R_t S$$
$$+ \text{Ɑ} \left(\frac{\bar{t}}{t} \cdot \frac{\bar{z}}{z} \cdot \frac{\bar{y}}{y} \cdot \frac{\bar{x}}{x} \right) S;$$

but, if S is definite positive, $\frac{\bar{t}}{t} . \frac{\bar{z}}{z} . \frac{\bar{y}}{y} . \frac{\bar{x}}{x} S$ is definite negative. Hence its denumerant is zero, and

$$\text{Ɑ} S = \text{Ɑ} R_x S - \text{Ɑ} \frac{\bar{x}}{x} R_y S + \text{Ɑ} \frac{\bar{y}}{y} \frac{\bar{x}}{x} R_z S - \text{Ɑ} \frac{\bar{z}}{z} \frac{\bar{y}}{y} \frac{\bar{x}}{x} R_t S.$$

The same equation will subsist if S be definite neuter, but not if S be definite negative or indefinite.

It is not necessary in general that *all* the coefficient groups should be prime groups, or *all* of them distinct from one another. Great importance of this observation.

Depression of order of denumerants by one degree.

Depression by several degrees:—(1) By successive eductions. (2) By one compound eduction.

Observe that successive eduction can only finally conduct to equations which are simple resultants of the original system, being resultants of its resultants.

Allusion to fundamental theorem for depression by two degrees, viz.—

$$\text{Ɑ} R_{x.y} . S = \text{Ɑ} S + \text{Ɑ} . \frac{\bar{x}}{x} S + \text{Ɑ} . \frac{\bar{y}}{y} S + \text{Ɑ} . \frac{\bar{x}}{x} . \frac{\bar{y}}{y} S.$$

This equation is subject to the condition that the minor determinants of the matrix formed by the x and y coefficient groups conjoined shall form a prime group.

Observe the singular symbolical equations—

$$R_x = -\frac{1}{x}, \quad R_{x,y} = R_x . R_y = \frac{1}{x} \times \frac{1}{y}.$$

Notice that the lemma at p. 54 is true for systems of any order.

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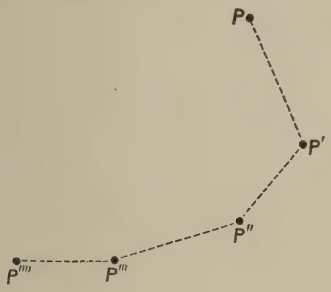
Problem of normal sequences stated. Its geometrical solution by means of the perspective to the cluster, (if a plane cluster) upon a line, (if a solid cluster) upon a plane, or (if a hypersolid cluster) upon a space.

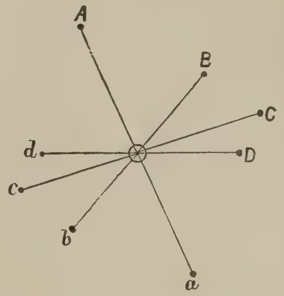
Binary Systems.

Geometrical representation of the successive systems

$$S\,.\,\frac{\bar{x}}{x}\,S,\quad \frac{\bar{y}}{y}\,\frac{\bar{x}}{x}\,S,\ \&c.,$$

in which S is supposed to be definite and positive. (See figure.)





Reversal of successive points in cluster, accompanied with parallel motion of primary.

The systems $\frac{\bar{x}}{x}\,.\,S,\ \frac{\bar{y}}{y}\,\frac{\bar{x}}{x},\ S$, &c. may be regarded as successive deformations of S, and then we may say the system, as it undergoes deformation, tends more and more to lose its positive character, until at length it becomes *neutral*, and immediately after changes into and continues *negative.*

The deformation may be commenced from either side. There are thus two courses of deformation.

If the primary were stationary, the deformed system in either course would become neuter after as many deformations as there are rays in S outside the beam, on that side of it from which the deformation proceeds.

The effect of the motion of the primary is to *accelerate* the tendency of the deformed system to become neuter.

If the primary be moved along the beam to a sufficient distance from the origin, the effect of such tendency will become at length insensible for such and for all greater distances.

The number of deformations in either course which may take place before the primary ray becomes denuded gives the number of terms in the development (by the eduction process) corresponding to that course.

When the primary is sufficiently remote, the sum of these numbers corresponding to the two courses of deformation will be the number of rays, *i.e.*, the number of variables in the system.

On account of the motion of the primary, the united sum may be *less* than this number; it cannot be greater.

Corollary.—If all the groups are prime groups, the denumerant of a binary system may be expressed by a number of simple denumerants, *not greater* than half the number of variables in the system.

The question of partitions, given in number and species, is expressed by a binary system satisfying these conditions.

Example.—To find the number of ways in which n can be made up of r values each limited to be either 1, 2, 3, or 4. This requires the denumeration of the system

$$x+2y+3z+4t = n,$$
$$x+\ y+\ z+\ t = r.$$

The *skew-matrix* of elimination becomes

$$\begin{array}{cccc|c} x & y & z & t & \\ \hline 0 & 1 & 2 & 3 & n-\ r \\ -1 & 0 & 1 & 2 & n-2r \\ -2 & -1 & 0 & 1 & n-3r \\ -3 & -2 & -1 & 0 & n-4r. \end{array}$$

If $n < r$, the system is neuter, and the number required is 0.

If $n > 2r$ but $< 2r$, the number required is the connumerant $\dfrac{n-r\,;}{1,\,2,\,3\,;}$ which is the same in value as the sum of the three complementary connumerants.

If $n > 2r < 3r$, the number required is the sum of the connumerants

$$\frac{n-r\,;}{1,\,2,\,3\,;}+\frac{n-2r\,;}{-1,\,1,\,2\,;}, \quad \text{or, if we please,} \quad \frac{4r-n\,;}{1,\,2,\,3}+\frac{3r-n\,;}{-1,\,1,\,2}.$$

If $n > 3r$ but $< 4r$, the number required is $\dfrac{4r-n\,;}{1,\,2,\,3\,;}$, which is the same in value as the sum of the three complementary denumerants.

If $n > 4$, the system is again neuter, and its denumerant is zero.

By aid of the above expressions we may give the general analytical representation of the number of partitions of the kind proposed.

These expressions, it will be observed, are discontinuous, the particular one to be employed depending on the comparative values of n and r. It may be shown, however, that they, as it were, melt or modulate into one another, so not only at the mere limiting values of n, which separate the several formulæ, but also for a short distance beyond these limits, either of the conterminous expressions may be used indifferently.

The proof of this depends upon the proposition that the coefficient of t^n in the *ascending* expansion of

$$\frac{1}{(1-t^a)(1-t^b)(1-t^c)\dots(1-t^k)}$$

treated *as a function of* n remains fixed at zero, when n is made to become any negative number whose absolute value is inferior to

$$a+b+c+\dots+k.$$

The truth of this proposition follows immediately from a theorem of Mr. Cayley relating to the development of any rational fraction $\frac{1}{\phi t}$ in its two forms. If $\theta(n)$ is the fractional value of the coefficient of t^n in the positive, and $\pi(n)$ the fractional value of the coefficient of t^n in the negative, expansion of $\frac{1}{\phi t}$, $\theta(n) \pm \pi(n) = 0$, the sign $+$ or $-$ being employed according as the degree of the rational function ϕt is an odd or an even number.

If the coefficient groups be arranged in natural order, the transformed systems will throughout remain definite.

The natural order for binary systems is indicated by the order in either direction in which the rays of the cluster succeed each other.

If the coefficients are all positive, this order will correspond with the order in which the variables must succeed each other, so that the ratios in each group of the coefficient out of one given equation with the corresponding coefficient out of the other may continually increase or decrease.

But, if the coefficients are positive and negative intermingled, the rule is that the determinants formed by the combination of any group with each in succession of those which follow must all bear the same sign; or, as we may express it, the algebraical sign arising from the contact of the groups, with due regard to antecedence, must be always the same.

If the system were indefinite, such a uniformity of signs could not be established by any arrangement of the groups whatever.

Example (1) :—

$$\left.\begin{array}{r} x+y+z = 50 \\ x+2y+3z = 120 \end{array}\right\}.$$ In this system all the groups are prime groups.

First solution.—

$$\begin{array}{ccc|c} x & y & z & \\ \hline 0 & 1 & 2 & 70 \\ -1 & 0 & 1 & 20 \\ -2 & -1 & 0 & 30. \end{array}$$

Observe that the coefficients form a skew-matrix.

$R_x . \frac{\bar{x}}{x} . S$ is $y+2z = 70$

$R_y . \frac{\bar{y}}{y} . \frac{\bar{x}}{x} . S$ is $(x+1)+z = 20$, or $x+z = 19$.

$R_z . \frac{\bar{z}}{z} . \frac{\bar{y}}{y} . \frac{\bar{x}}{x} . S$ is $2(x+1)+(y+1) = -30$, or $2x+y = -33$.

The desired denumerant will be $\frac{70;}{1,\,2;} - \frac{19;}{1,\,1}$ which is

$$36-20 = 16.$$

Second solution.—

$$\begin{array}{ccc|c} z & y & x & \\ \hline 0 & 1 & 2 & 30 \\ -1 & 0 & 1 & -20 \\ -2 & -1 & 0 & -70. \end{array}$$

$R_z . \frac{\bar{z}}{z} . S$ is $y+2x = 30$.

$R_y . \frac{\bar{y}}{y} . \frac{\bar{z}}{z} . S$ is $z+x = -21$.

$R_x . \frac{\bar{x}}{x} . \frac{\bar{y}}{y} . \frac{\bar{z}}{z} . S$ is $2z+y = -73$.

The desired denumerant will therefore be $\frac{30;}{1,\,2;} = 16$, as before.

Observe, that $C\!\!\!\mid S$ is the *sum* of the connumerants of $R_x S$, $R_y S$, $R_z S$, when these equations are written in the form in which they appear when the process of successive elimination is conducted by a uniform course of operations. This may be done in two ways, so as to give rise to two sets of equations differing from each other only in the signs. Observe, that the connumerant of $L = \pm c$ is not the same as of $-L = \mp c$; of that one of these equations, in which the constant term is negative, the connumerant being zero, but of the other the connumerant being generally different from zero.

Corollary.—The coefficient of $t^m . \tau^\mu$ in

$$\frac{1}{(1-t^a . \tau^\alpha)(1-t^b \tau^\beta)(1-t^c \tau^\gamma)}$$

is the sum of the coefficients of

$$\rho^{a\mu - \alpha m} \quad \text{in} \quad \frac{1}{(1 \sim \rho^{a\beta - b\alpha})(1 \sim \rho^{a\gamma - c\alpha})},$$

of

$$\rho^{b\mu - \beta m} \quad \text{in} \quad \frac{1}{(1 \sim \rho^{b\alpha - a\beta})(1 \sim \rho^{b\gamma - c\beta})},$$

and of

$$\rho^{c\mu - \gamma m} \quad \text{in} \quad \frac{1}{(1 \sim \rho^{c\alpha - a\gamma})(1 \sim \rho^{c\beta - b\gamma})},$$

subject to the interpretation that the preceding fractions are to be expanded all in terms of ascending, or all in terms of descending, powers of ρ, *provided* that the system

$$\left.\begin{aligned} ax + by + cz + dt &= m \\ \alpha x + \beta y + \gamma z + \delta t &= \mu \end{aligned}\right\}$$

is a definite non-negative system.

Allusion to Mr. Cayley's proof of the proposition in this form. The proposition is, of course, general, that the denumerant of a definite binary system with r variables, in which the groups are all prime groups, admits of a double mode of representation as the sum of the connumerants of its principal derivatives. In the one mode of representation, only a certain number, at *utmost*, of these connumerants, say p, can differ from zero; in the other mode, only a certain number, at utmost, say q, can differ from zero; and, as we shall have $p + q = r$, these two modes of representation may be termed complementary to each other.

The denumerant of the system

$$\left.\begin{aligned} ax + by + cz + \ldots + dt &= m \\ x + y + z + \ldots + t &= \mu \end{aligned}\right\}$$

may therefore be represented in two modes as the sum of simple denumerants.

This is the system the denumeration of which constitutes the problem of the resolution of integers into parts given in number and species. See Euler's *Second Memoir on Partitions.*

The theorem of eduction may be put under the following form :—

$$\reflectbox{D}S = \reflectbox{D}R_x \frac{\bar{x}}{x} S - \reflectbox{D}R_y \frac{\bar{y}}{y}\,\frac{\bar{x}}{x} S + \reflectbox{D}R_z \frac{\bar{z}}{z}\,\frac{\bar{y}}{y}\,\frac{\bar{x}}{x} S \mp \quad \&c.,$$

provided that S is a definite system in which the groups of coefficients appertaining to the several variables, or at least to so many of them as are included between x and the last of them which appears as a suffix in a non-zero term of the above expression, are all distinct prime groups.

2nd Example.—

$$\left.\begin{aligned} x+\ y+\ z &= 10 \\ x+2y+3z &= 40 \end{aligned}\right\}.$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline 0 & 1 & 2 & 30 \\ -1 & 0 & 1 & 20 \\ -2 & -1 & 0 & 10 \end{array} \qquad\qquad \begin{array}{c|ccc} & x & y & z \\ \hline -10 & 2 & 1 & 0 \\ -20 & 1 & 0 & -1 \\ -30 & 0 & -1 & -2 \end{array}$$

The solution corresponding to the right-hand matrix is evidently 0, the system in fact being neuter. The left-hand matrix gives the solution—

$$\frac{30\,;}{1,\,2\,;} + \frac{20\,;}{-1,\,1\,;} + \frac{10\,;}{-2,\,-1\,;}$$

$$= \frac{30\,;}{1,\,2\,;} - \frac{19\,;}{1,\,1\,;} + \frac{7\,;}{2,\,1\,;}$$

$$= 16-20+4 = 0, \text{ as before.}$$

Illustrate effect of taking the variables in abnormal order.

3rd Example.—

$$\begin{aligned} 2x+5y+2z-\ t &= m, \\ x+\ y-2z-2t &= m, \end{aligned}$$

the system will also be neuter, and we shall have

$$\frac{m\,;}{6,\,9,\,3\,;} + \frac{2m\,;}{-3,\,6,\,3\,;} + \frac{4m\,;}{-9,\,-12,\,3\,;} + \frac{m\,;}{-3,\,-6,\,-3\,;} = 0,$$

or

$$\frac{m\,;}{3,\,6,\,9\,;} - \frac{2m-3\,;}{3,\,3,\,6\,;} + \frac{4m-21}{3,\,9,\,12\,;} - \frac{m-12}{3,\,3,\,6\,;} = 0.$$

Allusion to importance and fertility of theory of neuter systems.

Example of a denumeration of a binary system containing unprime groups :—

$$x+\ y+4z+3t=m,$$
$$3x+2y+6z+3t=n,$$

$\frac{m}{n}$ being supposed to be intermediate between $\frac{1}{2}$ and $\frac{2}{3}$. The denumerant will be $\frac{3m-n\,;}{1,\,6,\,6\,;}-\frac{2m-n\,;}{-1\,;\,2\,;\,3\,;}$.

Cæsura (definition of, and how determined), accidental and universal, distinguished.

FIFTH LECTURE.*

Eduction and Reduction.

The cæsura for equation-systems generally falls after that coefficient group subsequent to the introduction of which, in the eduction process, the depressed systems whose denumerants are to be taken *must* cease to be positive, so that they may be neglected. It is determined for binary systems by the relation of the ratios of the terms in the coefficient groups to that of the terms in the constant group; the determinant formed by the apposition of the constant group with any group on one side of the cæsura being positive, on the other side negative.

The point after which the terms in an eduction process can be neglected *may* (if the constant terms are sufficiently small) be attained before the cæsura is reached. Such a point may be termed a turning-point, or pause. There may thus (in the case of binary systems) be two turning-points or pauses on each side of the cæsura corresponding to the two courses of eduction, but either or both of them may, and in general will, coincide with the cæsura.

For greater simplicity, we may suppose the constant terms given in ratio only, and not in magnitude, so as to obviate the necessity of paying any attention to the accidental pauses as distinguishable from the cæsura. The cases where they are so distinguishable are always exceedingly limited in number. Their existence arises solely from the fact of the introduction of $-x-1$, $-y-1$, &c., and not $-x$, $-y$, &c., in lieu of x, y, in applying the method of eduction.

A *per-reducible* binary system is one in which *all* the coefficient groups are *prime* groups *distinct* from each other.

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* Delivered at King's College, London, June, 1859.

Being prime and distinct, none of them can be in syzygy. Such a system admits of a double process of eduction, giving rise in general to two distinct forms of solution. But it may happen, in some very special cases, that these two solutions are identical in form as well as in value.

Example.—The system

$$\left.\begin{array}{l} x+3y+7z+9t=5i \\ x+2y+4z+5t=3i \end{array}\right\}$$

gives rise to the bordered matrix

$$\begin{array}{rrrr|r} 0 & 1 & 3 & 4 & 2i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -i \\ -4 & -3 & -1 & 0 & -2i \\ \cline{1-4} -2i & -i & i & 2i & \end{array}$$

Here each solution is the same, viz., $\dfrac{2i;}{1,3,4;}+\dfrac{i;}{-1,2,3;}$, meaning

$$\frac{2i;}{1,3,4;}-\frac{i-1;}{1,2,3.}$$

But, if the constant terms in the above system were $11i$ and $7i$ respectively, the bordered matrix would be

$$\begin{array}{|rrrr|r} \cline{1-4} 0 & 1 & 3 & 4 & 4i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -5i \\ -4 & -3 & -1 & 0 & -8i \\ \cline{1-4} \multicolumn{1}{r}{-4i} & -i & 5i & \multicolumn{1}{r}{8i} & \end{array}$$

giving rise to the two equal sums of connumerants,

$$\frac{4i;}{1,3,4;}+\frac{i;}{-1,2,3;} \quad \text{and} \quad \frac{8i;}{1,3,4;}+\frac{5i;}{-1,2,3;}.$$

In this example the matrix happens to be *persymmetrical*, which is the reason of the denumeratives being the same in each solution.

This is avoided in the example below of the system

$$\left.\begin{aligned} x+2y+\ z+\ t &= 7i \\ x+3y+2z+4t &= 12i \end{aligned}\right\},$$

for which the bordered matrix is

$$\begin{array}{cccc|c} 0 & 1 & 1 & 3 & 5i \\ -1 & 0 & 1 & 5 & 3i \\ -1 & -1 & 0 & 2 & -2i \\ -3 & -5 & -2 & 0 & -16i \\ \hline -5i & -3i & 2i & 16i & \end{array}$$

giving rise to the two equivalent solutions

$$\frac{5i\,;}{1,\,1,\,3\,;}+\frac{3i\,;}{-1,\,1,\,5\,;}\quad\text{and}\quad\frac{16i\,;}{2,\,5,\,3\,;}+\frac{2i\,;}{-2,\,1,\,1\,;},$$

meaning

$$\frac{5i\,;}{1,\,1,\,3\,;}-\frac{3i-1\,;}{1,\,1,\,5\,;}\quad\text{and}\quad\frac{16i\,;}{2,\,3,\,5\,;}-\frac{2i-2\,;}{1,\,1,\,2\,;}.$$

A *simply-reducible* system is one for which the coefficient groups are prime and distinct on *one* side of the cæsura only.

Example.—

$$\left.\begin{aligned} x+2y &= 4m \\ x+4y &= 5m \end{aligned}\right\}.$$

The eduction from the x side gives rise to the equation of $2y=m$, of which the denumerant is $\frac{m\,;}{2\,;}$. This is the true solution, whereas the eduction from the y side gives rise to the denumerant of $2x=6m$, *i.e.*, 1, which is a false solution, owing to the group (2, 4) being a non-prime group.

If in a binary system the groups, which are either non-prime or repeated, or non-prime and repeated, represent ratios (between quantities given in algebraical sign) which are all less or all greater than the corresponding ratio of the constant terms, the system is still depressible by eduction commenced from that side of the system on which the groups of the kind mentioned do not fall.

Corollary.—A single non-prime group, or a single sequence of any number of identical groups, can in no case hinder a binary system from being soluble by eduction.

The above remark is true also *a fortiori* for ultra-binary systems.

It should be noticed that $\left.\begin{matrix} a \\ 0 \end{matrix}\right\}$ is a non-prime group unless $a = \pm 1$.

[For non-prime we may in future use the term composite.]

Example.—
$$\left.\begin{aligned} 3x+2y+z+\ t &= i \\ 2z+3t &= i \end{aligned}\right\}.$$

Here the coefficient groups of x and y are both of them composite; but, the cæsura falling between y and z, the denumerant required will be the sum of the connumerants of the two resultants in respect to t and z, *i.e.*,

$$\frac{2i\,;}{1,\,6,\,9\,;} + \frac{i\,;}{-1,\,4,\,6\,;},$$

meaning
$$\frac{2i\,;}{1,\,6,\,9\,;} - \frac{i-1\,;}{1,\,4,\,6\,;}$$

If a system is affected with composite or repeated groups on *each* side of the cæsura, its denumeration may be made to depend on systems where such groups exist on only one side of their respective cæsuras.*

Example.—
$$\left.\begin{aligned} 10x+2y+3z &= \ \ 5i \\ 15x+4y+9z &= 11i \end{aligned}\right\}, \text{ which call } S.$$

If we form a ternary system as follows:—

$$\left.\begin{aligned} 10x+2y+3z\qquad &= \ \ 5i \\ 15x+4y+9z\qquad &= 11i \\ px+qy+rz-t &= -\,m \end{aligned}\right\}, \text{ which call } S',$$

where p, q, r, m are any positive integers whatever, it is apparent that the omni-positive solutions of S' may be found from the omni-

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* In certain special cases the composite groups may be reduced in number by substituting a connective of the equations in lieu of one of them, as in the example

$$\left.\begin{aligned} 10t-7z-8y &= 5i \\ 9z+2y &= 7i \end{aligned}\right\},$$

which is apparently irreducible, but which, put under the equivalent form

$$\left.\begin{aligned} 10t-7z-\ 8y &= \ \ 5i \\ 15t-6z-11y &= 11i \end{aligned}\right\},$$

becomes simply-reducible.

positive solutions of S, and to each of the latter will correspond one, and only one, of the former. Hence the denumerant of S is the same as the denumerant of S', and, if p, q, r are so chosen that 10, 15, p ; 2, 4, q ; 3, 9, r are all prime groups, $\text{Ɑ}S'$, and therefore $\text{Ɑ}S$, may be made to depend on the denumerants of a certain set of new binary systems obtained by the eduction of S'. Thus let $p=1$, $q=1$, $r=1$, $m=0$, so that the auxiliary equation becomes

$$x+y+z-t=0.$$

R_xS' may be represented by

$$\left.\begin{array}{r}10t-8y-7z=5i\\15t-11y-6z=11i\end{array}\right\},$$

R_yS' by

$$\left.\begin{array}{r}2t+8x+z=5i\\4t+11x+5z=11i\end{array}\right\},$$

R_zS' by

$$\left.\begin{array}{r}3t+7x-y=5i\\9t+6x-5y=11i\end{array}\right\},$$

R_tS' by

$$\left.\begin{array}{r}10x+2y+3z=5i\\15x+4y+9z=11i\end{array}\right\},\text{ being the original system }S.$$

It will be seen therefore that

$$R_xS'\,;\quad \frac{\bar{x}}{x}R_yS'\,;\quad \frac{\bar{y}}{y}\cdot\frac{\bar{x}}{x}R_zS'\,;\quad\text{and}\quad \frac{\bar{z}}{z}\cdot\frac{\bar{y}}{y}\cdot\frac{\bar{x}}{x}R_tS'$$

respectively represent the systems following:—

$$\left.\begin{array}{r}10t-8y-7z=5i\\15t-11y-6z=11i\end{array}\right\},\qquad(1)$$

$$\left.\begin{array}{r}2t+z-8x=5i+8\\4t+5z-11x=11i+11\end{array}\right\},\qquad(2)$$

$$\left.\begin{array}{r}3t+y-7x=5i+6\\9t+5y-6x=11i+1\end{array}\right\},\qquad(3)$$

$$\left.\begin{array}{r}-10x-2y-3z=5i+15\\-15x-4y-9z=11i+28\end{array}\right\}.\qquad(4)$$

All these four systems are definite: in the first of them the natural order of the groups is

$$\begin{pmatrix} 10 & -7 & -8 \\ 15 & -6 & -11 \end{pmatrix};$$

in the others the natural order is that in which they are written. The first two only will be definite-positive, the last will be definite-negative, and the last but one neuter if $i>0$, negative if $i=0$, and the denumerant required will be the difference between the denumerants of the two systems

$$\left.\begin{aligned} 10p-7q-\ 8r &= \ 5i \\ 15p-6q-11r &= 11i \end{aligned}\right\},$$

$$\left.\begin{aligned} 2p+\ q-\ 8r &= \ 5i+\ 8 \\ 4p+5q-11r &= 11i+11 \end{aligned}\right\}.$$

In each of these systems there is one, but only one, of the original non-prime groups, and no new ones have been introduced.

Consequently they admit of being depressed, and the final result will be an aggregate of simple denumerants.

If we had applied to S' a different course of eduction as follows :—

$$R_z S', \quad \frac{\bar{z}}{z} R_y S', \quad \frac{\bar{z}}{z}\,\frac{\bar{y}}{y} R_x S', \quad \frac{\bar{z}}{z}\,\frac{\bar{y}}{y}\,\frac{\bar{x}}{x} R_t S',$$

it may easily be seen that all these systems likewise would be definite, and only the first of them definite-positive. Hence a second solution of the question will be $\text{Ɑ} R_z S'$, *i e.*,

$$\frac{5i,\ 11i\ ;}{3,\ 9\ ;\ 7,\ 6\ ;\ -1,\ -5\ ;},$$

which is of a depressible form, there being only one *affected* group (3, 9), and may be educed into a linear function of simple denumerants.

Dispersion process defined.

Cases which resist its application.

Theorem.—The denumeration of any equation system whatever may be made to depend upon the denumeration of systems that shall contain no composite groups, and at most only one set of repeated groups, and which will consequently be depressible.

Proof of this theorem in case of binary systems.

Definition of meaning of kG, and of $kG \pm lG'$, where G, G' represent any two coefficient groups of a system, and k, l are any two integers.*

Lemma 1.—A system S, containing the coefficient group G, may be made to depend for its denumeration upon systems in each of which the coefficient groups are the same as in S, except that kG takes the place of G.

Lemma 2.—A system S, containing the groups G and G', may be made to depend upon two systems, in one of which the coefficient groups are the same as in G, with the exception that H replaces G, and in the other the same as in G, with the exception that H replaces G', where H is $G-G'$ or $G'-G$.

Note that, if, instead of the coefficient group G in any definite system, first any other group H and then $-H$ be substituted, one *at least* of these substitutions must leave the deformed system definite.

Lemma 3 (*Corollary to Lemma* 1).—Any equation-system may be made to depend for its denumeration on equation-systems in each of which one of the equations has all its coefficients positive units.

It follows from this lemma that, if a binary equation-system is free from syzygy (*i.e.*, from equalities of ratios between the coefficients of different variables), its denumeration may be made to depend upon that of systems which [their coefficient groups being all different and of the form $(a, 1)$] are per-reducible. But, if there be e sets of syzygies in the given system, there will be e sets of repetitions in the groups $(a, 1)$ in each of the deduced systems.

Lemma 4.—In the case immediately above supposed, the e sets of syzygetic groups in the deduced systems may be replaced by e other syzygetic sets of groups of which all but one are of the form $(a, 1)$, $(a, 1)$, ... ; $(b, 1)$, $(b, 1)$, $(b, 1)$, ..., &c., and that one of the form $(\sigma, 0)$; $(\sigma, 0)$; $(\sigma, 0)$, ...

Lemma 5.—Any system of the form last supposed may (by virtue of Lemma 2) be replaced by two, in one of which $\pm(a-k\sigma, 1)$ takes the place of $(a, 1)$, and in the other $\pm(a-k\sigma, 1)$ takes the place of

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* In a definite ternary system, where all the coefficient groups are prime groups, it may be shown that the only possible cases of syzygy are where $F = G$ or $F+G = H$ (F, G, H denoting coefficient groups of the system).

$(\sigma, 0)$, k being so chosen that $(a-k\sigma, 1)$ is distinct from every other coefficient group associated with it in the same system.

Lemma 6.—Hence, by *repeated* application of this last process of replacement, the number of syzygetic groups in the deduced systems may be continually reduced until we arrive at systems in one class of which all the groups $(\sigma, 0)$ have disappeared, and in the other class of which all the syzygetic groups except those of the form $(\sigma, 0)$ have disappeared.

Lemma 7.—Hence, so long as e is greater than 1, the deduced systems will eventually none of them contain more than $(e-1)$ sets of groups in syzygy, and thus we must eventually arrive at systems in none of which will be found more than a single set of groups in syzygy, which may be taken indifferently of the form $(a, 1)$ or $(a, 0)$.

Consequently the denumeration of every binary system, if free of syzygies, may be made to depend on the denumeration of per-reducible systems; and, if not free of syzygies, on the denumeration of simply reducible systems.

A similar demonstration may be extended to systems of a higher order than the second. Consequently every denumerant of an order higher than the first may be made to depend on denumerants of a lower order, and eventually upon simple denumerants.

EXAMPLES OF REDUCTION OF PERSYZYGETIC SYSTEMS.

Let the given system S be

$$\left.\begin{array}{ll} x+y & = m \\ & z+t = n \end{array}\right\},$$

and suppose m not less than n. Making

$$x+z-u = 0,$$

we obtain the systems

$$\left.\begin{array}{ll} u+y-z & = m \\ & z+t = n \end{array}\right\},$$

which is $R_x S'$, say T; and

$$\left.\begin{array}{ll} u+t+x & = (n-1) \\ & x+y = m \end{array}\right\},$$

which is $\overset{\bar{x}}{\underset{x}{=}} R_z S'$, say U.

Since $\overset{\bar{x}}{\underset{x}{=}}\ \overset{\bar{z}}{\underset{z}{=}}\ S'$ contains the equation

$$-x-z-n=2,$$

the eduction is complete, and the required denumerant $=\text{Ɑ}T-\text{Ɑ}U.$

T arranged in natural order becomes

$$\begin{aligned} z+t \qquad\quad &= n, \\ -z \qquad +u+y &= m, \end{aligned}$$

the cæsura falling between t and u. Accordingly we obtain

$$\text{Ɑ}T=\frac{n+m\,;}{1,\,1,\,1\,;}+\frac{m\,;}{-1,\,1,\,1\,;}=\frac{n+m\,;}{1,\,1,\,1\,;}-\frac{m-1\,;}{1,\,1,\,1\,;}$$

$$=\frac{(n+m+1)(n+m+2)}{2}-\frac{m\,(m+1)}{2}.$$

In like manner

$$\text{Ɑ}U=\frac{n-1\,;}{1,\,1,\,1\,;}-\frac{n-1-m\,;}{-1,\,1,\,1\,;}=\frac{n-1\,;}{1,\,1,\,1\,;}=\frac{n\,(n+1)}{2},$$

for $n-1-m$ is negative. And we have

$$\text{Ɑ}S=\frac{(n+m+1)(n+m+2)}{2}-\frac{m\,(m+1)}{2}-\frac{n\,(n+1)}{2}$$

$$=\frac{2nm+2m+2n+2}{2}=(m+1)(n+1),$$

which is evidently the correct answer, being the product of the denumerants of the two given equations taken independently.

Second Example :— $\left.\begin{aligned} x+y+\theta \qquad\quad &= m \\ \theta+z+t &= n \end{aligned}\right\}.$

By the same method as above, we obtain

$$\text{Ɑ}S=\text{Ɑ}T-\text{Ɑ}U,$$

where T is $\left.\begin{aligned} z+t+\theta \qquad\quad &= n \\ -z \qquad +\theta+u+y &= m \end{aligned}\right\},$

and U is $\left.\begin{aligned} u+t+\theta+x \qquad &= n-1 \\ \theta+x+y &= m \end{aligned}\right\},$

the cæsura in T falling between θ and u, and in U between x and y. Hence

$$\alpha T = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3}{2,\,1,\,1,\,1\,;},$$

and $$\alpha U = \frac{n-1\,;}{1,\,1,\,1,\,1\,;}.$$

Thus $$\alpha S = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} - \frac{n-1\,;}{1,\,1,\,1,\,1\,;}.$$[*]

Example of a composite group and a syzygy falling on opposite sides of the cæsura. *Problem.*—To express the residue of q in respect to p as a linear function of simple denumerants.

If we call x the required residue, we have

$$x+py=q, \quad x<p,$$

or

$$\begin{aligned} x+py &= q, \\ x+z &= p-1. \end{aligned}$$

Hence the required residue is the denumerant of the system

$$\begin{aligned} py+t+u &= q-1, \\ t+u+z &= p-2, \end{aligned}$$

in which the coefficient groups are in natural sequence.

In its present form the system is irreducible, because the cæsura falls between y and t (observe that p, 0 is a non-prime group); but, by the method above given, the denumerant of this system, by virtue of the subsidiary equation

$$y+t=v,$$

becomes the difference between the denumerants of

$$\left.\begin{aligned} (1-p)\,t \quad +u+pv &= q-1 \\ t+z+u \quad &= p-2 \end{aligned}\right\}$$

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[*] The value of this expression will evidently be the sum

$$(m+1)(n+1)+mn+(m-1)(n-1)+\&\text{c.}+(m-n+1),$$

which is $$\left(m+1-\frac{n}{3}\right)\frac{(n+1)(n+2)}{2}.$$

and of $$\left.\begin{array}{rl}(1-p)\,y+v+u+t=(q-1)+(p-1) & =q+p-2 \\ y+z+u+t=(p-2)-1 & =p-3\end{array}\right\}.$$

The second system is neuter, for all the coefficient groups put in apposition with the constant group give determinants with negative values.

Hence the required expression is simply the denumerant of the first system, in which the cæsura falls between u and v. $(p,\,0)$ being a composite group, the eduction must be commenced from the t side, and accordingly we obtain the series

$$\frac{(q-1)+(p-1)(p-2)\,;}{(p-1),\,p,\,p\,;}+\frac{(q-1)\,;}{-(p-1),\,1,\,p\,;}+\frac{q-p+1\,;}{-p,\,-1,\,p\,;}$$

$$=\frac{p^2-3p+q+1\,;}{p-1,\,p,\,p\,;}-\frac{q-p\,;}{1,\,p-1,\,p\,;}+\frac{q-2p\,;}{1,\,p,\,p\,;}$$

as the expression required for the residue of q in respect to p.

SIXTH LECTURE.*

SIMPLE PARTITION.

Resolution of an integer into a defined number of parts.

With or without repetition.

$\dfrac{n-r\,;}{1,\,2,\,3,\,\ldots\,r\,;}$ expresses the r-ary partibility of n when repetitions are allowed;

$\dfrac{n-r\dfrac{r+1}{2}\,;}{1,\,2,\,3,\,\ldots\,r\,;}$ the same when repetitions are excluded.

Example: $n=7,\ \ r=3$.

Proof of the above formulæ by Ferrers' method.

When n is great compared with r, these two functions approach to a ratio of equality.

The generating function for partitions without repetition.

Indefinite resolution of numbers with and without repetition.

Generating functions for both these kinds of indefinite resolutions.

Euler's *Series Mirabilis*, and its application.

Remark on indefinite *partition* with the elements 1, 2, 4, 8, &c.

Partition or composition. Partible number. Elements.

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* Delivered at King's College, London, July 4th, 1859.

Construction of equation-system whose denumerant is the number of compositions of n with unrepeated elements.

The negation of the *possibility* (for integers) of the equation $x^i+y^i=z^i$ capable of being transformed into the affirmation of an analytical *identity* by the method of partitions.

Resolution of integers into a given number of parts, how treated by Sir John Herschel and others.

Formula of reduction $\dfrac{n\,;}{1,2,\ldots k\,;}=\dfrac{n-k\,;}{1,2,\ldots k\,;}+\dfrac{n-k\,;}{1,2,\ldots (k-1)\,;}$.

Objections to this method :—(1) As inductive instead of direct (besides being limited to a mere special case of partition). (2) As excessively prolix and unmanageable. (3) As leading to an amorphous result.

Mr. Cayley's improved method. The true form of representation.

The lecturer's discovery of the general analytical solution.

The provisional method superseded.

FUNDAMENTAL THEOREM IN SIMPLE PARTITION.

Axiom.— $\dfrac{n\,;}{abc\ldots l\,;}=\dfrac{1}{abc\ldots l}\Sigma H_n(\alpha,\beta,\gamma,\ldots\lambda)$,

where $H_n(\alpha,\beta,\gamma,\ldots\lambda)$ indicates the sum of the homogeneous powers and products of $\alpha,\beta,\gamma,\ldots\lambda$ of the n^{th} degree, and $\alpha,\beta,\gamma,\ldots\lambda$ are respectively roots of

$$x^a=1,\quad y^b=1,\quad z^c=1,\quad\ldots\quad u^l=1.$$

Example.— $\dfrac{7\,;}{2,3\,;}=\frac{1}{6}\Sigma\,(\rho^7+\rho^6\sigma+\ldots+\sigma^7)$,

where Σ includes six sums corresponding to the following six systems of values ρ,σ; viz.,

$$1,1;\quad -1,1;\quad 1,\rho;\quad -1,\rho;\quad 1,\rho^2;\quad -1,\rho^2,$$

ρ meaning a root of $\rho^2+\rho+1=0$.

In general,

$$H_n(p,q)=\frac{p^{n+1}}{p-q}+\frac{q^{n+1}}{q-p},$$

$$H_n(p,q,r)=\frac{p^{n+2}}{(p-q)(p-r)}+\frac{q^{n+2}}{(q-p)(q-r)}+\frac{r^{n+2}}{(r-p)(r-q)},$$

$$H_n(p,q,r,s)=\frac{p^{n+3}}{(p-q)(p-r)(p-s)}+\&\text{c.}+\&\text{c.}+\&\text{c.}$$

In applying this formula to the preceding axiom, several or all of the quantities p, q, r, &c., will become equal *inter se*, because the equations $a^\alpha = 1$, $b^\beta = 1$, $c^\gamma = 1$, ... have the root unity in common, and will have other roots in common unless $\alpha, \beta, \gamma, \dots$ are all prime to each other.

The value of $\Sigma \dfrac{\phi p_1}{(p_1-p_2)(p_1-p_3)\dots(p_1-p_{e+1})}$,

when $$p_1 = p_2 = \dots p_{e+1},$$

is $$\frac{1}{1.2.3\dots e}\left(\frac{d}{dp}\right)^e \phi p.$$

Every distinct root of $x^m = 1$, where m is the least common multiple of $a, b, c, \dots l$, furnishes a distinct expression to the sum and gives rise to a separate term, in the complete analytical expression for $\dfrac{n;}{a, b, c, \dots l;}$. Such a term is called a wave. Reason for this name.

There are as many waves as distinct factors in $a, b, c, \dots l$; every such factor as q giving rise to a term W_q.

If $a, b, c, \dots l$ become the series of natural numbers $1, 2, 3, \dots r$, the number of waves is r.

The value of W_q for $\dfrac{n;}{a, b, c, \dots l;}$ is the coefficient of $\dfrac{1}{t}$ in

$$\frac{1}{a\,b\,c\dots l}\Sigma\frac{(\rho e^t)^n}{\left[1-(\rho e^t)^{-a}\right]\left[1-(\rho e^t)^{-b}\right]\dots\left[1-(\rho e^t)^{-l}\right]},$$

where ρ is a *primitive* root of $\rho^q = 1$, *i.e.*, a root not belonging to $\rho^{q/i} = 1$.

The only cases where the quantity under the sign of summation reduces to a single term is when $q = 1$, for which case $\rho = 1$, and when $q = 2$, for which case $\rho = -1$.

W_1 considered. It is non-periodic. It is the coefficient of t^n in the development of $\dfrac{\phi(t)}{(1-t)^?}$, when the Eulerian function

$$\frac{1}{(1-t^a)(1-t^b)\dots(1-t^l)}$$

is supposed capable of being represented under the form

$$\frac{\phi t}{(1-t)^{?}}+\frac{\psi t}{\frac{1-t^a}{1-t}\,\frac{1-t^b}{1-t}\cdots\frac{1-t^l}{1-t}}.$$

It is also the *mean* of the m algebraical forms, m being the least common multiple of $a, b, c, \ldots l$, which represent $\frac{n\,;}{a,\,b,\,c,\,\ldots\,l\,;}$ when n is made to go through the m forms

$$\frac{km\,;}{a,\,b,\,c,\,\ldots\,l\,;},\quad \frac{km+1\,;}{a,\,b,\,c,\,\ldots\,l\,;},\ \ldots\ \frac{km+(m-1)\,;}{a,\,b,\,c,\,\ldots\,l\,;}.$$

It is therefore the mean value of $\frac{n\,;}{a,\,b,\,c,\,\ldots\,l\,;}$.

W_2 is $(-)^n B$, where B is the coefficient of $\frac{1}{t}$ in

$$\frac{1}{a\,b\,c\,\ldots\,kl}\,\frac{1}{(1-e^{-at})(1-e^{-bt})\ldots(1+e^{-kt})(1+e^{-lt})},$$

$a, b, \ldots$ being the odd, and $\ldots k, l$ the even, integers among $a, b, c, \ldots k, l$.

If the first wave is A, $A+B$ will be the mean of $\frac{n\,;}{a,\,b,\,c,\,\ldots\,l\,;}$ for even values of n, and $A-B$ the mean of the same for odd values of n.

Observe that the degree in n of W_1 is one unit less than the number of the elements $a, b, c, \ldots l$; in the algebraical part of W_2 is one unit less than the number of even elements among $a, b, c, \ldots l$, and in general W_q is one unit less than the number of elements which contain q as a factor.

Provisional notation co_{-1}, co_r explained.

The equations

$$co_l\,\phi\,(t)=co_{l+\omega}\left[t^{\omega}\phi\,(t)\right],\quad co_{2t}\,\phi\,(t^2)=co_t\,\phi\,(T)$$

identically true.

Mode of developing

$$\frac{(\rho e^t)^n}{\left[1-(\rho e^t)^{-a}\right]\left[1-(\rho e^b)^{-b}\right]\ldots\text{ to } i \text{ terms}},$$

under the form $\rho^n\,.\,e^{nt-R}$; where

$$R=\Sigma\log\left[1-(\rho e^t)^{-a}\right].$$

This an essential part of the theorem.

The expression for $\dfrac{1}{1-ke^u}$ in terms of u being known, $\log(1-ke^u)$ is known by integration from the identity

$$\frac{d}{du}\log(1-ke^u) = \frac{-ke^u}{1-ke^u} = 1 - \frac{1}{1-ke^u},$$

so that in the first and second waves the only numerical constants to be determined are the numbers of Bernouilli.

Thus when $\rho = 1$, corresponding to W_1, R becomes

$$\Sigma\left\{\log(at) - \frac{at}{2} + \frac{B_1}{2^2}a^2t^2 - \frac{B_2}{2.3.4^2}a^4t^4 + \frac{B_3}{2.3.4.5.6^2}a^6t^6, \text{ \&c.}\right\}$$

$$= \log(a_1a_2\ldots a_i) + i\log t - \tfrac{1}{2}\Sigma a.t + \frac{B_1}{2^2}(\Sigma a^2)\,t^2 - \frac{B_2}{2.3.4^2}(\Sigma a^4)\,t^4 \pm \text{\&c.},$$

so that $nt - R$ becomes

$$-\log(a_1a_2\ldots a_i) - i\log t + \left\{(n+\tfrac{1}{2}\Sigma a)\,t - \frac{B_1}{2^2}(\Sigma a^2)\,t^2 \pm \text{\&c.}\right\},$$

and
$$e^{nt-R} = \frac{t^{-i}}{a_1a_2\ldots a_i}\,e^{\nu t-(B_1/2^2)\,s_2t^2+\text{\&c.}};$$

and, finally,
$$W_1 = \frac{1}{a_1a_2\ldots a_i}\,co_{i-1}\left\{e^{\nu t-\frac{1}{24}s_2t^2+\frac{1}{2880}s_4t^4-\frac{1}{181440}s^6t^6\pm}\right\};$$

where
$$\nu = n + \tfrac{1}{2}\Sigma a.$$

In like manner, when $\rho = -1$, corresponding to W_2, if $a_1, a_2, \ldots a_e$ are the even, and $b_1, b_2, \ldots b_\omega$ the odd, elements,

$$R = \Sigma\log(1-e^{-at}) + \Sigma\log(1+e^{-bt}),$$

and we shall obtain

$$W_2 = \frac{(-)^n}{2^\omega a_1a_2\ldots a_e}\,co_{e-1}\left\{e^{\nu t-\frac{1}{24}(s_2+3\sigma_2)+\frac{1}{2880}(s_4+15\sigma^4)\pm\text{\&c.}}\right\},$$

where
$$\nu = n + \tfrac{1}{2}(a_1+a_2+\ldots+a_e+b_1+b_2+\ldots b_\omega),$$

$$s_l = \Sigma a^l, \quad \sigma_l = \Sigma b^l.$$

CALCULATION OF MEAN VALUES FOR ANY GIVEN NUMBER OF ELEMENTS.

Example 1.—To find the mean value of $\dfrac{n;}{a,\,b,\,c,\,d;}$.

This will be the coefficient of t^3 in

$$\frac{1}{abcd}\left\{e^{\nu t}\times e^{-\frac{1}{24}s_2t^2}\right\}$$

$$=\frac{1}{abcd}\,co_{+3}\left\{\begin{array}{l} 1+\nu t+\dfrac{\nu^2t^2}{1.2}+\dfrac{\nu^3t^3}{1.2.3}+\ldots \\ \times \\ 1-\dfrac{1}{24}\,s_2t^2 \end{array}\right\}$$

$$=\frac{\nu}{abcd}\,co_{+1}\left\{\left(1+\frac{\nu^2t}{1.2.3}\right)\left(1-\frac{1}{24}\,s_2t\right)\right\}$$

$$=\frac{\nu}{abcd}\left\{\frac{\nu^2}{6}-\frac{s_2}{24}\right\}.$$

Example 2.—To find the mean value of $\dfrac{n\,;}{a,\,b,\,c,\,d,\,e\,;}$.

This will be the coefficient of t^4 in

$$\frac{1}{abcde}\left\{e^{\nu t}\times e^{\frac{1}{24}s_2t^2}\times e^{\frac{1}{2880}s_4t^4}\right\}$$

$$=\frac{1}{abcde}\,co_{+2}\left\{\begin{array}{lll} 1+\dfrac{\nu^2}{1.2} & t & +\dfrac{\nu^4}{1.2.3.4}t^2 \\ \times & & \\ 1-\dfrac{1}{24} & s_2t & +\dfrac{1}{1152}\,s_2^2t^2 \\ \times & & \\ 1+\dfrac{1}{2880}\,s_4t^2 & & \end{array}\right\}$$

$$=\frac{1}{abcde}\left\{\frac{\nu^4}{24}-\frac{s_2}{24}\,\nu^2+\left(\frac{s_2^2}{1152}+\frac{s_4}{2880}\right)\right\}.$$

EXAMPLES OF ARITHMETICAL CALCULATION OF SIMPLE DENUMERANTS.

Example 1.—To find the complete expression for $\dfrac{n\,;}{1,\,2,\,3\,;}$

$$W_1=\frac{1}{1.2.3}co_2\left\{e^{(n+3)\,t}\times e^{-\frac{14}{24}t^2}\right\}$$

$$=\frac{1}{1.2.3}\,co_1\left\{1+\frac{(n+3)^2}{2}\,t\right\}\left\{1-\tfrac{14}{24}t\right\}\text{ (since }1+4+9=14)$$

$$=\tfrac{1}{12}\left\{(n+3)^2-\tfrac{7}{12}\right\};$$

$$W_2 = co_{-1}\,\frac{(-)^n\,e^{nt}}{(1+e^{-t})(1+e^{-3t})(1-e^{-2t})}$$

$$= \frac{(-)^n\,1}{8} = \tfrac{1}{8}\left\{\frac{n\,;}{2\,;} - \frac{n-1\,;}{2}\right\},$$

$$W_3 = co_{-1}\,\Sigma\,\frac{\rho^n\,.\,e^{nt}}{(1-\rho^{-1}e^{-t})(1-\rho^{-2}e^{-2t})(1-e^{-3t})} \quad (\text{where } \rho^2+\rho+1=0)$$

$$= \tfrac{1}{3}\left\{\frac{\rho^n}{(1-\rho)(1-\rho^2)} + \frac{\rho'^n}{(1-\rho')(1-\rho'^2)}\right\}$$

$$= \tfrac{1}{9}\,(\rho^n+\rho'^n)$$

$$= \tfrac{1}{9}\left\{2\,\frac{n\,;}{3\,;} - \frac{n-1\,;}{3\,;} - \frac{n+1\,;}{3\,;}\right\}.$$

Thus the complete analytical value of $\dfrac{n\,;}{1,\,2,\,3\,;}$ is

$$\frac{(n+3)^2}{12} - \tfrac{7}{144} + \tfrac{1}{8}\left\{\frac{n\,;}{2\,;} - \frac{n-1\,;}{2;}\right\} + \tfrac{1}{9}\left\{2\,\frac{n\,;}{3\,;} - \frac{n-1\,;}{3\,;} - \frac{n+1\,;}{3\,;}\right\}.$$

Since $\qquad \tfrac{7}{144}+\tfrac{1}{8}+\tfrac{2}{9} = \tfrac{57}{144} < \tfrac{1}{2},$

the arithmetical value of $\dfrac{n\,;}{1,\,2,\,3\,;}$ is the nearest integer to $\dfrac{(n+3)^2}{12}$, as had been early observed under a different form of statement by Mr. De Morgan.

Example 2.—The same process applied to $\dfrac{n\,;}{1,\,4,\,7\,;}$ will give

$$W_1 = \frac{1}{2\times4\times7}\left\{\left(n+\frac{1+4+7}{2}\right)^2 - \frac{1+16+49}{24}\right\}$$

$$= \tfrac{1}{56}\left\{(n+6)^2 - \tfrac{11}{4}\right\},$$

$$W_2 = \frac{(-)^n}{2^2\,.\,4} = \tfrac{1}{16}\left\{\frac{n\,;}{2\,;} - \frac{n-1\,;}{2\,;}\right\},$$

$$W_4 = co_{-1}\,\Sigma\,\frac{i^n e^{nt}}{(1-i^{-1}e^{-t})(1-i^{-7}e^{-7t})(1-e^{-4t})} \quad (\text{where } i^2+1=0)$$

$$= \tfrac{1}{4}\left\{\frac{i^n}{(1-i^3)(1-i)} + \frac{i'^n}{(1-i'^3)(1-i')}\right\}$$

$$= \tfrac{1}{8}\left\{i^n + i'^n\right\}$$

$$= \tfrac{1}{8}\left\{2\,\frac{n\,;}{4\,;} - 2\,\frac{n-2\,;}{4\,;}\right\}$$

$$= \tfrac{1}{4}\left\{\frac{n\,;}{4\,;} - \frac{n-2\,;}{4\,;}\right\}.$$

$$W_7 = co_{-1}\,\Sigma\,\frac{\theta^n e^{nt}}{(1-\theta^6 e^{-t})(1-\theta^3 e^{-nt})(1-e^{-7t})}$$

$$(\text{where } \theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1=0)$$

$$= \tfrac{1}{7}\Sigma\,\frac{\theta^n}{(1-\theta^3)(1-\theta^6)}$$

$$= \tfrac{1}{49}\Sigma\,\theta^n\,(1-\theta)(1-\theta^2)(1-\theta^4)(1-\theta^5)$$

$$= \tfrac{1}{49}\left\{\theta^n + \theta^{n+5} - 2\theta^{n+1} - 2\theta^{n+6}\right\}$$

$$= \tfrac{1}{7}\left\{\frac{n\,;}{7\,;} - 2\,\frac{n+1\,;}{7\,;} - 2\,\frac{n+4\,;}{7\,;} + \frac{n+5\,;}{7\,;} + 2\,\frac{n+6\,;}{7\,;}\right\}.$$

Thus the complete value of $\frac{n;}{1,\,4,\,7;}$, expressed in terms exclusively of

$$\nu = n+6,$$

is the following :—

$$\tfrac{1}{56}\left\{\nu^2 - \tfrac{11}{4}\right\} + \tfrac{1}{16}\left\{\frac{\nu\,;}{2\,;} - \frac{\nu-1\,;}{2\,;}\right\} + \tfrac{1}{4}\left\{\frac{\nu-2\,;}{4\,;} - \frac{\nu\,;}{4\,;}\right\}$$

$$+\tfrac{1}{7}\left\{2\,\frac{\nu\,;}{7\,;} + \left(\frac{\nu+1\,;}{7\,;} + \frac{\nu-1\,;}{7\,;}\right) - \left(2\,\frac{\nu+2\,;}{7\,;} + 2\,\frac{\nu-2\,;}{7\,;}\right)\right\}.$$

The limiting values of the sum of the second, third, and fourth waves for any value of n will be

$$\frac{1}{16} + \frac{1}{4} + \frac{2}{7} = \frac{67}{112}$$

on the positive side, and

$$+\frac{1}{16}-\frac{1}{4}-\frac{2}{7}=\frac{-53}{112}$$

on the negative side.

Hence the difference between the exact value and $\frac{\nu^2}{56}$ must lie between $\frac{123}{224}$ and $\frac{-117}{224}$.

So that in the greatest number of cases the *nearest* integer to $\frac{(n+6)^2}{56}$ gives the value of $\frac{n\,;}{1,\,4,\,7\,;}$; and the result can never be in error by more than a single unit.

An analogous approximate form of representation can be made for the number of modes of composing an integer with any number of elements mutually prime to each other.

Observe in the foregoing expression that the form $\frac{\nu+i\,;}{q\,;}$ is always paired with $\frac{\nu-i\,;}{q\,;}$: $\frac{\nu\,;}{q\,;}$ $\Big($which in the actual case under consideration is $\frac{\nu\,;}{7\,;}\Big)$ affords no exception, for this may be expressed as

$$\tfrac{1}{2}\left\{\frac{\nu+0\,;}{q\,;}+\frac{\nu-0\,;}{q\,;}\right\}.$$

Neither does $\frac{\nu-r\,;}{2r\,;}$ $\Big($in the actual case $\frac{\nu-1\,;}{2\,;}\Big)$, for this may be expressed as

$$\tfrac{1}{2}\left\{\frac{\nu-r\,;}{2r\,;}+\frac{\nu+r\,;}{2r\,;}\right\}.$$

The sign of the pairing may be positive or negative according to a rule which the exhibition of the result worked out in the following examples will render clear.

Example 3.—The denumerant $\frac{n\,;}{1,\,3,\,5\,;}$, expanded in a similar manner, gives rise to the following expression :—

$$\tfrac{1}{30}\left\{(\nu)^2-\tfrac{35}{24}\right\}+\tfrac{2}{9}\left\{\frac{\nu+\frac{3}{2}\,;}{3\,;}+\frac{\nu-\frac{3}{2}\,;}{3\,;}-\frac{\nu+\frac{1}{2}\,;}{3\,;}-\frac{\nu-\frac{1}{2}\,;}{3\,;}\right\}$$
$$+\tfrac{1}{5}\left\{\frac{\nu+\frac{1}{2}\,;}{5\,;}+\frac{\nu-\frac{1}{2}\,;}{5\,;}-\frac{\nu+\frac{3}{2}\,;}{5\,;}-\frac{\nu-\frac{3}{2}\,;}{5\,;}\right\}.$$

And, since $\frac{2}{9}+\frac{1}{5}+\frac{7}{144}=\frac{339}{720}<\frac{1}{2}$,

the arithmetical value of $\frac{n\,;}{1,3,5\,;}$ is always the *nearest integer* to $\frac{(2n+9)^2}{120}$.

This arithmetical mode of statement, how capable of extension to any set of elements following the natural order of the prime numbers, and to other cases.

Example 4.—The denumerant $\frac{n\,;}{1,2,3,4,5,6,7\,;}$ in its expanded form is expressed by the following function of ν, which here represents $n+14$, viz.,

$$\frac{1}{725760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2-\frac{190325}{126}\right\}$$

$$+\frac{1}{768}\left\{\frac{\nu^2}{2}-\frac{77}{6}\right\}\left(\frac{\nu\,;}{2\,;}-\frac{\nu-1\,;}{2\,;}\right)$$

$$+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,;}\right\}\nu-\frac{5}{972}\left\{\left(\frac{\nu\,;}{3\,;}+\frac{\nu\,;}{3\,;}\right)-\left(\frac{\nu-1;}{3\,;}+\frac{\nu+1\,;}{3\,;}\right)\right\}$$

$$+\frac{1}{64}\left\{\left(\frac{\nu+2\,;}{4\,;}+\frac{\nu-2\,;}{4\,;}\right)-\left(\frac{\nu\,;}{4\,;}+\frac{\nu\,;}{4\,;}\right)\right\}$$

$$+\frac{1}{25}\left\{\left(\frac{\nu+1\,;}{5\,;}+\frac{\nu-1\,;}{5\,;}\right)-\left(\frac{\nu+2\,;}{5\,;}+\frac{\nu-2\,;}{5\,;}\right)\right\}$$

$$+\frac{1}{36}\left\{\left(\frac{\nu+3\,;}{6\,;}+\frac{\nu-3\,;}{6\,;}\right)-\left(\frac{\nu\,;}{6\,;}+\frac{\nu\,;}{6\,;}\right)+\left(\frac{\nu-2;}{6;}+\frac{\nu+2\,;}{6\,;}\right)\right.$$
$$\left.-\left(\frac{\nu-1;}{6;}+\frac{\nu+1\,;}{6\,;}\right)\right\}$$

$$+\frac{1}{49}\left\{3\left(\frac{\nu\,;}{7\,;}+\frac{\nu;}{7;}\right)-\left(\frac{\nu+1\,;}{7\,;}+\frac{\nu-1\,;}{7\,;}\right)-\left(\frac{\nu+2\,;}{7\,;}+\frac{\nu-2\,;}{7\,;}\right)\right.$$
$$\left.-\left(\frac{\nu+3\,;}{7;}+\frac{\nu-3\,;}{7\,;}\right)\right\}.\text{*}$$

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* The arithmetical value of $\frac{\nu-14\,;}{1,2,3,4,5,6,7\,;}$ is obviously the nearest integer to $\frac{1}{5760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2\right\}+\frac{1}{1536}\left(\frac{\nu\,;}{2}-\frac{\nu-1\,;}{2\,\cdot}\right)\nu^2+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,\cdot}\right\}\nu.$

Observation.—Provided that $\frac{-p\,;}{i\,;}$ shall be understood to signify the same thing as $\frac{p\,;}{i\,;}$, every wave in the above expansion remains entirely unaltered when ν becomes $-\nu$.

A priori view of the form of such expansions of $\frac{n\,;}{a, b, c, \ldots l\,;}$.

First, the algebraical part is, as it ought to be, a *homogeneous* function of n; a, b, c, ... l.

Secondly, the change of ν into $-\nu$ either leaves the expansion absolutely unaltered, or unaltered save as to algebraical sign.

This depends on the theory of the *denumerantive functions* as distinguishable from denumerants. The latter discontinuous quantities, the former continuous.

Binary denumerants have in general several functions attached to them, viz., one less than the number of their denominatives.*

All generated forms have arithmetical and functional values.

Example.—The form u_n generated by

$$\frac{1}{(1-x)^2} = u_0 + u_1 x + \ldots u_n x^n + \&c.$$

Property of the denumerantive function $\phi\,(n, a, b, c, \ldots l)$ to $\frac{n\,;}{a, b, c, \ldots l\,;}$; viz., that

$$\phi\,(n, a, b, c, \ldots l) = \pm\,\phi\,(n', a, b, c, \ldots l),$$

if

$$n + n' = -a - b - c \ldots - l,$$

the + sign being used when the number of elements is odd, and the negative sign when it is even.

This explains the pairing of the terms observed in $\frac{n\,;}{1, 2, 3, \ldots 7\,;}$. Great importance of this fact of pairing.

The number of modes of dividing n into seven parts is represented by the above formula, viz., with repetitions and zero values of parts allowed by making

$$\nu = n + 14,$$

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* This is when the form zero is not counted as a function. Zero occurs as a form once only in simple, but twice over in binary denumerants.

with repetitions and zero values disallowed by making

$$\nu = n - 14,$$

with repetitions allowed, but zero values disallowed, by making

$$\nu = n + 7.$$

And so in general with the values $n + \frac{(r-1)\,r}{4}$; $n - \frac{(r-1)\,r}{4}$; $n - r$ respectively substituted for ν.

Mr. Kirkman's representation of partitions to the modulus 7.

Example 5.—Expansion of $\frac{n:}{1,2,3,4,5,6:}$ as a function of n.

$$\tfrac{1}{17280}\left\{\frac{\nu^5}{5} - \frac{91}{6}\nu^3 + \frac{9191}{48}\right\} + \tfrac{1}{768}\left\{\frac{\nu+\frac{1}{2}:}{2:} - \frac{\nu-\frac{1}{2}:}{2:}\right\}(\nu^2 - \tfrac{161}{12})$$

$$+\tfrac{1}{162}\left\{\left[\left(\frac{\nu+\frac{3}{2}:}{3:} + \frac{\nu-\frac{3}{2}:}{3:}\right) - \left(\frac{\nu+\frac{1}{2}:}{3:} + \frac{\nu-\frac{1}{2}:}{3:}\right)\right]\nu + \left(\frac{\nu+\frac{1}{2}:}{3:} - \frac{\nu-\frac{1}{2}:}{3:}\right)\right\}$$

$$+\tfrac{1}{32}\left\{\left(\frac{\nu+\frac{3}{2}:}{4:} - \frac{\nu-\frac{3}{2}:}{4:}\right) + \left(\frac{\nu+\frac{1}{2}:}{4:} - \frac{\nu-\frac{1}{2}:}{4:}\right)\right\}$$

$$+\tfrac{1}{25}\left\{\left(\frac{\nu-\frac{1}{2}:}{4:} - \frac{\nu+\frac{1}{2}:}{4:}\right) + \left(\frac{\nu-\frac{3}{2}:}{4:} - \frac{\nu+\frac{3}{2}:}{4:}\right)\right\}$$

$$+\tfrac{1}{18}\left\{\left(\frac{\nu+\frac{3}{2}:}{6:} - \frac{\nu-\frac{3}{2}:}{6:}\right) + \tfrac{1}{36}\left(\frac{\nu+\frac{5}{2}:}{6:} - \frac{\nu-\frac{5}{2}:}{6:}\right) + \left(\frac{\nu+\frac{1}{2}:}{6:} - \frac{\nu-\frac{1}{2}:}{6:}\right)\right\},$$

where $\nu = n + \frac{21}{2}$.

Observe the substitution of the colon for the semicolon above and below the line in the fraction-form to distinguish a denumerantive function from a denumerantive proper. The arithmetical value of the foregoing is the nearest integer to the sum of its first, second, and third waves; and in the two latter it is only necessary to retain those terms which contain ν^2 and ν respectively.

On the expression for the number of waves when the denominatives of a denumerant or the elements of a partition are given.

On the blending of waves, and its advantages in some cases, as when the elements are all prime to each other without all being absolute primes.

Easy mode of deducing the fundamental theorem, by the application of a formula in the calculus of residues to the Eulerian. Its capital importance in the theory of partitions.

Close of the analytical portion of the course.

SEVENTH LECTURE.*

The representation of systems of linear equations by clusters of points recalled.

A single equation by a cluster of points in a line, of two simultaneous equations by a cluster of points in a plane, three simultaneous equations by a cluster of points in space—geometrical criterion between definite and indefinite systems.

The linear cluster which corresponds to a single equation is unique in form, not so the plane cluster which corresponds to two, or the solid cluster which corresponds to three equations. One arbitrary parameter enters into the former, three into the latter.

All such clusters balance about their respective origins with the same weights at corresponding points.

Clusters so related might be termed homobaric.

Mechanical representation of the property of homobarism by a series of jointed parallelograms having two axes in common (see Fig. *A*).

Criterion between definite and indefinite systems recalled.

"To determine the chance that three points thrown anywhere within a parallelogram may contain the centre."

Solution of this problem by theory of definite and indefinite binary equation systems with three variables alluded to.

The chance is $\frac{5}{32}$ths against the points including the centre, whatever the form or dimensions of the parallelogram.

Hence the chance is $\frac{5}{32}$ths against three points capable of being taken anywhere in an indefinite plane including a *given* fourth point in the same plane.

But it would be incorrect to infer from this that the chance of some three out of four points (capable of being taken anywhere in an indefinite plane), including the fourth, is $\frac{5}{8}$; it will be much less than this.

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* Delivered at King's College, London, July 11th, 1859.

Explanation of this seeming paradox. Geometrical and analytical modes of treating this second question alluded to.

Experimental method of verification. New game of odd and even.

The natural order of the variables in a single homonymous equation recalled.

Unless definite there is no natural order.

The importance of obtaining such natural orders to the theory of compound partition, viz., in applying the process of eduction. Example of natural and disturbed order.

General *analytical* condition of normal sequence.

Difficulty of seeing any natural order among the rays of a solid cluster; it will presently appear that such orders do exist, but that instead of one natural order there are several; the number depending (1) upon the number of points in the cluster, (2) upon the mode in which the rays are grouped, subject to the observation that the distinct modes of grouping in the view of this theory are always limited in number, and determinable *a priori.*

Passage by perspective from the grouping of rays in a plane to the grouping of points in a line; from the grouping of rays *in solido* to the grouping of points in a plane; and from the grouping of rays in plu-space to the grouping of points *in solido.*

Observe that in studying the character of an equation-system no attention need in the first instance to be paid to the primary, because the process of eduction concerns the variables only, and not the constant terms in the system; by the act of taking the perspective, the origin no longer appears—thus, nothing is left but a perspective cluster or group of points, as many in number as the variables of the system.

Theorem.—The number of classes of definite binary systems of linear equations for any given number of variables is *one*, because there is but one species of arrangements of a given number of points in a line. The number of classes of definite ternary systems of equations with r variables is the number of distinct modes of grouping together r points in a plane; the number of classes of definite quaternary systems of equations with r variables is the number of distinct modes of grouping together r points in space.

Observe the fact of space being made subservient through the method of perspective to systems of linear equations greater in number than the so-called dimensions of space.

In determining the natural order in a binary system, the perspective group may be substituted for the cluster, provided the line of projection cuts all the rays on the same side of the origin, so that a line through the origin parallel to the line of projection falls *outside* the cluster; but, if this condition is not observed, the order will be disturbed. (See Fig. *B*.)

In like manner, for a ternary system, the plane of projection must be supposed to be drawn parallel to a plane through the origin external to the solid cluster.

Observation on Plane and Solid Groups, considered as representing definite Ternary and Quaternary Equation Systems respectively.— If we suppose a ternary system of which one of the equations is of the form

$$x+y+z+\ldots u-1=0,$$

the others being

$$ax+by+cz+\ldots+lu-m=0,$$

$$\alpha x+\beta y+\gamma z+\ldots+\lambda u-\mu=0,$$

such a system may evidently be represented by a group of points in a plane whose coordinates are

$$(a, \alpha)(b, \beta)(c, \gamma) \ldots (l, \lambda);\ (m, \mu),$$

and

$$x, y, z, \ldots u;\ -1,$$

and will be the weights to be placed at these points respectively in order to *balance* each other.

Moreover, if we start with any *definite* ternary system, we may substitute for one of the equations in it a homonymous equation reducible to the form of

$$x'+y'+z' \ldots +u'-1=0,$$

on taking

$$x', y', z', \ldots u',$$

all homonymous multiples of

$$x, y, z, \ldots u.$$

Consequently, the *form* of the plexus of principal derivatives which depends essentially only on the relations of algebraical signs in the coefficients of these derivatives will be the same whether the system be considered as involving explicitly $x, y, z, \ldots u$, or $x', y', z', \ldots u'$, and, consequently, every definite ternary system of equations whatever may be represented in its essential properties of form by a plane group of points, in lieu of a solid cluster of rays. And in like manner, without going from hypersolidity to the solid, we see that any definite quaternary system may be represented by a group of points *in solido.*

The property indicated of the self-balancing group *in plano* being substitutable for the solid cluster with its centre of gravity at the origin may be deduced easily from this more general theorem, that if two groups of weighted points are in perspective about a given point G, and the weights at the corresponding points in the two groups are in the inverse ratios of their distances from G, if one of them has its centre of gravity at G, the other also will have its centre of gravity there. Hence, if one of these groups be considered as a derivative from the first, and all the points of the derivative group be brought to lie within the same plane, it must become self-balancing, since otherwise a plane group of statical points would have its centre of gravity outside the plane.

Notice that the geometrical construction for determining whether a system of equations is definite or indefinite would fail for a quaternary system, but the analytical method operative through the principal plexus continues to hold good.

Ternary Systems, and Plane Groups.

Imagine a sphere to be drawn with the origin of a solid cluster as its centre; the general arrangement of the points on the sphere will correspond to the arrangement of the points on the perspective plane, and, when convenient to do so, the one may be substituted for the other.

Illustration by examples with four and five points.

Recall eduction and the condition of its giving rise to definite systems, namely, that the systems deduced by the successive deformations of the given system shall remain definite, *i.e.*, external to the centre. If a group of points on a sphere be contained within the boundary of a hemisphere, the centre will be external to such group; but, if the bounding contour of the group formed by arcs of great circles cover more than half the sphere, the centre will be contained within the group. The effect upon the spherical perspective of a cluster representing a ternary system of equations due to the change of a variable x into $-x$ is to make the point corresponding to the coefficients of x pass to the opposite end of the diameter passing through it; such a change may be termed a *reversal* of the point, and the point so obtained the opposite of the original point. The problem of normal orders for ternary systems may therefore be stated geometrically as follows :—

A given number of points being contained within a hemisphere, to discover what orders of sequence of these points will possess this property that on the first, second, third, &c., to the last of them, one after the other undergoing *reversal*, the transformed group shall never occupy more than half the surface of the sphere.

From this it follows that, if

$$xyz \ldots tuv$$

be a normal sequence, $\qquad vut \ldots xyz$

will be so likewise, for, if we denote the opposite points to

$$xyz \ldots uv \quad \text{by} \quad x'y'z' \ldots u'v',$$

respectively, it is clear that, if the groups

$$\left.\begin{array}{l} x'yz \ \ldots\ uv \\ x'y'z \ \ldots\ uv \\ x'y'z' \ \ldots\ uv \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x'y'z' \ \ldots\ u'v \end{array}\right\}$$

are respectively contained, each within their own hemispheres, the groups

$$\left.\begin{array}{l} v'u \ \ldots\ zyx \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ v'u' \ \ldots\ zyx \\ v'u' \ \ldots\ z'yx \\ v'u' \ \ldots\ z'y'x \end{array}\right\}$$

will each also be contained in hemispheres opposite to the former, taken in reverse order.

The *contour* of a spherical group defined.

What is meant by a peripheral and what by an internal point to a group on a sphere.

Again, to obtain the law of normal sequences, we have the following propositions :—

(1) Any sequence $x, y, z, t, \ldots u, v, w$ of points in a sphere will be a normal order of sequence, provided the following condition is satisfied, viz., that, on joining those points with each other in the order of their succession by arcs of great circles, the broken line or spherical zig-zag so formed shall be capable at every one of its angular points of being divided into two parts by a great circle which does not cut the line at any other point; for evidently in such case, if we draw a great circle through a point u, which does not cut any of the sides, or any other angle except u, the points x, y, z, t being all reversed, will lie together with v, w in a hemisphere bounded by the great circle so drawn.

(2) A normal sequence of points in a group cannot be bounded at either extremity by an interior point of the group. For, on joining the opposite of such exterior point with the closed figure surrounding that point, we evidently obtain a figure clasping the hemisphere bounded by the great circle perpendicular to the diameter through these points, and stretching into the hemisphere beyond. On the other hand, a normal order may always be commenced from any peripheral point in the group at pleasure, for, if u, v, y, z, x, t be any group contained within a hemisphere H, and u a point in the contour, it is apparent that u, v, y, z, x, t, u' will also be contained within the same hemisphere H, so that, in fact, one way of characterizing a normal sequence would be as a sequence in which each point in turn becomes a peripheral point, alike when all the points preceding as when all the points following it are *reversed*.

(3) No arc joining y, z, two consecutive points in a normal sequence $x, y, z, t, u, v, w, \omega$, can cut uv any other such arc, for it is clear that, if yz crosses uv, x will be contained within the triangle $y'uv$ (y' meaning the opposite points to y), and, consequently, z will not be external to $tuvw\omega x'y'$, as it must be if the given order is normal.

(4) It follows also, as an immediate corollary from 2, that no point t can be contained within the contour of xyz or within that of $uvw\omega$.

Hence (5) it follows from (3) and (4) that the two spherical areas bounded respectively by the contours of xyz, $uvw\omega$ have no part whatever in common, and, consequently, may be separated by a great circle drawn through t.

Hence, combining the conclusions of (1) and (5), we arrive at the theorem that the sole necessary and sufficient condition for determining $x, y, z, t, u, v, w, \omega$ to be in normal sequence upon a sphere is that through any point as t a great circle can be drawn upon the sphere not cutting this line in any other point; and, consequently, the sole necessary and sufficient condition for a number of points in a plane group being in normal sequence is that the zigzag formed by drawing straight lines from any one point to the next in the sequence shall be capable of being cut in twain at any of its angles by a right line.

General definition of a diatomic line continuous or discontinuous in a *plane* or in *space* (see plate).

The condition of normal sequence may be extended from plane to solid groups, *i.e.*, from ternary to quaternary equation systems, the sole necessary and sufficient condition for determining a normal order of points, as well *in solido* as *in plano*, being that the zigzag following the succession of the points in the order shall be a *diatomic* line.

In order to depress a ternary system so as to make its denumerant depend upon binary denumerants, we must be able to form orders of normal sequence among its variables.

Every such order will or may furnish two distinct forms of solution, provided the requisite conditions of relative primeness and asyzygeticism are satisfied.

The easiest way of determining such normal orders is by means of diatomic lines drawn from point to point of the representative plane group.

Every ternary system may be identified by means of its principal plexus, as will presently be shown with some specific form of group, corresponding in number to the number of the variables in the system. It becomes necessary, therefore, to facilitate the solution of the problem of denumeration of ternary systems, to classify and register the distinct forms of arrangement of plane groups (and in like manner, in order to make denumerants of the fourth order depend upon those of the third, we must begin with classifying and registering the various dispositions of which a given number of points is susceptible in space).

Plane Groups.

For three points only one species of arrangement is possible, and all the orders are normal orders.

For four points two distinct arrangements only are possible, viz., of four points external to one another, or three points with a fourth point in the interior.

Morph defined—its geometrical and analytical meaning.

Exclusion of syzygetic cases.

The morph corresponding to the one case (see plate) will be the following :—

$$xy : \quad yz : \quad zt : \quad tx :$$

$$x : z \quad y : t$$

The morph corresponding to the other case (see plate) will be

$$zt : \quad xt : \quad yt :$$

$$y : z \quad z : x \quad x : y$$

Accordingly, the plexus of principal derivatives to every definite ternary system with four variables, omitting the constant terms, will be of one or the other of the two following forms :—

$$\left.\begin{array}{ll} ax+by=? & lx-mz=? \\ cy+dz=? & ny-pt=? \\ ez+ft=? & \\ gx+ht=? & \end{array}\right\} \text{1st form,}$$

$$\left.\begin{array}{ll} az+bt=? & \alpha y-\beta z=? \\ cx+dt=? & \gamma z-\delta x=? \\ ey+ft=? & \epsilon x-\phi y=? \end{array}\right\} \text{2nd form.}$$

The constants are of course not independent of each other, nor are the *signs* of the quantities left unexpressed, but symbolized by ?

independent. How to assign the laws of the dependence is not material to our immediate object.

By means of these two forms, we can by examination of the plexus of principal derivatives to a ternary system of linear equations with four variables, ascertain whether it corresponds to the form of group in Figs. 2, 3, or to that in Fig. 4.

The normal orders belonging to these groups respectively are shown in the plate.

If we proceed now to the groupings of five points, we shall find that the only essentially distinct arrangements are the following:—

Five points all external to each other.

Four points with one inside.

Three points with two inside.

No differences of any other kind than those above denoted will give rise to any difference in the character of the morph resulting. The three morphs corresponding to these three arrangements are the following:—

$$\left.\begin{array}{lllll} ztu: & tux: & uxy: & xyz: & yzt: \\ zt:x & tu:y & ux:z & xy:t & yz:u \end{array}\right\} \text{1st form,}$$

$$\left.\begin{array}{llll} ztu: & xtu: & xyu: & yzu: \\ & zu:x & tu:y & \\ yz:t & xt:z & xy:t & xy:z \end{array}\right\} \text{2nd form,}$$

$$\left.\begin{array}{lll} ztu: & xtu: & ytu: \\ uz:y & uz:x & yu:x \\ yt:z & xt:z & xt:y \\ & xz:y & \end{array}\right\} \text{3rd form.}$$

And by simply observing whether the principal plexus has three, four, or five homonymous derivatives, the perspective representation of any definite ternary system of linear equations with five variables can be identified with one or the other of the Figures 5–7, 8–11, or 12–15. The diatomic zigzags expressing the normal orders to these figures are given in the plate.

So far we have found that the number of morphs has followed the progression 1; 2; 3; arising from the fact that there are no essential differences of position of a point or a couple of points within a triangle or of one point within a quadrilateral. Very different is the case for a figure of six points, corresponding to a definite ternary system, with six variables. The following cases arise :—

(1) The six points may be at the angles of a hexagon.

(2) Five of the six points may be at the angles of a pentagon, and the interior point may occupy any one of three essentially distinct regions, each such position giving rise to a distinct species of morph confined to the inner.

These three distinct kinds of position defined.

(3) Four of the points may be external, and the pair of points within occupy six distinct kinds of position.

These six different relative positions described.

Finally, three of the six points may be external, so that the figure may be viewed as a triangle within a triangle, and there will be six distinct relative positions of triangles so related.

These six different relative positions described.

There are thus in all sixteen different classes of arrangement of six points in a plane, and therefore sixteen different classes of ternary systems with six variables.

The normal orders for each of these sixteen cases have been completely worked out by Captain Noble, R.A., and the numbers for each figure attached to them in the plate (16–31).

On the arrangement of plane groups in natural families.

Classes belong to the same natural family which are capable of appearing simultaneously in the same course of the same process of eduction.

Any two classes of systems belong to the same family which may be obtained from one or other by altering the signs concurrently of one or more of such of the coefficient groups as may be so altered without the system ceasing to be definite. If in the morph of any class we make any letter pass from right to left of the colon, and *vice versa* throughout, and after such change the morph still contains characters of the form $xyz \ldots u :$ corresponding to homonymous principal derivatives, the morph so obtained will belong to the same family with that from which it is derived. Thus from one morph all others of the same family may be derived by simple inspection and transposition.

Conversion defined.

Scales of derivation in general are divaricative, but for principal family are lineal.

Example four-, five-, and six-point systems.

How to determine *a priori* what letters in a given morph are convertible.

Examples in six-point systems.

The number of families for six-point arrangements is four.

The two classes of four-point systems, and the three classes of five-point systems, belong respectively to a single family. Proof.

Numerical and natural modes of classifying groups contrasted.

The principal class and principal family of any r-point group defined.

The tactical rule which serves to define any normal order in the principal class. Example in five-point system. Example in six-point system.

In four- and five-point systems all the classes belong to the principal family, there being no other.

The numerical system of arrangement in families gives rise to a new question in the partition of numbers.

Thus a seven-point system, and an eight-point system arranged after the numerical system, consist respectively of families which may be typified as follows:—

$$7 \quad 6,1 \quad 5,2 \quad 4,3 \quad 3,4 \quad 3,3,1$$

$$8 \quad 6,2 \quad 5,3 \quad 4,4 \quad 4,3,1 \quad 3,5 \quad 3,4,1 \quad 3,3,2$$

Theorem.—All classes of the same family may be derived from one another by perspective projection.

Conversion balls and their use.

Theorem.—Normal orders are orders of perspective sequence (see plate).

Application of perspective regions to finding normal orders by exhaustive method.

The position of the eye must be external to the group.

The entire plane outside the group may be divided into as many distinct perspective regions as there are normal orders (see plate).

A ship tacking along a diatomic zigzag is continually making angular way in reference to a point taken anywhere in some determinate region.

Normal orders of points in space are also orders of double perspective sequence, a line of view and planes of light being substituted for the point of view and rays of light.

Four-, five-, and six-point systems in space, like three-, four-, and five-point systems in planes, are reducible respectively to one, two, and three classes.

The classes of quaternary point-systems like those of ternary, and by the same method may be arranged in natural families.

Reasons for believing a higher or more complex colligation of classes possible for quaternary systems.

Although the geometry of dispositions does not explicitly recognise distinctions grounded on magnitude, still the relations which it contemplates must admit of quantitative discrimination.

The *cæsura* in the *eduction* process following any normal order of the variables; how determined geometrically for ternary or quaternary systems by the principle of *denudation;* conformity of this with the rule for binary systems.

How the neutral region in a normal order which does not exist for binary arises for ternary and higher systems (see plate).

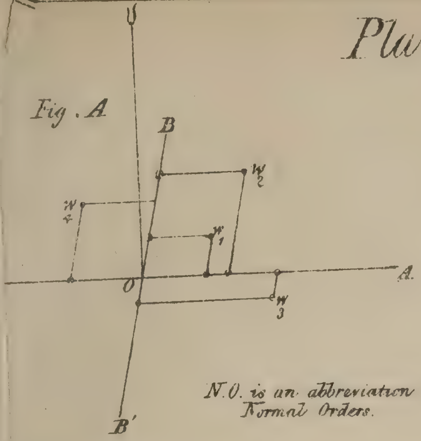
Example.—The distances from each other of four points in a plane being given (six quantities connected by one equation) it must be possible to form one or more rational functions of these quantities of which the values as positive or negative must serve to discriminate between the two kinds of disposition in which four points may be grouped.

General character of the new geometry of disposition.

" It is the theory of permutation in space."—*Cayley.*

Plate 1

Fig . A



N.O. is an abbreviation for Normal Orders.

ipal Family

17.

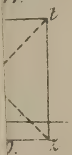


Fig .18.

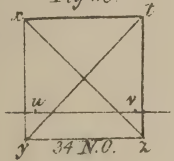
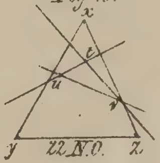
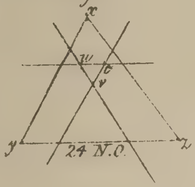


Fig 19.



nd Family

Fig. 21.



rd Family

Fig. 24.

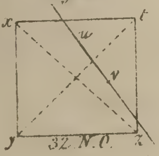


Fig. 25.

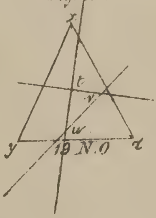
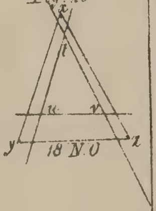


Fig. 26



Family.

Fig. 29.

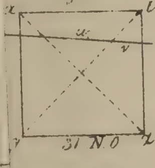


Fig 30.

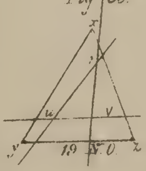


Fig. 31.

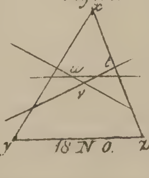
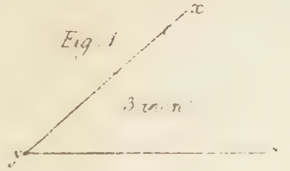


Plate 1

Normal Orders
3 Pts.

Fig. 1



Normal Orders 4 Pts

Principal Case
8 N.O.

Fig. 2.

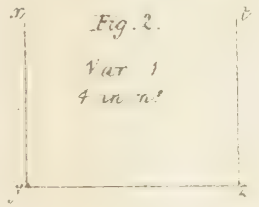
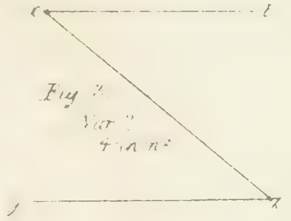
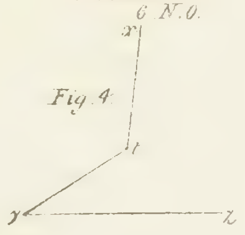


Fig. 3



3 Pts and
1 internal
6 N.O.

Fig. 4



Normal Orders 5 Pts

Principal Case 15 N.O.

Fig. 5

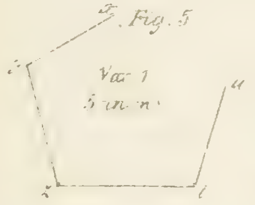


Fig. 6

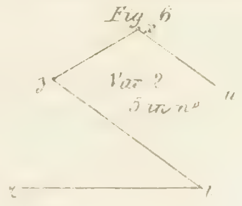
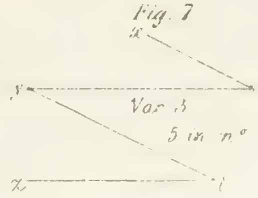


Fig. 7



Four Pts and 1 internal Pt 16 N.O.

Fig 8

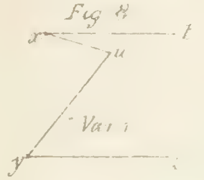


Fig 9

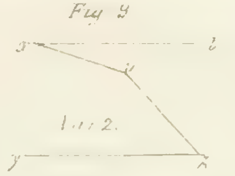


Fig 10

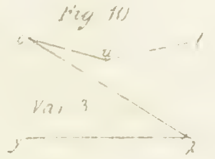
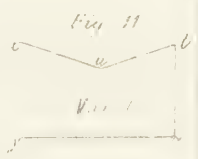


Fig 11



Three Pts and 2 internal Pts 11 N.O.

Fig 12

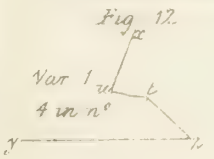


Fig 13

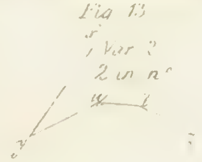


Fig 14

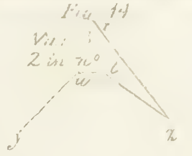
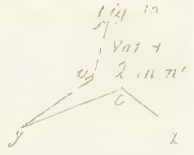
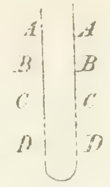


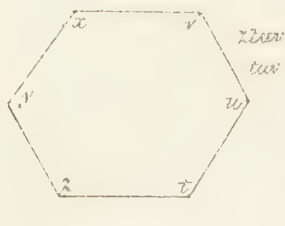
Fig 15



Scale of Derivation to principal family in 6 point system



A

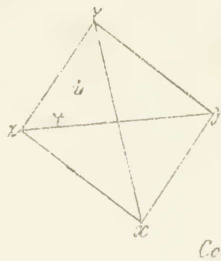


Morph

ztuv. tuvx: uvxy. vxyz xyzt: yztu.
tuv:y uvx:z vxy:t xyz:u yzt:v ztu:x
yz vu zt:vx tu:xy

Convertibles x, y, z, t, u v. Convert x.

C

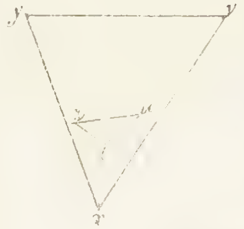


Morph

tuvx: tuxz. tuxy tuyv:
tux:y tuv:x
yuv:t xz.uv vu:xy xyu:x
zt:yv ztx:v yxt:v xt:xy
vz:xy

Convertibles x, y, v z. Convert z.

B

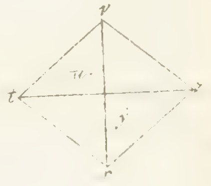


Morph

ztux: ztuy. xtuv:
yz.uv xz.uv xu.yz
yxt:v xzt:v yzt:x
tuv:y tuv:x xtu:y
xt:yv yz xv yuv:x

Convertibles x, y v   Convert y.

D

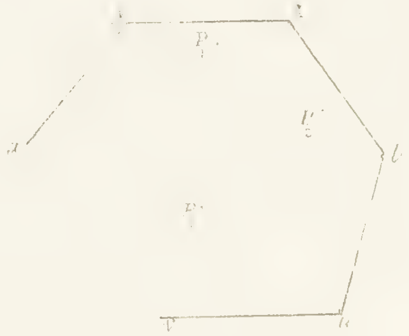


Morph

uvt: uytx. uyxz: uyzv:
tu:yx vu:xy
tuv.z tuv:x vuz.x tux:z
vyz:t txy v xyz t xyz:v
tx:vz

Convertibles x, z, v, t

Denudation Illustrated



Normal Order Punctuated

Primary at P1 xv ztuv

P2 xy; z, tuv

P3 xy, zt, uv

Fig. B

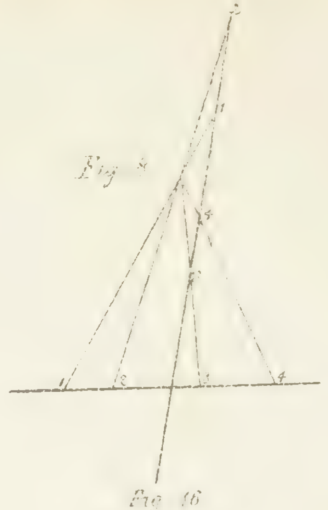
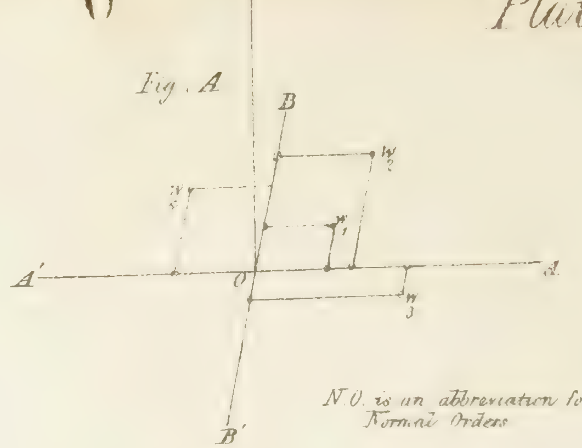


Fig. A



N.O. is an abbreviation for Normal Orders

Principal Family

Fig 16

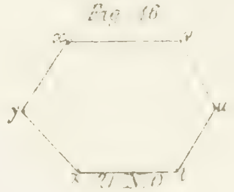


Fig 17

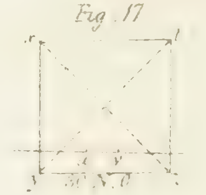


Fig 18

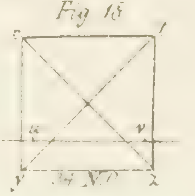
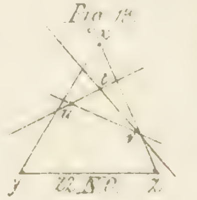


Fig 19



Second Family

Fig 20

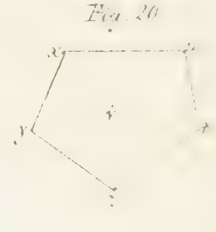
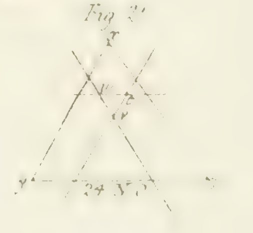


Fig 21



Third Family

Fig 22

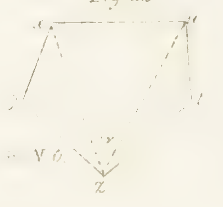


Fig 23

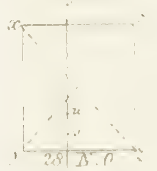


Fig 24

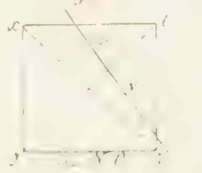


Fig 25

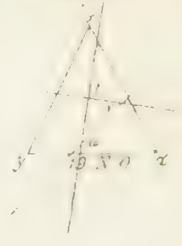
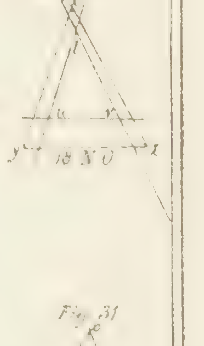


Fig 26



Fourth Family

Fig 27

Fig 28

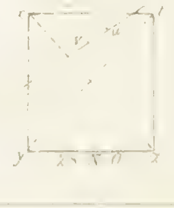


Fig 29

Fig 30

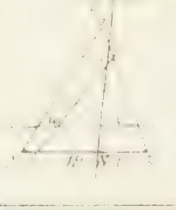


Fig 31

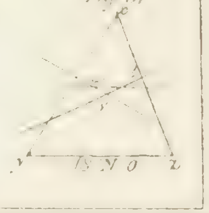
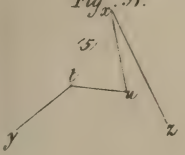


Plate 2.

Fig. 54.



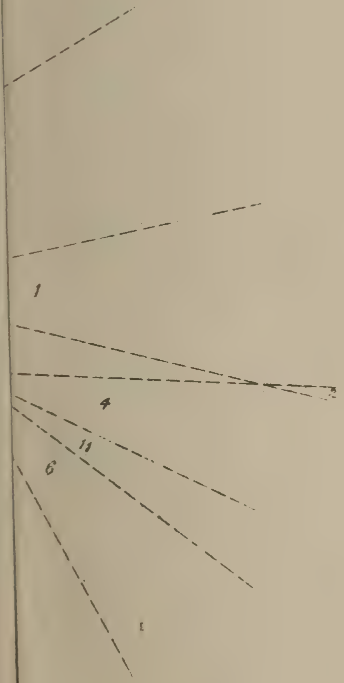


Fig. 59.

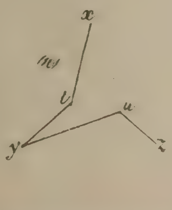


Fig. 60.

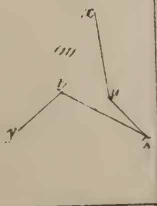


Plate 2.

Coiled Lines or Lines of Abnormal Order.

Fig. 32.

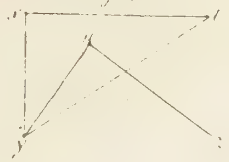


Fig. 33.

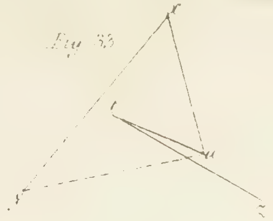


Fig. 34.

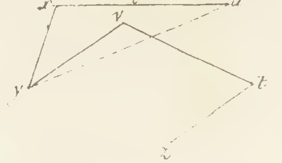
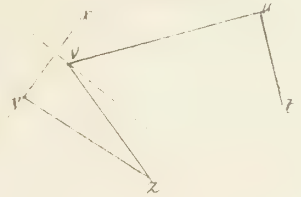


Fig. 35.



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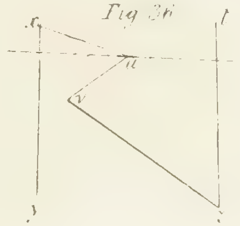
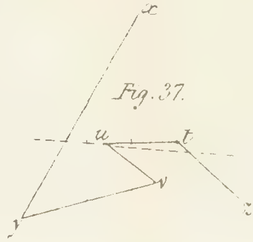


Fig. 37.



True Zig-zags or Lines of Normal Order.

Fig. 38.

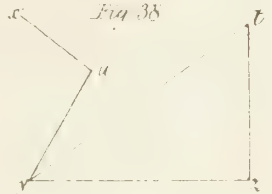


Fig. 39.

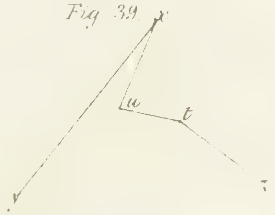


Fig. 40.

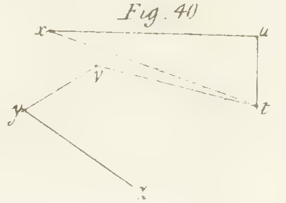


Fig. 41.

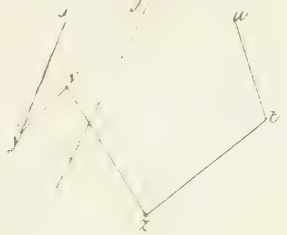


Fig. 42.

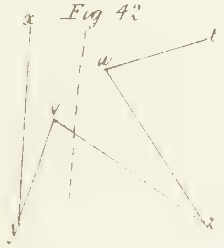


Fig. 43.

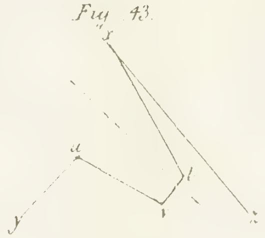
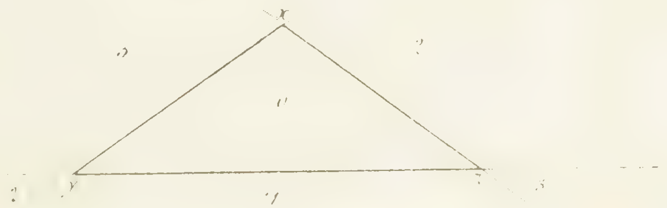


Fig. 44.



Perspective Regions of Normal Orders.

Fig. 45.

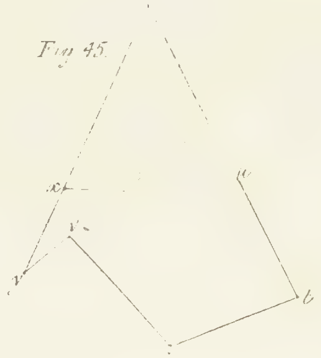


Fig. 46.

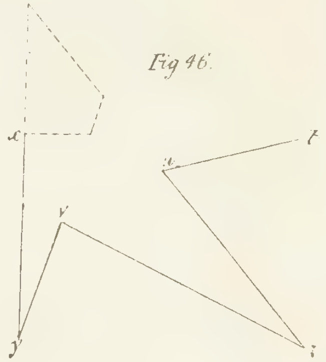
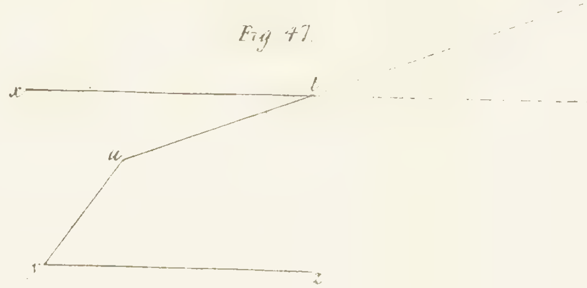
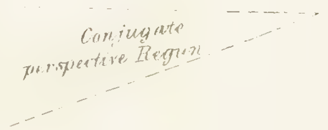


Fig. 47.





Crossed Lines or Lines of Abnormal Order.

Fig. 48.

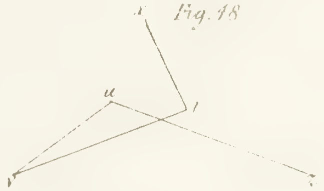


Fig. 49.

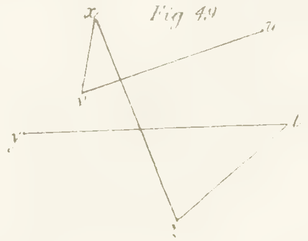


Fig. 50.

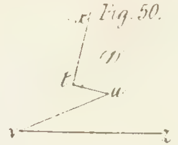


Fig. 51.

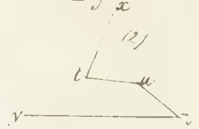


Fig. 52.

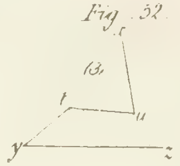


Fig. 53.

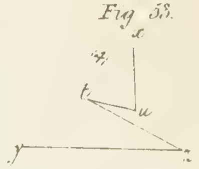


Fig. 54.

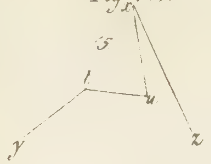


Fig. 61.

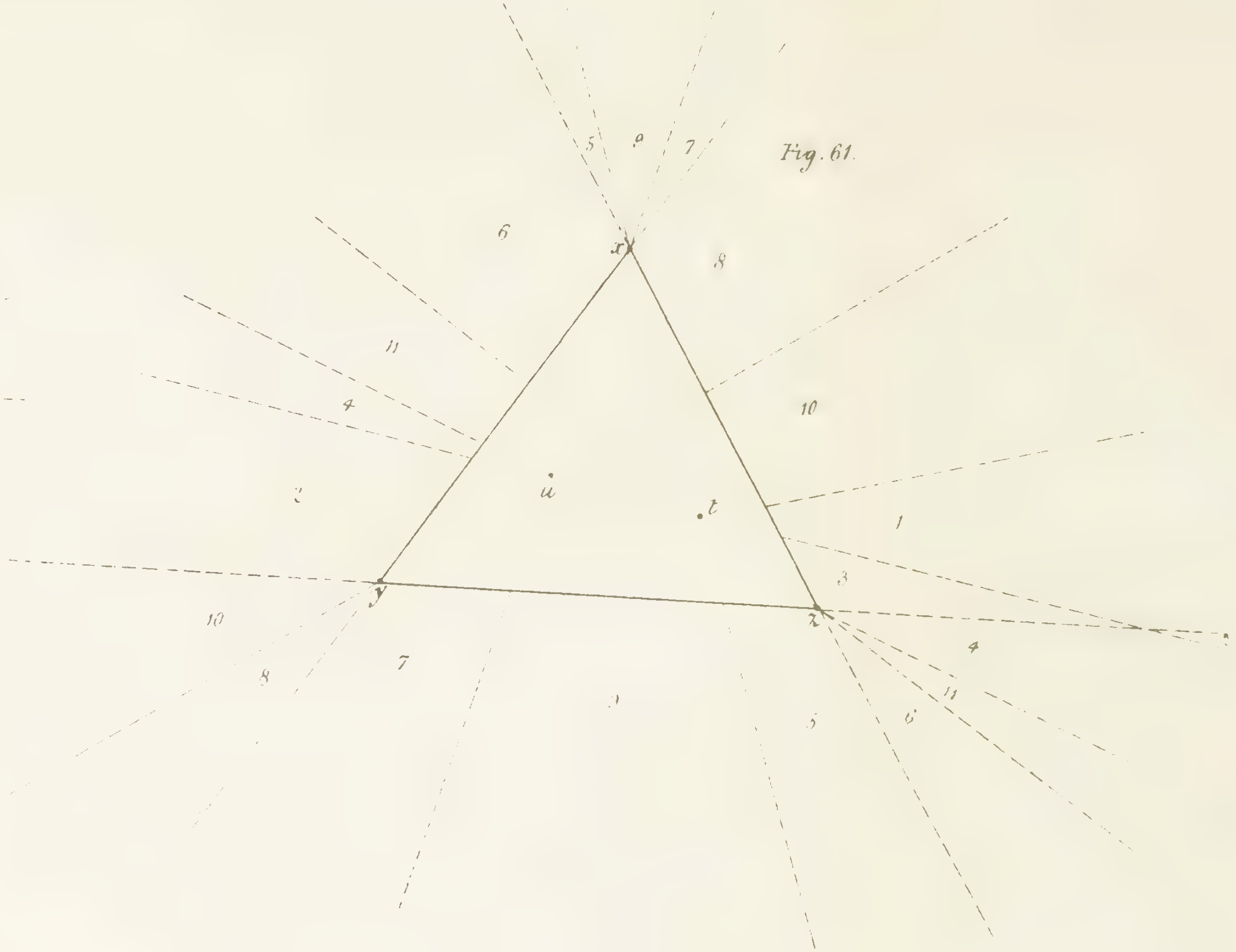


Fig. 55.

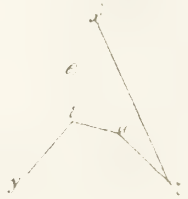


Fig. 56.

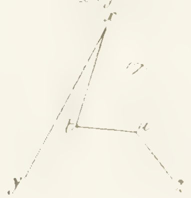


Fig. 57.

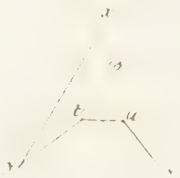


Fig. 58.

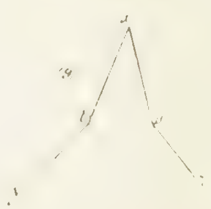


Fig. 59.

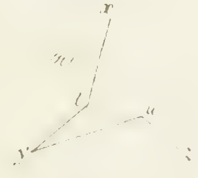
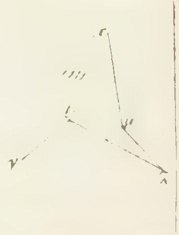


Fig. 60.



The Fluted Vibrations of a Circular Vortex Ring with a Hollow Core. *By* H. S. Carslaw, B.A. Emmanuel College, Cambridge.

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Introduction.

1. The discussion of the circular vortex ring of finite cross-section was initiated by Hicks in his papers in *Phil. Trans.*, 1884–5. In his presidential address to Section A of the British Association, in 1895, he called attention to the incorrectness of his result for the fluted vibrations of a "solid" ring, and accepted the correction given by Basset in his *Hydrodynamics*, §§ 326–340. The error referred to is only for this particular ring; whereas, if Basset's work is correct, a mistake must also have crept into his earlier investigation in respect to the simple hollow ring treated of independently by Pocklington (*Phil. Trans.*, 1895), and with the same results.

In this paper I have attacked the general problem afresh. By using the velocity potential in the disturbed motion I have made sure of giving to it its proper acyclic and irrotational nature. My result gives, as special cases, those of Basset. Further, I consider the analogous problem in two dimensions, and find between the two period equations the same similarity as exists between Basset's equation and that obtained by Lord Kelvin for the columnar vortex in *Phil. Mag.*, 1880. This agreement is to be expected in both cases—since the physical aspects of the problem of a ring with a very large aperture and small cross-section resemble those of a cylindrical column in an infinite liquid—and affords a verification of my work for the ring.

The notation adopted by Basset lends itself most easily to the work required, so that it is necessary, first of all, to work out in its terms the problem of the steady motion; this involves only an adaptation of Hicks' analysis.

The Notation.

2. Consider the transformation

$$z + i\varpi = a \tan \frac{\xi + i\eta}{2}.$$

This gives $$e^{i\xi-\eta} = -\left(\frac{\varpi-a-iz}{\varpi+a-iz}\right),$$

i.e., $$\xi = \pi-(\theta_1-\theta_2), \quad \eta = \log\left(\frac{r_2}{r_1}\right).$$

Thus $\eta =$ constant is the system of coaxal circles having A, B for limiting points,

$\xi =$ constant, the orthogonal system;

also along AB on the upper side, $\xi = 0$,

along $-\infty A$ and $B\infty$, $\xi = \pi$,

along BA on the lower side, $\xi = 2\pi$;

further along OZ, $\eta = 0$,

at A, $\eta = -\infty$,

at B, $\eta = +\infty$;

while $$\varpi = \frac{a\sinh\eta}{\cosh\eta+\cos\xi} = \frac{aS'}{C+c},$$

$$z = \frac{a\sin\xi}{\cosh\eta+\cos\xi} = \frac{as}{C+c},$$

and $$ds^2 = h^2(d\xi^2+d\eta^2) = \left(\frac{a}{C+c}\right)^2(d\xi^2+d\eta^2),$$

where $$S = \sinh\eta, \quad s = \sin\xi, \text{ \&c.}$$

Here we have a notation which lends itself at once to the problem of approximately circular vortex rings.

The Steady Motion.

3. We can reduce the problem of a ring moving forward steadily in an infinite liquid with velocity V to that of a stream flowing with velocity V past a stationary ring by impressing on the whole liquid a velocity $-V$.

Thus, since the motion is symmetrical with respect to a line passing through the centre of the ring, and perpendicular to its plane, we may use Stokes' current function and the notation of § 2.

If the steady motion is possible, the current function must satisfy the following equations:—

$$\left(\frac{\partial^2}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}+\frac{\partial^2}{\partial z^2}\right)\psi_2 = 0, \qquad \text{outside the ring,} \qquad (1)$$

$$\left(\frac{\partial^2}{\partial\varpi^2}-\frac{1}{\varpi}\frac{\partial}{\partial\varpi}+\frac{\partial^2}{\partial z^2}\right)\psi_1 = M\varpi^2, \text{ inside,} \qquad (2)$$

where
$$\omega = -\frac{M}{2}\varpi, \tag{3}$$

and we use a right-handed system of axes;

$$\psi_2 = -\tfrac{1}{2}V\varpi^2 \text{ at infinity}, \tag{4}$$

$$\psi = \text{constant at the surface of the ring}; \tag{5}$$

and the pressure must be continuous at the outer and constant at the inner surface. (6)

4. In accordance with Hicks' method, we assume as the equations of the cross-section of the ring

$$\kappa = e^{-\eta} = b_1(1+\alpha_2\cos 2\xi + \ldots), \tag{7}$$

$$\kappa = e^{-\eta} = b_2(1+\beta_1\cos\xi+\beta_2\cos 2\xi+\ldots), \tag{8}$$

where we shall consider β_n and α_n of order b_1^n or b_2^n, and b_1, b_2 are small, because η is nearly infinite at the surface of the ring if the cross-section is small compared to the aperture.

Further we take, following Basset's work, for our ψ,

$$\psi_2 = -\tfrac{1}{2}V\varpi^2 + (2b_2)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}\sum_0^\infty A_n'\left(\frac{b_2}{\kappa}\right)^{n+\frac{1}{2}}R_n\cos n\xi, \tag{9}$$

$$\begin{aligned}\psi_1 = {} & \tfrac{1}{8}M\varpi^4 + (2b_2^5)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}} \\ & \times\sum_0^\infty\left[A_n\left(\frac{b_2}{\kappa}\right)^{n+\frac{1}{2}}R_n + B_n\left(\frac{\kappa}{b_2}\right)^{n-\frac{1}{2}}T_n\right]\cos n\xi,\end{aligned} \tag{10}$$

R_n and T_n being the modified toroidal functions used by Basset.

Here we shall take $\dfrac{A_n'}{A_{n-1}'}$, $\dfrac{A_n}{A_{n-1}}$, and $\dfrac{B_n}{B_{n-1}}$ to be of order b. We have now to determine what orders we shall give to A_0', A_0, and B_0.

To find information as to these, we must consider the different circulations, as they are important features of our problem.

Let μ_2 be the circulation outside the ring,
μ' ,, ,, at the outer surface due to the vortex filaments alone,
μ_1 ,, ,, at the inner surface.

We find, as in Hicks,

$$\mu_2 = -\frac{\pi}{a}\left(A_0' - A_1' b_2 + A_2' b_2^2 - \&c.\right), \tag{11}$$

$$\mu' = -4\pi M a^3 \left[(b_2^2 - b^2)\left\{1 + 4(b_1^2 + b_2^2)\right\} + \frac{\beta_1^2}{2} b_2^2 + \beta_1 b_1 b_2^3\right] + \ldots$$

$$\doteqdot -4\pi M a^3 (b_2^2 - b_1^2). \tag{12}$$

Then, if there were no added circulations these two expressions for μ_2 and μ' would have to be the same, so we are led to take A_0' to be of the order Mb^2.

Now, for μ_1 we have the equation

$$\mu_1 = \int_0^{2\pi} \left[\frac{1}{\varpi}\frac{\partial \psi_1}{\partial \eta}\, d\xi + \left(-\frac{1}{\varpi}\frac{\partial \psi_1}{\partial \xi}\right)\frac{d\eta}{d\xi}\, d\xi\right]$$

along $\kappa = b_1 (1 + \alpha_2 \cos 2\xi + \ldots)$.

Evaluating this, we find, correct to the lowest order,

$$\mu_1 = -\frac{\pi}{a}\left[4Mb_1^2 a^4 (1 + 4b_1^2) + \frac{A_0}{b_2^2} - \frac{A_1}{b} + \&c.\right]. \tag{13}$$

Thus we see that, if the inner circulation were zero, A_0 would have to be of the order Mb^4. We therefore take A_0 to be of order Mb^4.

Also B_0 must be of order Mb^2. We are led to this by considering the case when there is a "solid" core; if this condition were not satisfied, the origin, *i.e.* the point $\kappa = 0$, would, in fact, be a singular point in the flow.

ψ_2 = *Constant at the Outer Surface.*

5. We are now in a position to calculate the values of ψ to any approximation. For the work on vibrations, we shall need the circumstances of the steady motion, and this calculation we shall have to carry to the order Mb^3. We shall for the future regard Mb^2, that is A_0', as of the order b^0.

We have

$$\psi_2 = -\tfrac{1}{2} V \varpi^2 + (2b_2)^{-\frac{1}{2}} (C + c)^{-\frac{1}{2}} \sum_0^\infty A_n' \left(\frac{b_2}{\kappa}\right)^{n+\frac{1}{2}} R_n \cos n\xi.$$

We know $\varpi = a(1 - 2\kappa\cos\xi + 2\kappa^2 \cos 2\xi \ldots)$;

thus $\varpi^2 = a^2 (1 + 2\kappa^2 - 4\kappa\cos\xi + 6\kappa^2\cos 2\xi + \ldots)$.

Also $$\frac{1}{(C+c)^{\frac{1}{2}}} = (2\kappa)^{\frac{1}{2}}\left[1+\tfrac{1}{4}\kappa^2-(\kappa+\tfrac{3}{8}\kappa^3)\cos\xi+\tfrac{3}{4}\kappa^2\cos 2\xi\,\ldots\right],$$

$$R_0 = -\tfrac{1}{2}(L-2)-\tfrac{1}{8}\kappa^2(L+1),$$

$$R_1 = \quad \tfrac{1}{2}-\tfrac{3}{4}\kappa^2(L-\tfrac{1}{2}),$$

where $$L = \log\left(\frac{4}{\kappa}\right).$$

Thus, to our order,

$$\psi_2 = -\tfrac{1}{2}Va^2(1-4\kappa\cos\xi)+A_0'(1-\kappa\cos\xi)\left[-\tfrac{1}{2}(L-2)\right]$$
$$+A_1'\left(\frac{b_2}{\kappa}\right)\tfrac{1}{2}\cos\xi.$$

Therefore, when $$\kappa = b_2(1+\beta_1\cos\xi+\ldots),$$

$$\psi_2 = -\tfrac{1}{2}Va^2-\frac{A_0'}{2}(L_2-2)+\left[2Va^2+\tfrac{1}{2}A_0'(L_2-2)+\frac{\beta_1}{b_2}\frac{A_0'}{2}+\frac{A_1'}{2b_2}\right]b_2\cos\xi$$
$$+\text{higher terms.}$$

Therefore, to our order, ψ_2 is constant at the surface if

$$2Va^2+\tfrac{1}{2}A_0'(L_2-2)+\frac{A_0'}{2}\,\frac{\beta_1}{b_2}+\frac{A_1'}{2b_2} = 0, \tag{14}$$

where $$L_2 = \log\left(\frac{4}{b_2}\right).$$

ψ_1 = *Constant at the Outer Surface.*

6. Putting Q for $Ma^4b_2^2$, we may write equation (10) in the form

$$b_2^2\psi_1 = \tfrac{1}{8}Q\left(\frac{\varpi}{a}\right)^4+(2b_2)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}\sum_0^\infty\left[A_n\left(\frac{b_2}{\kappa}\right)^{n+\frac{1}{2}}R_n\right.$$
$$\left.+B_n\left(\frac{\kappa}{b_2}\right)^{n-\frac{1}{2}}T_n\right]\cos n\xi.$$

Since we want ψ_1 to the order b^1, we must take the right-hand side to the order b^3.

Remembering that, while A_0 is of order b^2, B_0 is of order b^0, we find, to our approximation,

$$b_2^2\psi_1 = \tfrac{1}{8}Q+B_0+(Q+\tfrac{1}{2}G)\kappa^2-\tfrac{1}{2}A_0(L-2)$$
$$+\cos\xi\left[\frac{\kappa}{2}A_0(L-2)+\tfrac{1}{2}A_1\frac{b_2}{\kappa}-\kappa G-\left\{\tfrac{1}{4}(G+H)+\tfrac{9}{2}Q\right\}\kappa^3\right]$$
$$+\cos 2\xi\,\tfrac{1}{2}H\kappa^2+\cos 3\xi\,K\kappa^3.$$

Following Basset, we have here put

$$G = Q + B_0 - 3\,\frac{B_1}{b_2},$$

$$H = 5Q + \tfrac{3}{2}B_0 - \tfrac{3}{2}\,\frac{B_1}{b} + \tfrac{15}{4}\,\frac{B_1}{b_2^2},$$

$$K = -5Q - \tfrac{5}{8}B_0 + \tfrac{9}{16}\,\frac{B_1}{b_2} - \tfrac{15}{16}\,\frac{B_2}{b_2^2} + \tfrac{35}{16}\,\frac{B_3}{b_2^3}.$$

Thus, when $\kappa = b_2\,(1 + \beta_1 \cos \xi + \beta_2 \cos 2\xi + \dots) = \kappa_2$,

to our order,

$$b_2^2 \psi_1 = p_0 + p_1 \cos \xi + p_2 \cos 2\xi + p_3 \cos 3\xi,$$

where

$$p_1 = -Gb_2 - \left[\tfrac{1}{4}\,(G+H) + \tfrac{9}{2}Q\right] b_2^3 + 2\beta_1 b_2^2\,(Q + \tfrac{1}{2}G) - \tfrac{1}{2}Gb_2\beta_2 + \tfrac{1}{2}Hb_2^2\beta_1$$
$$+ A_0\left[\frac{\beta_1}{2} + \tfrac{1}{2}\,(L_3 - 2)\,b_2\right] + \frac{A_1}{2},$$

$$p_2 = -G\,\frac{b_2}{2}\,\beta_1 + \tfrac{1}{2}Hb_2^2,$$

$$p_3 = -G\,\frac{b_2}{2}\,\beta_2 + \tfrac{1}{2}Hb_2^2\beta_1 + \tfrac{1}{16}Kb_2^3.$$

Then ψ_1 is constant at κ_2, if

$$p_1 = 0 = p_2 = p_3.$$

From $p_1 = 0$, since all the terms excepting the first are at any rate of order b^3, G must be of order b^2 at any rate.

From $p_2 = 0$, H must be of order b^2 at any rate; and from $p_3 = 0$, so must K.

Thus, to the first order,

$$\left.\begin{aligned} G &\equiv Q + B_0 - \tfrac{3}{2}\,\frac{B_1}{b_2} = 0 \\ H &\equiv 5Q + \tfrac{3}{2}B_0 - \tfrac{3}{2}\,\frac{B_1}{b_2} + \tfrac{15}{4}\,\frac{B_2}{b_2^2} = 0 \\ -K &\equiv 5Q + \tfrac{5}{8}B_0 - \tfrac{9}{16}\,\frac{B_1}{b_2} + \tfrac{15}{16}\,\frac{B_2}{b_2^2} - \tfrac{35}{16}\,\frac{B_3}{b_2^3} = 0 \end{aligned}\right\}. \tag{15}$$

These three equations give B_1, B_2, and B_3 in terms of Q and B_0.

Also, from $p_1 = 0$,

$$-\frac{G}{b_2^2} - \tfrac{9}{2}Q + \frac{2\beta_1}{b_2}Q + \frac{A_0}{b_2^2}\left[\frac{\beta_1}{2b_2} + \tfrac{1}{2}(L_2 - 2)\right] + \frac{A_1}{2b_2^3} = 0. \qquad (16)$$

ψ_1 = *Constant at the Inner Surface.*

7. We have, from § 6, to order κ^3 in right-hand side

$$b_2^2\psi_1 = \tfrac{1}{8}Q + B_0 + Q\kappa^2 - \tfrac{1}{2}(L-2)A_0$$
$$+ \cos\xi\left[-\kappa G - \tfrac{9}{2}Q\kappa^3 + \tfrac{1}{2}(L-2)\kappa A_0 + \frac{A_1}{2}\frac{b_2}{\kappa}\right].$$

Thus, at $\kappa = b_1(1 + \alpha_2\cos 2\xi + \ldots)$,

to this order,

$$b_2^2\psi_1 = \tfrac{1}{8}Q + B_0 + Qb_1^2 + \cos\xi\left[-Gb_1 - \tfrac{9}{2}Qb_1^3 + \tfrac{1}{2}A_0(L_1 - 2)b_1 + \frac{A_1}{2}\left(\frac{b_2}{b_1}\right)\right];$$

and therefore to our order ψ_1 is constant if

$$-Gb_1 - \tfrac{9}{2}Qb_1^3 + \tfrac{1}{2}A_0(L_1 - 2)b_1 + \frac{A_1}{2}\left(\frac{b_2}{b_1}\right) = 0, \qquad (17)$$

where $L_1 = \log\left(\dfrac{4}{b_1}\right)$.

The Pressure.

8. We have now to examine the pressure conditions.

The pressure equations are as follows:—

$$\frac{p}{\rho_2} + \tfrac{1}{2}q^2 = \frac{\Pi}{\rho_2} + \tfrac{1}{2}V^2, \text{ where } \Pi \text{ is the pressure at } \infty, \qquad (18)$$

$$\frac{p}{\rho_1} + \tfrac{1}{2}q^2 - \frac{Q}{a^4 b_2^2}\psi_1 = \text{a constant}. \qquad (19)$$

Also

$$q^2 = \left(\frac{\kappa}{\varpi h}\right)^2\left\{\left(\frac{\partial\psi}{\partial\kappa}\right)^2 + \left(\kappa^{-1}\frac{\partial\psi}{\partial\xi}\right)^2\right\}. \qquad (20)$$

(i.) Let us take the inner surface where p is constant.

We have then to order b^0 at least

$$\frac{\kappa}{\omega h}\frac{\partial\psi}{\partial\kappa} = \frac{Q}{2a^2b_2^2}\left[2\kappa + \frac{A_0}{2Q\kappa} + \cos\xi\left(\frac{A_0}{2Q}(L-2) + 3\frac{A_0}{2Q}\right.\right.$$
$$\left.\left. - \frac{A_1}{2Q}\frac{b_2}{\kappa^2} - \tfrac{11}{2}\kappa^2 - \frac{G}{Q}\right)\right], \qquad (21)$$

and

$$\varpi h\frac{\partial\psi}{\partial\xi} = -\frac{Q\sin\xi}{4a^2b_2^2}\left[\frac{A_0}{Q}(L-2) + \frac{A_1}{Q}\frac{b_2}{\kappa^2} - 9\kappa^2 - 2\frac{G}{Q}\right]. \qquad (22)$$

The first gives a term of order b^{-1}; the second has its most important term of order b^0.

Therefore, in finding q^2 to the two most important terms, we may omit $\left(\frac{1}{\varpi h}\frac{\partial\psi}{\partial\xi}\right)^2$, and deal only with $\left(\frac{\kappa}{\varpi h}\frac{\partial\psi}{\partial\kappa}\right)^2$.

Further, we have already made ψ_1 constant at the inner surface. Thus it follows from this pressure equation that we simply have

$$\frac{A_0}{2Q}(L_1+1)-\frac{A_1}{2Qb_1}\left(\frac{b_2}{b_1}\right)-\frac{11}{2}b^2-\frac{G}{Q}=0. \tag{23}$$

(ii.) Pressure continuous at the outer surface.

Using the values for $\frac{\kappa}{\varpi h}\frac{\partial\psi}{\partial\kappa}$ and $\frac{1}{\varpi h}\frac{\partial\psi}{\partial\xi}$, we find for the pressures at the outer surface to our order

$$\frac{p}{\rho_2}+\frac{A_0'}{8a^4b_2}\left[2Va^2+\frac{A_0'}{2}\left(L_2+1-\frac{\beta_1}{b_2}\right)-\frac{A_1'}{2b_2}\right]\cos\xi+\text{constant terms}$$
$$=\frac{\Pi}{\rho_2}+\tfrac{1}{2}V^2,$$

$$\frac{p}{\rho_1}+P\cos\xi+\text{constant terms}=\text{a constant},$$

where P stands for

$$-8\beta_1b_2^2-\frac{A_0^2}{2Q^2}\frac{\beta_1}{b_1^2}+2\left(2b_2+\frac{A_0}{2Qb_2}\right)\left(\frac{A_0}{2Q}(L_2+1)-\frac{11}{2}b_2^2-\frac{A_1}{2Qb_1}-\frac{G}{Q}\right)$$
$$-4\left(\frac{A_0}{Q}\beta_1+\frac{A_0}{Q}b_2(L_2-2)+\frac{A_1}{Q}-9b_2^3-2b_2\frac{G}{Q}\right).$$

Therefore we must have

$$\rho_2A_0'\left[2Va^2+\frac{A_0'}{2}\left(L_2+1-\frac{B_1}{b_2}\right)-\frac{A_1'}{2b_2}\right]=8a^4b_2\rho_1P. \tag{24}$$

We might here get an equation between the Π, V, and the other constants, but in our work this is not required.

9. We have now got all the information desired. The only point left undetermined is the condition the pressure at infinity must satisfy that the hollow may be present. This is not a matter with which in the present investigation we shall be concerned.

For convenience the equations already obtained are here collected, and it will be seen they give us all the information as to the steady motion.

In every case the equations are correct to the lowest order only.

$$\mu_2 = -\frac{\pi}{a} A_0', \tag{25}$$

$$\mu' = -\frac{4\pi}{a} Q \Big[1 - \Big(\frac{b_1}{b_2}\Big)\Big]^2, \tag{26}$$

$$\mu_1 = -\frac{4\pi}{a} Q \Big(\frac{b_1}{b_2}\Big)^2 - \frac{\pi}{a}\,\frac{A_0}{b_2^2}, \tag{27}$$

$$2Va^2 + \frac{A_0'}{2}\Big[(L_2-2) + \frac{\beta_1}{b_2}\Big] + \frac{A_1'}{2b_2} = 0, \tag{28}$$

$$-G - \tfrac{9}{2}Qb_2^2 + 2\beta_1 b_2 Q + \frac{A_0}{2}\Big(L_2 - 2 + \frac{\beta_1}{b_2}\Big) + \frac{A_1}{2b_2} = 0, \tag{29}$$

$$-G - \tfrac{9}{2}Qb_1^2 + \qquad \frac{A_0}{2}(L_1-2) + \frac{A_1}{2b_1}\Big(\frac{b_2}{b_1}\Big) = 0, \tag{30}$$

$$-G - \tfrac{11}{2}Qb_1^2 \qquad + \frac{A_0}{2}(L_1+1) - \frac{A_1}{2b_1}\Big(\frac{b_2}{b_1}\Big) = 0, \tag{31}$$

$$\begin{aligned} &\rho_2 A_0'\Big[2Va^2 + \frac{A_0'}{2}\Big(L_2 + 1 - \frac{\beta_1}{b_2}\Big) - \frac{A_1'}{2b_2}\Big] \\ &= \rho_1 \frac{Q^2}{b_2^3}\Big[-8\beta_1 b_2^2 - \frac{A_0^2}{2Q^2}\frac{\beta_1}{b_2^2} + 2\Big(2b_2 + \frac{A_0}{2Qb_2}\Big)\Big(\frac{A_0}{2Q}(L_2+1) - \tfrac{11}{2}b_2^2 - \frac{A_1}{2Qb_1} - \frac{G}{Q}\Big) \\ &\qquad - 4\Big(\frac{A_0}{Q}\beta_1 + \frac{A_0}{Q}b_2(L_2-2) + \frac{A_1}{Q} - 9b_2^3 - 2b_2\frac{G}{Q}\Big)\Big]. \end{aligned} \tag{32}$$

From (30) and (31), we find

$$Qb_1^2 = \tfrac{3}{2}A_0 - \frac{A_1}{b_1}\Big(\frac{b_2}{b_1}\Big).$$

Therefore we have A_1 in terms of Q and A_0, and (27) gives A_0. Then, from either (30) or (31), we find G, and, subtracting (29) and (30), we find β_1. Substituting in (32), we find V.

E.g., if $\mu_1 = 0$, we get

$$\begin{aligned} \frac{2\beta_1}{b_2}\Big[1 - \Big(\frac{b_1}{b_2}\Big)^2 &= \tfrac{9}{2}\Big[1 - \Big(\frac{b_1}{b_2}\Big)^2\Big] + 2\Big(\frac{b_1}{b_2}\Big)^2 \log\Big(\frac{b_1}{b_2}\Big) \\ &\qquad - \tfrac{7}{2}\Big(\frac{b_1}{b_2}\Big)^2\Big[1 - \Big(\frac{b_1}{b_2}\Big)^2\Big]. \end{aligned} \tag{33}$$

Further, if $b_1 = 0$, we get the well-known result that

$$\beta_1 = \tfrac{9}{4} b_2,$$

since $$\mathrm{Lt}_{x=0}\,(x \log x) = 0.$$

We shall want later the tangential velocity at the surfaces in the steady motion.

From our work we see that to the lowest order they are given by

$$-\frac{\kappa}{\varpi h}\,\frac{\partial\psi}{\partial\kappa}.$$

Referring to the results in § 8, and equations (25), (26), (27), we get to the lowest order

$$U_1 = \text{tangential velocity at} \quad \kappa_1 = \frac{\mu_1}{4\pi a b_1}, \tag{34}$$

$$U' = \quad ,, \quad ,, \quad \text{just inside } \kappa_2 = \frac{\mu_1 + \mu'}{4\pi a b_2}, \tag{35}$$

$$U_2 = \quad ,, \quad ,, \quad \text{just outside } \kappa_2 = \frac{\mu_2}{4\pi a b_2}. \tag{36}$$

Also
$$\left.\begin{aligned} \frac{\partial U_1}{\partial\kappa} &= -\frac{U_1}{b_1} - \frac{2Q}{a^2 b_2^2} \\ \frac{\partial U'}{\partial\kappa} &= -\frac{U'}{b_2} - \frac{2Q}{a^2 b_2^2} \\ \frac{\partial U_2}{\partial\kappa} &= -\frac{U_2}{b_2} \end{aligned}\right\}. \tag{37}$$

The Fluted Vibrations.

10. To find the small oscillations, we assume that the equations of the inner and outer surfaces may be given by

$$\kappa = b_1\,(1 + \alpha e^{\iota(n\xi + \lambda t)}), \tag{38}$$

$$\kappa = b_2\,(1 + \beta e^{\iota(n\xi + \lambda t)}), \tag{39}$$

where α and β are small, the order of their smallness being determined later, n a positive integer; and we want to find an equation for λ. If we find an equation whose roots are real under all circumstances, the small oscillations are always possible, and the assumption as to the form of the surface is justified. If λ is only real under certain circumstances this restricts our fluid motion for stability to these.

The motion of the liquid is that which subsists when there is no vibration, together with small changes introduced by the vibration. The former we have been already investigating. The latter we may approach more logically through the velocity potential, because the disturbance gives an acyclic irrotational motion.

11. Let ϕ_2 and ϕ_1 be the velocity potentials of the disturbed motion. Then we shall take

$$\phi_2 = (C+c)^{\frac{1}{2}} A P_n e^{\iota(n\xi+\lambda t)}, \tag{40}$$

$$\phi_1 = (C+e)^{\frac{1}{2}} (B P_n + C Q_n)\, e^{\iota(n\xi+\lambda t)}, \tag{41}$$

where P_n and Q_n are the toroidal functions of order n given by Hicks, or in Basset, §§ 273–282; A, B, C small constants.

These are acyclic, and the disturbance is very small when we go to infinity.

Now $$P_n \propto \kappa^{-(n-\frac{1}{2})}, \quad Q_n \propto \kappa^{n+\frac{1}{2}};$$

also $$(C+c)^{\frac{1}{2}} \doteqdot (2\kappa)^{-\frac{1}{2}},$$

when κ is small; therefore, when κ is small, we may take, for ϕ_2 and ϕ_1,

$$\phi_2 = A\left(\frac{b_2}{\kappa}\right)^n e^{\iota(n\xi+\lambda t)}, \tag{42}$$

$$\phi_1 = \left[B\left(\frac{b_2}{\kappa}\right)^n + C\left(\frac{\kappa}{b_2}\right)^n\right] e^{\iota(n\xi+\lambda t)}, \tag{43}$$

where the A, B, C are now different constants, and this is correct to the lowest order.

We want now to find corresponding expressions for the stream functions χ_1, χ_2.

For a point outside the ring, we have, following Basset,

$$\chi_2 = -\int_1^\kappa \frac{\varpi}{\kappa}\frac{\partial\phi_2}{\partial\xi}\,d\kappa + \int_0^\xi \kappa\varpi\frac{\partial\phi_2}{\partial\kappa}\,d\xi.$$

Therefore, retaining only the most important terms, we have

$$\chi_2 = ia\left(\frac{b_2}{\kappa}\right)^n A\, e^{\iota(n\xi+\lambda t)}. \tag{44}$$

For a point inside we have to subdivide the integrals, and we find, to the lowest order,

$$\chi_1 = ia\left[B\left(\frac{b_2}{\kappa}\right)^n - C\left(\frac{\kappa}{b_2}\right)^n\right] e^{i(n\xi+\lambda t)}, \tag{45}$$

where we have omitted a term in $e^{i\lambda t}$, as it may be counterbalanced by the arbitrary function of the time in the pressure equation.

12. The bounding surfaces must always contain the same particles. Thus, if we put

$$F_1 \equiv -\kappa + b_1\left(1 + \alpha e^{\iota(n\xi+\lambda t)}\right) = 0,$$

$$F_2 \equiv -\kappa + b_2\left(1 + \beta e^{\iota(n\xi+\lambda t)}\right) = 0,$$

we must have at each

$$\frac{\partial F}{\partial t} + \left(\frac{1}{\varpi h}\frac{\partial \Psi}{\partial \eta}\right)\left(\frac{1}{h}\frac{\partial F}{\partial \xi}\right) + \left(-\frac{1}{\varpi h}\frac{\partial \Psi}{\partial \xi}\right)\left(-\frac{1}{h}\frac{\partial F}{\partial \eta}\right) = 0,$$

where $$\Psi = \psi + \chi.$$

Taking F_2, this gives

$$\iota h b_2 \beta \lambda e^{\iota(n\xi+\lambda t)} - \iota n b_2 \beta e^{\iota(n\xi+\lambda t)}\left[\frac{\kappa}{\varpi h}\left(\frac{\partial \psi_2}{\partial \kappa} + \frac{\partial \chi_2}{\partial \kappa}\right)\right] - \frac{\kappa}{\varpi h}\left(\frac{\partial \psi_2}{\partial \xi} + \frac{\partial \chi_2}{\partial \xi}\right) = 0.$$

If we assume A and β very small compared to b, and look on the cross-section of the ring in steady motion as an exact circle, and the normal velocity there as zero, we shall be able to omit the third and fourth terms in this equation, and to divide out by $e^{\iota(n\xi+\lambda t)}$.

This gives $$A = -2iab_2\beta\left(U_2 + 2ab_2\frac{\lambda}{n}\right). \qquad (46)$$

Further, from F_2, just inside the ring, we find, in the same way,

$$B - C = -2\iota ab_2\beta\left(U' + 2ab_2\frac{\lambda}{n}\right); \qquad (47)$$

and from F_1, at the inner surface,

$$B\left(\frac{b_2}{b_1}\right)^n - C\left(\frac{b_1}{b_2}\right) = -2iab_1\alpha\left(U_1 + 2ab_1\frac{\lambda}{n}\right). \qquad (48)$$

The Pressure.

13. The equations that the pressures satisfy are

$$\frac{p}{\rho_2} + \dot{\phi}_2 + \tfrac{1}{2}q^2 = \text{an arbitrary function of time} = f_2(t), \text{ say}, \quad (49)$$

$$\frac{p}{\rho_1} + \dot{\phi}_1 + \tfrac{1}{2}q^2 - M(\psi_1 + \chi_1) = \quad ,, \qquad ,, \qquad = f_1(t), \text{ say}. \quad (50)$$

Thus, at the surface

$$\kappa = b_1 (1 + \alpha e^{\iota(n\xi+\lambda t)}),$$

since we must have $\qquad p = \text{constant},$

$$f_1(t) = \text{const.} + i\lambda \left[B \left(\frac{b_2}{b_1} \right)^n + C \left(\frac{b_1}{b_2} \right)^n \right] e^{i(n\xi+\lambda t)}$$

$$+ U_1 \left(\frac{\partial U_1}{\partial \kappa} \delta\kappa - \frac{1}{2a^2} \frac{\partial \chi_1}{\partial \kappa} \right) - \frac{Q}{a^4 b_2^2} \left(\frac{\partial \psi_1}{\partial \kappa} \delta\kappa + \chi_1 \right),$$

because the other terms are either of higher order than these, or they are constant, from the conditions to which we have bound our current functions.

But, putting $\qquad \dfrac{b_2}{b_1} = x,$

$$\frac{\partial \chi_1}{\partial \kappa} = -\frac{1}{2ab_1} in (Cx^{-n} + Bx^n) e^{\iota(n\xi+\lambda t)};$$

also $\qquad Bx^n + Cx^{-n} = \dfrac{2ia}{x^n - x^{-n}} \left[2b_2 \beta V' - b_1 \alpha V_1 (x^n + x^{-n}) \right],$

where $\qquad V_1 = U_1 + 2ab_1 \dfrac{\lambda}{n},$ &c.,

further $\qquad \dfrac{\partial \psi_1}{\partial \kappa} \delta\kappa + \chi_1 = b_1 \alpha \, 4a^3 \dfrac{\lambda}{n} e^{i(n\xi+\lambda t)};$

therefore this pressure condition gives

$$b_1 \alpha \left[\left(U_1 \frac{\partial U_1}{\partial \kappa} - 4 \frac{Q}{ab_1} x^{-2} \frac{\lambda}{n} \right) (x^n - x^{-n}) + \frac{n}{b_1} (x^n + x^{-n}) V_1^2 \right] = 2\beta n x V_1 V'. \tag{51}$$

We obtain another equation from the continuity of pressure at the outer surface.

To the first order

$$f_1(t) = \frac{p}{\rho_1} + i\lambda (B + C) e^{\iota(n\xi+\lambda t)} + U' \left(\frac{\partial U'}{\partial \kappa} \delta\kappa - \frac{1}{2a^2} \frac{\partial \chi_1}{\partial \kappa} \right)$$

$$- \frac{Q}{a^4 b_2^2} \left(\frac{\partial \psi_1}{\partial \kappa} \delta\kappa + \chi_1 \right) + \tfrac{1}{2} U'^2 - \frac{Q}{a^4 b_2^2} \psi_1,$$

$$f_2(t) = \frac{p}{\rho_2} + \iota\lambda A e^{\iota(n\xi+\lambda t)} + U_2 \left(\frac{\partial U_2}{\partial \kappa} \delta\kappa - \frac{1}{2a^2} \frac{\partial \chi_2}{\partial \kappa} \right) + \tfrac{1}{2} U_2^2.$$

Therefore we must have

$$\rho_2 \left[i\lambda A + U_2 \left(\frac{\partial U_2}{\partial \kappa} b_2 \beta + \frac{niA}{2ab_2} \right) \right]$$

$$= \rho_1 \left[i\lambda\,(B+C) + U' \left(\frac{\partial U'}{\partial \kappa} b_2 \beta + \frac{ni}{2ab_2} (B+C) \right) \right.$$

$$\left. + \frac{Q}{a^4 b_2^2} \Big(b_2 \beta \,.\, 2a^2 U' - ia\,(B-C) \Big) \right],$$

the other terms having been already adjusted in the steady motion.

Putting in the values for A, B, and C, this gives

$$b_2\beta \left[\left(x^n - \frac{1}{x^n} \right) \left\{ \frac{n\sigma}{b_2} V_2^2 + \sigma U_2 \frac{\partial U_2}{\partial \kappa} - \left(U' \frac{\partial U'}{\partial \kappa} - 4 \frac{Q}{ab_2} \frac{\lambda}{n} \right) \right\} \right.$$

$$\left. + \left(x^n + \frac{1}{x^n} \right) \frac{n}{b_2} V'^2 \right] = \frac{2n}{x_2} b_1 a V_1 V'. \qquad (52)$$

From (51) and (52) we find at once

$$\left[\left(x^n - \frac{1}{x^n} \right) \left\{ n\sigma V_3^2 + b_3 \left(\sigma U_2 \frac{\partial U_2}{\partial \kappa} - U' \frac{\partial U'}{\partial \kappa} \right) + 4 \frac{Q}{a} \frac{\lambda}{n} \right\} \right.$$

$$\left. + nV'^2 \left(x^n + \frac{1}{x^n} \right) \right]$$

$$\times \left[\left(x^n - \frac{1}{x^n} \right) \left(b_1 U_1 \frac{\partial U_1}{\partial \kappa} - 4 \frac{Q}{a} \frac{1}{x^2} \frac{\lambda}{n} \right) + nV_1^2 \left(x^n + \frac{1}{x^n} \right) \right]$$

$$= 4n^2 V_1^2 V'^2. \qquad (53)$$

Using the results of § 9 (37), this becomes

$$\left[\left(x^n - \frac{1}{x^n} \right) \left(n\sigma V_2^2 - \sigma U_2^2 + U'^2 - \frac{\mu' b_2}{2\pi a} \frac{V'}{b_2^2 - b_1^2} \right) + \left(x^n + \frac{1}{x^n} \right) nV'^2 \right]$$

$$\times \left[\left(x^n - \frac{1}{x^n} \right) \left(-U_1^2 + \frac{\mu' b_1}{2\pi a} \frac{1}{b_2^2 - b_1^2} V_1 \right) + \left(x^n + \frac{1}{x^n} \right) nV_1^2 \right]$$

$$= 4n^2 V_1^2 V'^2.$$

If we now put u_1 for $\dfrac{U_1}{2ab_1}$, &c.; also p for $(x^n - x^{-n})$, q for

(x^n+x^{-n}); and remember that

$$\frac{\mu'}{4a\pi}=b_2U'-b_1U_1,$$

and divide all through by $16a^4b_1^2b_2^2$, we shall get, for the final form of our period equation,

$$\Big\{p\Big[\sigma\Big\{n\Big(u_2+\frac{\lambda}{n}\Big)^2-u_2^2\Big\}+u'^2-2\Big(u'+\frac{\lambda}{n}\Big)\frac{b_2^2u'-b_1^2u_1}{b_2^2-b_1^2}\Big] \\ +nq\Big(u'+\frac{\lambda}{n}\Big)^2\Big\}$$

$$\times\Big\{p\Big[-u_1^2+2\Big(u_1+\frac{\lambda}{n}\Big)\frac{b_2^2u'-b_1^2u_1}{b_2^2-b_1^2}\Big]+nq\Big(u_1+\frac{\lambda}{n}\Big)^2\Big\}$$

$$=4n^2\Big(u_1+\frac{\lambda}{n}\Big)^2\Big(u'+\frac{\lambda}{n}\Big)^2. \qquad (54)$$

Before discussing this equation, and considering the question of the reality or unreality of its roots, I propose to consider the analogous problem in columnar vortices, and to see whether we can, from the period equation for that case, verify the result just obtained.

The Columnar Vortex with a Hollow Core.

14. We shall now examine the case of a cylindrical vortex column in an infinite liquid, there being a hollow, in steady motion bounded by the surface $r=a$, and the surface $r=b$ separating the rotationally from the irrotationally moving liquid. This column is to be at rest in the liquid.

Dealing with the current function, we have the equations

$$\left.\begin{array}{ll}\dfrac{\partial^2}{\partial r^2}\psi+\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}+2\zeta=0, & a<r<b \\ \dfrac{\partial^3}{\partial r^2}\psi+\dfrac{1}{r}\dfrac{\partial\psi}{\partial r}\qquad\quad=0, & b<r<\infty\end{array}\right\}. \qquad (55)$$

$$\left.\begin{array}{l}\psi_1=A\log r+B-\tfrac{1}{2}\zeta r^2 \\ \psi_2=C\log r+D\end{array}\right\} \qquad (56)$$

satisfy these equations.

They give $\psi =$ constant at $r = a$, $r = b$.

Let μ_1, μ', and μ_2 be the circulations at $r = a$, just inside $r = b$, and just outside $r = b$.

Then
$$\mu_1 = \int_0^{2\pi} \left(-\frac{\partial \psi_1}{\partial r}\right) r\, d\theta = 2\pi\,(-A + \zeta a^2),$$
$$\mu' = 2\pi\,(-A + \zeta b^2),$$
$$\mu_2 = -2\pi C.$$

Thus
$$A = \frac{\zeta}{2}\,(a^2 + b^2) - \frac{\mu_1 + \mu'}{4\pi},$$
$$C = \qquad\qquad -\frac{\mu_2}{2\pi}.$$

B and D are unimportant.

The pressure equation when ψ is independent of θ is
$$\frac{1}{\rho}\,\frac{\partial p}{\partial r} = \frac{v^2}{r}.$$

Therefore
$$\frac{p}{\rho_1} = \tfrac{1}{2}\zeta^2\,(r^2 - a^2) - 2\zeta A \log\left(\frac{r}{a}\right) - \frac{A}{2}\left(\frac{1}{r^2} - \frac{1}{a^2}\right),$$
where we have made $p = 0$ at $r = a$.

Also outside
$$\frac{p}{\rho_2} = -\frac{C}{2r^2} + \frac{\Pi}{\rho_2},$$
if Π is the pressure at infinity, and the pressure at infinity must be such that
$$\Pi = \frac{C\rho_2}{2b^2} + \tfrac{1}{2}\zeta^2\rho_1\,(b^2 - a^2) - 2\zeta a\rho_1 \log\left(\frac{b}{a}\right) + \frac{A}{2}\,\rho_1\,\frac{b^2 - a^2}{a^2 b^2}. \tag{57}$$

This equation (57) gives the relation that must hold between Π and the other factors of the problem $(\mu_1, \mu', \mu_2)(\rho_1, \rho_2)(\zeta, a, b)$ to allow the hollow to exist.

15. Suppose this motion to be disturbed, and the equations of the bounding surfaces to be, at time t,
$$\left.\begin{aligned} r &= a\,(1 + \alpha\, e^{\iota(n\theta + \lambda t)}) \\ r &= b\,(1 + \beta\, e^{\iota(n\theta + \lambda t)}) \end{aligned}\right\}. \tag{58}$$

Let the velocity potentials for the disturbed motion be

$$\left.\begin{aligned}\phi_1 &= \left[P\left(\frac{r}{b}\right)^n + Q\left(\frac{b}{r}\right)^n\right] e^{i(n\theta+\lambda t)}\\ \phi_2 &= \qquad\qquad R\left(\frac{b}{r}\right)^n \quad e^{\iota(n\theta+\lambda t)}\end{aligned}\right\}, \tag{59}$$

where α, β, P, Q, and R are small quantities whose squares and products are to be neglected.

We have the corresponding current functions for the disturbance alone,

$$\left.\begin{aligned}\chi_1 &= -i\left[P\left(\frac{r}{b}\right)^n - Q\left(\frac{b}{r}\right)^n\right] e^{i(n\theta+\lambda t)}\\ \chi_2 &= iR\left(\frac{b}{r}\right)^n e^{i(n\theta+\lambda t)}\end{aligned}\right\}, \tag{60}$$

and the current function giving the complete motion we are taking, as in § 12, to be

$$\Psi = \psi + \chi.$$

16. The surfaces
$$r = a\left(1+\alpha e^{\iota(n\theta+\lambda t)}\right),$$
$$r = b\left(1+\beta e^{\iota(n\theta+\lambda t)}\right),$$

contain the same particles throughout the motion.

The equation we have to satisfy is then of the type

$$\frac{\partial F}{\partial t} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\Psi\right)\frac{\partial F}{\partial r} + \left(-\frac{\partial}{\partial r}\Psi\right)\frac{1}{r}\frac{\partial F}{\partial\theta} = 0.$$

Neglecting the squares, &c., of the small quantities, and putting in

$$U_1 = -\frac{\partial\psi_1}{\partial r},$$

when $r = a$, and

$$U_1 + \frac{a\lambda}{n} = V_1,$$

just as in § 13, we find

$$\left.\begin{aligned}P\left(\frac{a}{b}\right)^n - Q\left(\frac{b}{a}\right)^n &= ia\alpha V_1\\ P - Q &= ib\beta V'\\ -R &= ib\beta V_2\end{aligned}\right\}. \tag{61}$$

We shall require the following results, which are deduced at once from § 14:—

$$\left.\begin{aligned} U_1 &= \frac{\mu_1}{2a\pi}, & \frac{\partial U_1}{dr} &= 2\zeta - \frac{U_1}{a}, \\ U' &= \frac{\mu_1+\mu'}{2b\pi}, & \frac{\partial U'}{dr} &= 2\zeta - \frac{U'}{b}, \\ U_2 &= \frac{\mu_2}{2b\pi}, & \frac{\partial U_2}{dr} &= \quad - \frac{U_2}{b}. \\ \text{Also} \qquad 2\zeta &= \frac{\mu'}{\pi} \quad \frac{1}{b^2-a^2} & &= \frac{2}{b^2-a^2}(bU'-aU_1). \end{aligned}\right\} \quad (61')$$

The Pressure Equations.

17. The work here is perfectly analogous to that in § 13, though simpler, because of the nature of our current function.

(i.) At the inner surface $p=0$. Therefore, leaving out the constant terms, we find from

$$\frac{p_1}{\rho_1} + \dot{\varphi}_1 + \tfrac{1}{2}q^2 + 2\zeta\Psi = f_1(t),$$

to the lowest order

$$0 = i\lambda e^{i(n\theta+\lambda t)}\left[P\left(\frac{a}{b}\right)^n + Q\left(\frac{b}{a}\right)^n\right] + U_1\left[\frac{\partial U_1}{\partial r}\,.\,a\alpha e^{\iota(n\theta+\lambda t)} - \frac{\partial \chi_1}{\partial r}\right]$$
$$+2\zeta(-U_1\delta r + \chi_1).$$

Substituting the values for χ_1 and $\dfrac{\partial\chi_1}{\partial r}$, we have

$$i\,\frac{n}{a}\left[P\left(\frac{a}{b}\right)^n + Q\left(\frac{b}{a}\right)^n\right]V_1 + a\alpha U_1\frac{\partial U_1}{\partial r} + 2\zeta a^2\alpha\frac{\lambda}{n} = 0;$$

and therefore, from (61),

$$a\alpha\left\{\left(x^n - \frac{1}{x^n}\right)\left(U_1\frac{\partial U_1}{\partial r} + 2\zeta a\,\frac{\lambda}{n}\right) + \frac{n}{a}V_1^2\left(x^n + \frac{1}{x^n}\right)\right\} = 2nx\beta\, V_1V', \quad (62)$$

where
$$\frac{b}{a} = x.$$

(ii.) At the outer surface p is continuous. The condition being applied here just as in the three-dimensional case, we find in the end the following equation which must be satisfied:—

$$\sigma\left[inR\left(U_2 + \frac{\lambda b}{n}\right) + b\beta U_2\frac{\partial U_2}{\partial r}\right]$$
$$= (P+Q)\,i\,\frac{n}{b}\left(U' + \frac{\lambda b}{n}\right) + b\beta\left(U'\frac{\partial U'}{\partial r} + \frac{b\lambda}{n}\right),$$

where $$\sigma = \rho_2/\rho_1.$$

Putting in the values found for P, Q, and R in (61), this gives

$$b\beta\left[\left(x^n - \frac{1}{x^n}\right)\left(\sigma\frac{n}{b}V_2^2 + \sigma U_2\frac{\partial U_2}{\partial r} - U'\frac{\partial U'}{\partial r} - 2\zeta\frac{b\lambda}{n}\right) + \frac{n}{b}\left(x^n + \frac{1}{x^n}\right)V'^2\right]$$

$$= 2a\frac{n}{x}V_1V'. \tag{63}$$

From equations (62) and (63), we have the following determining λ,

$$\left\{\left(x^n - \frac{1}{x^n}\right)\left[\sigma nV_2^2 + b\left(\sigma U_2\frac{\partial U_2}{\partial r} - U'\frac{\partial U'}{\partial r}\right) - 2\zeta b^2\frac{\lambda}{n}\right] + n\left(x^n + \frac{1}{x^n}\right)V'^2\right\}$$

$$\times\left\{\left(x^n - \frac{1}{x^n}\right)\left[aU_1\frac{\partial U_1}{\partial r} + 2\zeta a^2\frac{\lambda}{n}\right] + n\left(x^n + \frac{1}{x^n}\right)V_1^2\right\} = 4n^2V_1^2V'^2. \tag{64}$$

Using the results of § (17) for $\dfrac{\partial U_1}{\partial r}$, &c., this becomes

$$\left\{\left(x^n - \frac{1}{x^n}\right)\left(\sigma nV_2^2 - \sigma U_2^2 + U'^2 - \frac{b\mu'}{\pi}\frac{1}{b^2-a^2}V'\right) + n\left(x^n + \frac{1}{x^n}\right)V'^2\right\}$$

$$\times\left\{\left(x^n - \frac{1}{x^n}\right)\left(-U_1^2 + \frac{a\mu'}{\pi}\frac{1}{b^2-a^2}V_1\right) + n\left(x^n + \frac{1}{x^n}\right)V_1^2\right\} = 4n^2V_1^2V'^2.$$

But $$\frac{\mu'}{2\pi} = b^2\frac{U_1'}{b} - a^2\frac{U'}{a};$$

therefore, putting $$\frac{U_1}{a} = u_1, \quad \frac{U'}{b} = u', \text{ \&c.,}$$

and dividing by a^4b^4, we find for our period equation

$$\left\{p\left[\sigma\left\{n\left(u_2 + \frac{\lambda}{n}\right)^2 - u_2^2\right\} + u'^2 - 2\left(u' + \frac{\lambda}{n}\right)\frac{b^2u' - a^2u_1}{b^2-a^2}\right]\right.$$

$$\left. + nq\left(u' + \frac{\lambda}{n}\right)^2\right\}$$

$$\times\left\{p\left[-u_1^2 + 2\left(u_1 + \frac{\lambda}{n}\right)\frac{b^2u' - a^2u_1}{b^2-a^2}\right] + nq\left(u_1 + \frac{\lambda}{n}\right)^2\right\}$$

$$= 4n^2\left(u_1 + \frac{\lambda}{n}\right)^2\left(u' + \frac{\lambda}{n}\right)^2. \tag{65}$$

On comparing this equation (65) with the period equation (54) found for the vortex ring vibrations the agreement is evident.

The Period Equation.

18. After this confirmation of the investigation of §§ 11–13, we may proceed to examine the period equation found for the vortex ring with a view to the conditions necessary for stability.

The equation in the general case is a quartic, and its discussion would be laborious in the extreme. However, by simplifying the details of the problem, we may get instructive information bearing on the general features of the motion.

Let there be a hollow, but the circulation round the hollow zero.

Thus $$u_1 = 0, \quad \mu_1 = 0.$$

The equation then becomes

$$4\lambda\,(\lambda + nu')^2 = \left(q\lambda + 2\,\frac{x^2}{x^2-1}\,pu'\right)$$
$$\times \left\{ p \left[\sigma \left\{ (\lambda + nu_2)^2 - nu_2^2 \right\} - 2v\lambda\,\frac{x^2}{x^2-1} - \frac{x^2+1}{x^2-1}\,nu'^2 \right] + (\lambda + nv)^2 q \right\},$$

when we have omitted the factor λ from both sides.

Writing $q/p = r$, this gives

$$\lambda^3\,(1 + r\sigma) + 2\lambda^2 \left(nu' + \sigma u'\,\frac{x^2}{x^2-1} + r\sigma nu_2 \right)$$
$$+ \lambda \left[n^2u' + 4u'\,\frac{x^2}{x^2-1} \left(\sigma nu_2 - u'\,\frac{x^2}{x^2-1} \right) \right.$$
$$\left. + r \left(\sigma u_2^2 . n . n-1 - nu'^2\,\frac{x^2+1}{x^2-1} + 4nu'^2\,\frac{x^2}{x^2-1} \right) \right]$$
$$+ 2v\,\frac{x^2}{x^2-1} \left(\sigma u^2 . n . n-1 - nu'^2\,\frac{x^2+1}{x^2-1} + rn^2u'^2 \right)$$
$$= 0. \tag{66}$$

If this form of ring is stable when $\mu_1 = 0$ for this type of disturbance, we must have the roots of equation (66) all real.

Writing it in the form

$$\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$

we have as our conditions, both being necessary,

$$\text{(i.)}\quad b^2 - 3c > 0,$$
$$\text{(ii.)}\quad b^2c^2 - 4c^3 > d\,(4b^3 - 18bc + 27d). \tag{67}$$

Even in the simple form (66) the discussion is complicated, for we are to have every integral value of n at our disposal, and u', u_2 need be taken only in such a way as to satisfy the condition holding between them and the other constants of the problem.

Hollow very Small; x very Great.

19. If x is very great, we find

$$b = \frac{2}{1+\sigma}\left[u'(n+\sigma)+\sigma n u_2\right],$$

$$c = \frac{1}{1+\sigma}\left[u'^2(n^2+3n-4)+4\sigma n u' u_2+\sigma . n(n-1)u_2^2\right],$$

$$d = n(n-1)(\sigma u_2^2+u'^2)\frac{2u'}{1+\sigma}.$$

Put $\frac{u_2}{u'} = z$, and $\frac{\lambda}{u'} = y$; we thus find

$$y^3(1+\sigma)+2y^2(n+\sigma+\sigma n z)+y\left[n^2+3n-4+4\sigma n z+\sigma n(n-1)z^2\right] + 2n(n-1)(1+\sigma z^2) = 0. \quad (67)$$

This is the same as

$$(y+2)\left[y^2(1+\sigma)+2y(n . n-1+n\sigma z)+n(n-1)(1+\sigma z^2)\right] = 0, \quad (68)$$

which agrees with the equation

$$y^2(1+\sigma)+2y(n . n-1+n\sigma z)+n(n-1)(1+\sigma z^2) = 0, \quad (69)$$

found by Basset for the "solid" ring.

He discusses this equation (*Hydrodynamics*, § 228), and finds that its roots are real only if

$$\sigma \geqq 1 \quad \text{and} \quad z = 1;$$

i.e., when the hollow is very small there must be no slipping at the outer surface, and the density inside cannot be greater than that outside if there is no circulation round the inner surface.

We see also that when a hollow has just formed in this ring there are two new periods introduced in addition to those it holds in common with the "solid" ring.

Hollow Large; x nearly Unity.

20. In this case our equation for λ becomes, after some reduction and neglecting small terms,

$$\lambda^3+n\lambda^2(u'+2u_2)+n\lambda\left[(n-1)u_2^2+2nu'u_2\right]+n^2u'^2\left[(n-1)u_2^2-\tfrac{1}{2}\frac{u'^2}{\sigma}\right]=0;$$

and, with the same notation as before,

$$y^3+ny^2(1+2z)+nyz\left[(n-1)\,z+2n\right]+n^2\left[(n-1)\,z^3-\frac{1}{2\sigma}\right]=0. \quad (70)$$

(i.) When $\sigma=\infty$, we get

$$y^3+ny^2\,(1+2z)+nyz\left[(n-1)\,z+2n\right]+n^2\,(n-1)\,z^2=0, \quad (71)$$

and this equation is solvable at once, giving as roots

$$y=-z, \quad -z\,(n\mp\sqrt{n}). \quad (72)$$

The roots $\lambda=-u_2\,(n\pm\sqrt{n})$ agree with the values for the periods found by Basset for a hollow vortex. The roots being all real, we see that when the hollow is large and there is no circulation round the inner surface the ring is not stable unless the outside liquid has a density infinitely greater than that of the inside.

(ii.) When $z=1$, we can examine easily the conditions for the reality of the roots of the period equation.

$$\begin{aligned} b^2-3c &= 3^2n^2-3n\,(3n-1) \\ &= 3n. \end{aligned}$$

Therefore, when n is positive, $b^2-3c>0$.

Also

$$b^2c^2-4c^3-d\,(4b^3-18bc+27d)=n^2\left[4n-\frac{27n^2}{4\sigma}\right]. \quad (73)$$

If σ is very great, this is always positive; but, if σ is finite, it is possible to choose n so that the expression is negative, and thus it is only for σ very great that the case is stable for this kind of disturbance.

21. The general investigation of the reality of the roots of our period equation, either in the quartic form of (65), or in the cubic form of (66), for all possible values of σ, r, u_1, u', u_2, and n, is not obvious.

So far I have not been able to get any other definite results than those I have given above, which correspond to those given by Hicks (*Phil. Trans.*, 1885).

It is true that all have the same peculiarity, which causes the difficulty of accounting for the masses of the elements as compared with the ether on the vortex hypothesis. In no one of the cases I have obtained does our criterion of stability allow the density of the core to be greater than that of the fluid outside.

Yet, in the face of the period equation, it would be rash to say that this must hold in all cases. Though my work has not brought to light any arrangement of density and circulation which would satisfy the conditions for stability and yet allow for the excessive masses of the elements, the question is reduced to the consideration of this equation. At any rate, before we reject the vortex atom theory because of the difficulty in certain types of ring of accounting for the difference in mass between gross matter and the ether, we must have more proof that its roots are unreal, when we have that distribution of density for which this difference seems to call.

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*Note on the Symmetric Group.** *By* W. Burnside.

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The symmetric groups of three and four symbols are capable of abstract definition by the relations

$$A^2 = 1, \quad B^3 = 1, \quad (AB)^2 = 1,$$

and

$$A^2 = 1, \quad B^3 = 1, \quad (AB)^4 = 1,$$

respectively; while the alternating group of five symbols can be defined by

$$A^2 = 1, \quad B^3 = 1, \quad (AB)^5 = 1.$$

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* [On November 23rd, 1896, I received from Prof. E. H. Moore, of the University of Chicago, for communication to the Society, a paper dealing with the abstract definition of the symmetric and the alternating groups. As the results contained in Prof. Moore's paper are much more complete than mine, I asked permission from the Council to withdraw my communication. Its appearance in the *Proceedings* is due to the Council having expressed an opinion that it was desirable that both papers should be printed.—W. B.]

So far as I know, no attempt has hitherto been made to define in abstract form the symmetric group of more than four symbols, or the alternating group of more than five. For the investigations of the various types of group which can be constructed from given factor-groups, a definition of the factor-groups in abstract form is, except in the simplest cases, essential.

In the present note I have made one step towards this definition for the case of the symmetric group of n symbols, whatever n may be, by showing that, in addition to

$$S_2^2 = 1, \quad S_n^n = 1, \quad (S_n S_2)^{n-1} = 1,$$

$\frac{1}{2}(3n-10)$ or $\frac{1}{2}(3n-11)$ additional equations, according as n is even or odd, are certainly sufficient to insure that the group generated by S_2 and S_n shall be the symmetric group of n symbols. In the last paragraph I show that the alternating group of six symbols (the simple group of lowest order for which an abstract definition has not hitherto been given) is defined by

$$A^3 = 1, \quad S^5 = 1, \quad (AS)^3 = 1, \quad (AS^{-1})^4 = 1, \quad (AS^3AS^{-3})^2 = 1.$$

1. Let $s_2, s_3, \ldots, s_n$ represent the substitutions

$$(a_1 a_2), \ (a_1 a_2 a_3), \ \ldots, \ (a_1 a_2 a_3 \ldots a_{n-1} a_n).$$

Whatever n may be, it is easy to verify that

$$s_{n-1} = s_n^2 s_2 s_n^{-1};$$

and therefore that $s_3, s_4, \ldots, s_{n-1}$ can be expressed in terms of s_2 and s_n. The expressions thus obtained are rather complicated. It is simple, however, to verify directly that

$$s_n^{r+1} (s_n^{-1} s_2)^r$$

is a circular substitution of $n-r$ letters for all values of r from 0 to $n-2$; and that therefore $s_n^{r+1} (s_n^{-1} s_2)^r$ and s_{n-r} are conjugate substitutions in the symmetric group. The relations

$$s_3^3 = 1, \quad s_4^4 = 1, \quad \ldots, \quad s_{n-1}^{n-1} = 0,$$

are therefore equivalent to

$$\left[s_n^{r+1} (s_n^{-1} s_2)^r\right]^{n-r} = 1. \quad (r = 1, 2, \ldots, n-3)$$

The form of these relations is capable of considerable modification.

Thus $s_n^{-1}s_2$ is the inverse of $s_2 s_n$, a substitution of order $n-1$, so that

$$(s_n^{-1}s_2)^r = (s_2 s_n)^{n-r-1}.$$

For $n-r=3$, the relation is

$$\left[s_n^{-2}(s_2 s_n)^2\right]^3 = 1,$$

or
$$(s_n^{-2}s_2 s_n s_2 s_n)^3 = 1,$$

or
$$(s_n^{-1}s_2 s_n s_2)^3 = 1.$$

For $n-r=4$, it is $\left[s_n^{-3}(s_2 s_n)^3\right]^4 = 1,$

or
$$(s_n^{-2}s_2 s_n s_2 s_n s_2)^4 = 1\,;$$

and so on.

2. Let S_2 and S_{n-1} be two operations of orders 2 and $n-1$; and suppose that

$$f_1(S_2, S_{n-1}) = 1,\quad f_2(S_2, S_{n-1}) = 1,\ \ldots,\ f_i(S_2, S_{n-1}) = 1,$$

form a set of relations connecting S_2 and S_{n-1}, sufficient and necessary to insure that the group generated by S_2 and S_{n-1} shall be simply isomorphic with the group generated by the two substitutions s_2 and s_{n-1} (*i.e.*, with the symmetric group of $n-1$ symbols); S_2 and s_2, S_{n-1} and s_{n-1}, being corresponding operations.

Also let S_n be an operation of order n, connected with S_2 and S_{n-1} by the two relations

$$S_{n-1} = S_n^2 S_2 S_n^{-1}, \tag{1}$$

and
$$(S_n^{-2} S_2 S_n^2 S_2)^2 = 1. \tag{2}$$

We may then show that S_2 and S_n generate a group simply isomorphic with the symmetric group of n symbols.

To this end, we first prove that the cyclical group generated by S_n and the group generated by S_2 and S_{n-1} are permutable with each other; *i.e.*, that, if the operations of the latter group are represented by the symbols T_i, then, for every value of r and i, $S_n^r T_i$ can be expressed in the form $T_{i'} S_n^{r'}$. Since every T_i can be expressed in terms of S_2 and S_{n-1}, it is clearly sufficient to show that, for all values of r, $S_n^r S_2$ and $S_n^r S_{n-1}$ can each be expressed in the form $T_{i'} S_n^{r'}$.

Now, from (1), it follows that

$$S_n S_2 = S_{n-1}^{-1} S_n^2,$$

$$S_n^2 S_2 = S_{n-1} S_n,$$

$$S_n^{-1} S_{n-1} = S_{n-1}^{-1} S_n,$$

and $$S_n^{-2} S_{n-1} = S_2 S_n^{-1}.$$

Also $$S_n^r S_{n-1} = S_n^{r+2} S_2 S_n^{-r-2} S_n^{r+1}.$$

Hence, if $S_n^{r+2} S_2 S_n^{-r-2}$ belongs to the group generated by S_2 and S_{n-1} for all values of r from 1 to $n-3$, the two groups in question are permutable with each other. We shall prove that these conditions are satisfied by showing that

$$S_n^{r+2} S_2 S_n^{-r-2} = S_{n-1}^{r+1} S_2 S_{n-1}^{-r-1}, \qquad (r = 1, 2, \ldots, n-3)$$

or that $S_n^{-r-2} S_{n-1}^{r+1}$ is permutable with S_2.

Expressed at length this condition is

$$S_n^{-r-2} (S_n^2 S_2 S_n^{-1})^{r+1} S_2 (S_n^2 S_2 S_n^{-1})^{-r-1} S_n^{r+2} S_2 = 1,$$

or $$S_n^{-r-1} (S_n S_2)^{r+1} S_n^{-1} S_2 S_n (S_2 S_n^{-1})^{r+1} S_n^{r+1} S_2 = 1. \tag{3}$$

Now $$S_{n-1}^{-1} S_2 S_{n-1} S_2 = S_n S_2 S_n^{-2} S_2 S_n^2 S_2 S_n^{-1} S_2,$$

and, from (2), $$S_2 S_n^{-2} S_2 S_n^2 S_2 = S_n^{-2} S_2 S_n^2,$$

so that $$S_{n-1}^{-1} S_2 S_{n-1} S_2 = S_n S_n^{-2} S_2 S_n^2 S_n^{-1} S_2$$

$$= S_n^{-1} S_2 S_n S_2.$$

Also, since the group generated by S_2 and S_{n-1} is simply isomorphic with the substitution group of § 1 when $n-1$ is written for n,

$$(S_{n-1}^{-1} S_2 S_{n-1} S_2)^3 = 1.$$

Hence also $$(S_n^{-1} S_2 S_n S_2)^3 = 1.$$

Writing now the equation (3) in the form

$$S_n^{-r-1} (S_n S_2)^r S_n S_2 S_n^{-1} S_2 S_n S_2 S_n^{-1} S_2 S_n^{-1} (S_2 S_n^{-1})^{r-1} S_n^{r+1} S_2 = 1,$$

and using the result just obtained, we find that

$$S_n^{-r-1} (S_n S_2)^r S_2 S_n S_2 S_n^{-2} (S_2 S_n^{-1})^{r-1} S_n^{r+1} S_2 = 1,$$

or $$S_n^{-r-1}(S_nS_2)^{r-1}S_n^2S_2S_n^{-2}(S_2S_n^{-1})^{r-1}S_n^{r+1}S_2 = 1. \qquad (4)$$

This condition then must be satisfied for all values of r from 1 to $n-3$.

3. Since S_2 and S_{n-1} generate a group simply isomorphic with the group generated by the substitutions s_2 and s_{n-1}, it may be verified by forming the corresponding substitutions that

$$(S_{n-1}^rS_2S_{n-1}^{-r}S_2)^2 = 1. \qquad (r = 2, 3, \ldots, n-3)$$

Hence, writing for S_{n-1} its value in terms of S_n,

$$\left[S_n(S_nS_2)^rS_n^{-1}S_2S_n(S_2S_n^{-1})^rS_n^{-1}S_2\right]^2 = 1,$$

and, using in this the relation

$$(S_nS_2S_n^{-1}S_2)^2 = S_2S_nS_2S_n^{-1},$$

it becomes $$\left[S_n(S_nS_2)^{r-2}S_n^2S_2S_n^{-2}(S_2S_n^{-1})^{r-2}S_n^{-1}S_2\right]^2 = 1. \qquad (5)$$

For $r = 2$, this gives

$$(S_n^3S_2S_n^{-3}S_2)^2 = 1.$$

Again, using the relation

$$S_2S_n^2S_2S_n^{-2} = S_n^2S_2S_n^{-2}S_2,$$

derived from (2), in (5), it becomes

$$\left[S_n(S_nS_2)^{r-3}S_n^3S_2S_n^{-3}(S_2S_n^{-1})^{r-3}S_n^{-1}S_2\right]^2 = 1.$$

For $r = 3$, this gives

$$(S_n^4S_2S_n^{-4}S_2)^2 = 1.$$

This process may clearly be continued step by step so that, from the equation at the beginning of this paragraph with equation (2), we derive

$$(S_n^rS_2S_n^{-r}S_2)^2 = 1. \qquad (r = 3, 4, \ldots, n-2)$$

These equations are not all distinct, the values $r = t$ and $r = n-t$ clearly giving the same equation.

4. Returning to equation (4), we may now verify step by step that it is satisfied for all values of r concerned. Thus, if we put $r = 1$, it is satisfied identically.

Making use of the relation

$$S_2 S_n^2 S_2 S_n^{-2} = S_n^2 S_2 S_n^{-2} S_2,$$

(4) may be written

$$S_n^{-r-1} (S_n S_2)^{r-2} S_n^3 S_2 S_n^{-3} (S_2 S_n^{-1})^{r-2} S_n^{r+1} S_2 = 1,$$

and in this form it is satisfied identically for $r = 2$.

Using now the relation

$$S_2 S_n^3 S_2 S_n^{-3} = S_n^3 S_2 S_n^{-3} S_2,$$

obtained in the last paragraph, (4) becomes

$$S_n^{-r-1} (S_n S_2)^{r-3} S_n^4 S_2 S_n^{-4} (S_2 S_n^{-1})^{r-3} S_n^{r+1} S_2 = 1,$$

and it is satisfied identically for $r = 3$. Proceeding thus, we show by the aid of the results of the last paragraph that (4) is satisfied for all values of r up to $n-3$.

Hence the cyclical group generated by S_n is permutable with the group generated by S_2 and S_{n-1}, and therefore S_2 and S_n generate a group whose order is equal to or is a factor of $n!$. Now the substitutions s_2 and s_n generate a group of order $n!$ (the symmetric group of n symbols), and they satisfy all the relations satisfied by S_2 and S_n. Hence the order of the group generated by S_2 and S_n cannot be less than $n!$. It is therefore equal to $n!$; and at the same time it is proved that the group generated by S_2 and S_n is simply isomorphic with the substitution group generated by s_2 and s_n, *i.e.*, with the symmetric group of n symbols.

5. The i independent relations,

$$f_r(S_2, S_{n-1}) = 1, \qquad (r = 1, 2, \ldots, i)$$

will not be increased in number, when in each of them S_{n-1} is replaced by $S_n^2 S_2 S_n^{-1}$. In addition to these i relations, S_2 and S_n satisfy the two further equations

$$S_n^n = 1,$$

and

$$(S_n^{-2} S_2 S_n^2 S_2)^2 = 1.$$

Hence, if S_2 and S_{n-1}, the generators of the symmetric group of $n-1$ symbols, are connected by i independent relations, then S_2 and S_n, the generators of the symmetric group of n symbols, are connected by *at most* $i+2$ independent relations.

6. Now for the symmetric group of five symbols we may show directly that

$$S_2^2 = 1, \quad (S_5^{-1} S_2 S_5 S_2)^3 = 1, \quad (S_5 S_2)^4 = 1, \quad S_5^5 = 1, \qquad (6)$$

form a necessary and sufficient set of relations between S_2 and S_5. For this purpose, consider the operations A and B defined by

$$A = S_5^{-2}, \quad B = (S_5 S_2)^2.$$

Since $$S_5^{-1} S_2 S_5 S_2 = S_5^{-2} (S_5 S_2)^2,$$

A and B satisfy the relations

$$A^5 = 1, \quad B^2 = 1, \quad (AB)^3 = 1;$$

and therefore* A and B generate a group of order 60. Now this group contains the operation A^3B or $S_2 S_5 S_2$; and therefore it contains every operation of the group generated by S_2 and S_5 in which S_2 occurs an even number of times. The set of operations $S_2 T$, when for T is put in turn every operation of the group generated by A and B, must therefore give every operation of the group generated by S_2 and S_5 in which S_2 occurs an odd number of times. The order of the group generated by S_2 and S_5 is therefore 120. Now the substitutions s_2 and s_5 satisfy the relations which S_2 and S_5 satisfy and generate a group of order 120. Hence S_2 and S_5, when connected by the relations (6), generate a group simply isomorphic with the symmetric group of five symbols.

It remains to show that the relations (6) are independent. If they are not, either the second or third of the equations must be redundant. The second cannot be, for the relations

$$S^2 = 1, \quad S_5^5 = 1, \quad (S_2 S_5)^4 = 1,$$

do not define a group of finite order.†

Again, the relations

$$S_2^2 = 1, \quad S_5^5 = 1, \quad (S^{-1} S_2 S_5 S_2)^3 = 1,$$

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* Hamilton, *Phil. Mag.*, 1856, p. 446.

† Dyck, "Gruppentheoretische Studien," *Math. Ann.*, xx (1882), § 15.

certainly cannot define a group simply isomorphic with the symmetric group of five symbols, for the substitutions

$$\sigma_2 = (a_1 a_3)(a_2 a_5), \quad \sigma_5 = (a_1 a_2 a_3 a_4 a_5)$$

satisfy these relations, and they generate the alternating group of five symbols.

Hence, finally, the relations (6) are necessary as well as sufficient to define the symmetric group of five symbols.

7. If we now define S_r in terms of S_{r+1}, for all values of r, by the relation

$$S_r = S_{r+1}^2 S_2 S_{r+1}^{-1},$$

the symmetric group of five symbols is given by

$$S_2^2 = 1, \quad S_3^3 = 1, \quad S_4^4 = 1, \quad S_5^5 = 1.$$

Hence S_2 and S_6 will generate a group simply isomorphic with the symmetric group of six symbols, if

$$S_2^2 = 1, \quad S_3^3 = 1, \quad S_4^4 = 1, \quad S_5^5 = 1, \quad S_6^6 = 1,$$

and $$(S_6^{-2} S_2 S_6^2 S_2)^2 = 1.$$

Proceeding thus, step by step, we find that the relations

$$S_r^r = 1, \qquad (r = 2, 3, \ldots, n),$$

and $$(S_r^{-2} S_2 S_r^2 S_2)^2 = 1, \qquad (r = 6, 7, \ldots, n),$$

are sufficient to insure that S_2 and S_n shall generate a group simply isomorphic with the symmetric group of n symbols.

These relations may be simplified, and some of them may be shown to be redundant, as follows:—

The equation $$S_3^3 = 1$$

is equivalent, when $n = 5$, to

$$(S_5^{-1} S_2 S_5 S_2)^3 = 1.$$

Let us suppose that, when $n = r$, it is equivalent to

$$(S_r^{-1} S_2 S_r S_2)^3 = 1.$$

Writing in this $S_{r+1}^2 S_2 S_{r+1}^{-1}$ for S_r, it becomes

$$(S_{r+1} S_2 S_{r+1}^{-2} S_2 S_{r+1}^{2} S_2 S_{r+1}^{-1} S_2)^3 = 1\,;$$

and since, when $n = r+1$, one of the defining relations is

$$(S_{r+1}^{-2} S_2 S_{r+1}^{2} S_2)^2 = 1,$$

the previous equation becomes

$$(S_{r+1}^{-1} S_2 S_{r+1} S_2)^3 = 1.$$

Hence, for all values of n,

$$S_3^3 = 1$$

may be replaced by

$$(S_n^{-1} S_2 S_n S_2)^3 = 1. \qquad (7)$$

In the next place, it follows from § 3 that the relation

$$(S_{n-1}^{r} S_2 S_{n-1}^{-r} S_2)^2 = 1,$$

which holds for all values of r other than ±1, is equivalent to

$$(S_n^{r+1} S_2 S_n^{-r-1} S_2)^2 = 1.$$

Hence

$$(S_{n-r}^{-2} S_2 S_{n-r}^{2} S_2)^2 = 1$$

is equivalent to

$$(S_n^{-r-2} S_2 S_n^{r+2} S_2)^2 = 1.$$

The relations

$$(S_r^{-2} S_2 S_r^{2} S_2)^2 = 1, \quad (r = 6, 7, \dots, n)$$

may therefore be replaced by

$$(S_n^{-t} S_2 S_n^{t} S_2)^2 = 1. \quad (t = 2, 3, \dots, n-4) \quad (8)$$

Now

$$(S_n^{-t} S_2 S_n^{t} S_2)^2 = 1$$

and

$$(S_n^{-n+t} S_2 S_n^{n-t} S_2)^2 = 1$$

are equivalent, and therefore of the relations (8) only those for which t is not greater than $\frac{1}{2}n$ can be independent.

Finally, the equations

$$S_r^r = 1$$

may be expressed in a simple form in terms of S_n and S_2. Thus

$$S_{n-1}^{n-1} = 1$$

is the same as
$$(S_n^2 S_2 S_n^{-1})^{n-1} = 1,$$
and it is therefore equivalent to
$$(S_n S_2)^{n-1} = 1.$$

Again,
$$S_{n-2}^{n-2} = 1$$
is equivalent to
$$(S_{n-1} S_2)^{n-2} = 1,$$
or to
$$(S_n^2 S_2 S_n^{-1} S_2)^{n-2} = 1\,;$$
and
$$S_{n-3}^{n-3} = 1$$
is equivalent to
$$(S_{n-1}^2 S_2 S_{n-1}^{-1} S_2)^{n-3} = 1,$$
which, expressed in terms of S_n and S_2, is
$$(S_n^2 S_2 S_n S_2 S_n^{-1} S_2 S_n S_2 S_n^{-2} S_2)^{n-3} = 1.$$

Using (7) in this, it becomes
$$(S_n^2 S_n S_2 S_n^{-1} S_2 S_n^{-1} S_2)^{n-3} = 1,$$
or
$$(S_n^3 S_2 S_n^{-1} S_2 S_n^{-1} S_2)^{n-3} = 1.$$
Proceeding thus, step by step, we show that, for each value of r,
$$S_r^r = 1$$
is equivalent to
$$\left[S_n^{-r+1} (S_n^{-1} S_2)^{-r+1}\right]^r = 1,$$
or
$$\left[S_n^{-r+1} (S_2 S_n)^{r-1}\right]^r = 1, \tag{9}$$
or
$$\left[S_n^{-r+1} (S_n S_2)^{r-1}\right]^r = 1.$$

Collecting the results thus obtained, we find that the relations
$$\left[S_n^{-r+1} (S_n S_2)^{r-1}\right]^r = 1, \quad (r = 2, 3, \ldots, n)$$
and
$$(S_n^{-r} S_2 S_n^r S_2)^2 = 1, \quad \left(r = 2, 3, \ldots, \frac{n-1}{2} \text{ or } \frac{n}{2}\right)$$
are sufficient to insure that S_n and S_2 shall generate a group simply isomorphic with the symmetric group of n symbols. Besides the equations
$$S_2^2 = 1, \quad S_n^n = 1, \quad (S_n S_2)^{n-1} = 1,$$

giving the orders of the generating operations and of their product, there are thus *at most* $\frac{3n-10}{2}$ or $\frac{3n-11}{2}$ additional relations according as n is even or odd.

8. When $n = 6$, the relations are

$$S_2^2 = 1, \quad (S_6 S_2 S_6^{-1} S_2)^3 = 1, \quad (S_6^2 S_2 S_6^{-1} S_2)^4 = 1,$$
$$(S_6 S_2)^5 = 1, \quad (S_6^2 S_2 S_6^{-2} S_2)^2 = 1, \quad S_6^6 = 1.$$

It is easy to verify from these relations that $S_6 S_2 S_6^{-1}$ is permutable with $S_6^2 S_2 S_6^{-1} S_2$, so that

$$(S_6^2 S_2 S_6^{-1} S_2)^4 = 1$$

is equivalent to
$$\left[(S_6 S_2)^2 S_6^{-1} S_2\right]^4 = 1.$$

The alternating group of six symbols will consist of those operations of the symmetric group which, when expressed in terms of S_6 and S_2, contain an even number of factors. It may therefore be generated by S_6^2 and $S_6 S_2$.

If now we write
$$A = S_6^{-2}, \quad C = S_6 S_2,$$

all the defining relations of the symmetric group (the third being replaced by the equivalent form just found), except the first, may be written in terms of A and C. Moreover, since S_2 does not occur in the alternating group, the first of the above equations cannot be one of its defining relations. The defining relations of the alternating group of six symbols may therefore be written

$$A^3 = 1, \quad C^5 = 1, \quad (CAC)^3 = 1, \quad (C^2AC)^4 = 1,$$

and
$$(A^{-1}C^{-1}AC)^2 = 1.$$

By taking C^2 instead of C for a generating operation, these relations become

$$A^3 = 1, \quad C^5 = 1, \quad (AC)^3 = 1, \quad (AC^{-1})^4 = 1, \quad (AC^3A^{-1}C^{-3})^2 = 1.$$

On the Application of Jacobi's Dynamical Method to the General Problem of Three Bodies. *By* Ernest W. Brown.

Read November 12th, 1896. Received October 25th, 1896.

(i.) The connexion which exists between the solution of the problem of three bodies, as obtained by varying the elliptic elements and by the ordinary methods of continued approximation, has had new light thrown on it by the appearance of Professor Newcomb's memoir "Action of the Planets on the Moon."* This memoir, which is in reality an application of an earlier publication by the same writer, † contains a general method of treatment for the undisturbed and the disturbed problem of three bodies. Owing, however, to the use of Lagrange's method for the variation of the arbitrary constants, the explanation and development are somewhat long, and therefore it may not be without value to derive the same results by Jacobi's method, which gives them quite briefly and which also enables us to add one or two new properties of the canonical system of arbitrary constants to be used.

The manner of applying Jacobi's method is substantially the same as that which I have given earlier ‡ for the explanation of Delaunay's "Lunar Theory," and the similarity of some of the results will be readily noticed. For the sake of clearness, only three bodies are considered—the Earth, the Moon, and the Sun; but there is no difficulty in extending the method to any number of bodies by a proper choice of origin for the fixed rectangular coordinates. The transformations necessary for this purpose have been given by Radau § and also by Newcomb in the latter of the two papers referred to above, and they will not be repeated here. Besides developing the results obtained by Newcomb and obtaining certain new properties of the constants, I give a simple proof of the well-known theorem enunciated in § (vii.) below. No assumption is made with respect to the rectangular coordinates except that the series obtained for them are convergent; it is assumed, however, that the elements are expressible by convergent sums of periodic terms whose arguments are all sums of multiples of six angles of the form $l+bt$ (l, b constants), and that the coefficients are independent of the six constants l: this assumption, which only concerns the *form* of the expressions, has been usually made by previous writers, and is a deduction from the results obtained by continued approximation. Mention will be made of the remarkable results which Professor Newcomb obtains for the indirect action of a planet on the Moon, on account of its importance in the calculation of the secular accelerations of the mean motion, the perigee, and the node of the Moon's orbit.

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* *American Ephemeris Papers*, Vol. v., Part iii.

† "General Integrals of Planetary Motion," *Smithsonian Contributions*, Vol. xxi.

‡ *Proc. Lond. Math. Soc.*, Vol. xxvii., pp. 385–390.

§ "Sur une Transformation des Equations Différentielles de la Dynamique," *Ann. de l'Ecole Normale Supérieure*, Vol. v.

(ii.) The following scheme of notation will be used, the rectangular coordinates being referred to any rectangular axes fixed in direction :—

$x, y, z, r,$	Coordinates of the Moon referred to the Earth;
$x', y', z', r',$	,, ,, Sun ,, centre of mass of the Earth and Moon;
$r_1', \Delta,$	Distances of the Sun from the Earth and Moon;
$E, M, m',$	Masses of the Earth, Moon, and Sun, respectively;
$T,$	Kinetic energy of the system relative to its centre of mass;
$F,$	Potential energy of the system;
$H,$	Constant of energy;

$$\mu = \frac{EM}{E+M}; \quad \mu' = \frac{m'(E+M)}{E+M+m'}.$$

We then have

$$r_1'^2 = \left(x' + \frac{M}{E+M}x\right)^2 + \left(y' + \frac{M}{E+M}y\right)^2 + \left(z' + \frac{M}{E+M}z\right)^2,$$

$$\Delta^2 = \left(x' - \frac{E}{E+M}x\right)^2 + \left(y' - \frac{E}{E+M}y\right)^2 + \left(z' - \frac{E}{E+M}z\right)^2.$$

The x-coordinates of the Moon, Earth and Sun, referred to parallel axes through the centre of mass of the system, are respectively

$$\frac{Ex}{E+M} - \frac{m'x'}{E+M+m'}, \quad -\frac{Mx}{E+M} - \frac{m'x'}{E+M+m'}, \quad \frac{(E+M)\,x'}{E+M+m'},$$

with similar expressions for the other coordinates.

$$\text{Hence}\quad 2T = M\left\{\left(\frac{E\dot{x}}{E+M}-\frac{m'\dot{x}'}{E+M+m'}\right)^2+\ldots+\ldots\right\}$$

$$+E\left\{\left(\frac{M\dot{x}}{E+M}+\frac{m'\dot{x}'}{E+M+m'}\right)^2+\ldots+\ldots\right\}$$

$$+m'\left\{\left(\frac{(E+M)\,\dot{x}'}{E+M+m'}\right)^2+\ldots+\ldots\right\}$$

$$=\mu\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\mu'\,(\dot{x}'^2+\dot{y}'^2+\dot{z}'^2).$$

$$\text{Also}\quad F=\frac{EM}{r}+\frac{Em'}{r_1'}+\frac{Mm'}{\Delta}.$$

By Lagrange's method, the equations of motion are therefore

$$\mu\ddot{x}=\frac{\partial F}{\partial x},\quad \mu\ddot{y}=\frac{\partial F}{\partial y},\quad \mu\ddot{z}=\frac{\partial F}{\partial z};$$

$$\mu'\ddot{x}'=\frac{\partial F}{\partial x'},\quad \mu'\ddot{y}'=\frac{\partial F}{\partial y'},\quad \mu'\ddot{z}'=\frac{\partial F}{\partial z'}.$$

Four first integrals of this system are known. They are,

$$T-F=H,$$

$$\left.\begin{aligned}\mu\,(y\dot{x}-x\dot{y})+\mu'\,(y'\dot{x}'-x'\dot{y}')&=\text{const.},\\ \mu\,(z\dot{y}-y\dot{z})+\mu'\,(z'\dot{y}'-y'\dot{z}')&=\text{const.},\\ \mu\,(x\dot{z}-z\dot{x})+\mu'\,(x'\dot{z}'-z'\dot{x}')&=\text{const.}\end{aligned}\right\},\qquad(1)$$

the first being the energy equation and the three others the "integrals of areas."

Further, since F is homogeneous and of degree -1 with respect to the coordinates, we can obtain from the equations of motions the equation

$$\tfrac{1}{2}\frac{d^2}{dt^2}(\mu r^2+\mu' r'^2)-2T=-F=-T+H.$$

We shall see in §(vii.) that the coordinates may be expressed as sums of periodic terms. Let T_0, F_0 be the non-periodic parts of T, F when so expressed. The equations just written down show that

$$T_0=-H,\quad F_0=-2H.\qquad(2)$$

(iii.) In Delaunay's method, the body whose perturbations are to be considered is supposed to move in a varying elliptic orbit. The angular elements, l, g, h, are the mean anomaly, the distance from node to perigee, and the longitude of the node; the areal elements,

L, G, H_1* are functions of the mean motion (or major axis), the eccentricity, and the inclination, given by the equations

$$L = \sqrt{a(E+M)}, \quad G = L\sqrt{1-e^2}, \quad H_1 = G\cos i,$$

where
$$E+M = n^2a^3.$$

The equations satisfied by these elements are

$$\frac{dL}{dt} = \frac{\partial R_1}{\partial l}, \quad \frac{dG}{dt} = \frac{\partial R_1}{\partial g}, \quad \frac{dH_1}{dt} = \frac{\partial R_1}{\partial h};$$

$$\frac{dl}{dt} = -\frac{\partial R_1}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R_1}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R_1}{\partial H_1};$$

where
$$\mu R_1 = F - \mu\frac{E+M}{r} + \mu\frac{(E+M)^2}{2L^2}.\dagger$$

But, since the expressions for the velocities have the same form in disturbed and undisturbed motion, we have

$$\tfrac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{E+M}{r} - \frac{(E+M)^2}{2L^2}.$$

Hence
$$\mu R_1 = F - \tfrac{1}{2}\mu(\dot{x}^2+\dot{y}^2+\dot{z}^2).$$

Similar expressions will hold for the motion of the Sun round the centre of mass of the Earth and Moon, as disturbed by the Moon. Let accented letters refer to the elliptic elements of this orbit. If we put

$$R_2 = F - \mu\frac{E+M}{r} + \mu\frac{(E+M)^2}{2L^2} - \mu'\frac{E+M+m'}{r'} + \mu'\frac{(E+M+m')^2}{2L'^2},$$

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* The suffix is used to distinguish between this symbol and the constant of energy.

† The true disturbing function in its most general form is

$$\left(\frac{Em'}{r_1'} + \frac{Mm'}{\Delta}\right)\frac{E+M}{EM}.$$

If we expand this in powers of r/r', neglecting the ratio M/E and the irrelevant factor m'/r'^3, we obtain

$$m'\left(\frac{1}{\Delta'} - \frac{xx'+yy'+zz'}{r'^3}\right),$$

where
$$\Delta'^2 = (x-x')^2 + \ldots + \ldots;$$

this is the part of the disturbing function used by Delaunay. See Arts. 5, 7, 179 of my *Treatise on the Lunar Theory*.

and write L for μL, L' for $\mu' L'$, &c., the expression R_2 may be taken as the disturbing function for both orbits, and the equations for the elements will be

$$\frac{dL}{dt} = \frac{\partial R_2}{\partial l}, \ \ldots, \ \ldots; \quad \frac{dl}{dt} = -\frac{\partial R_2}{\partial L}, \ \ldots, \ \ldots;$$

$$\frac{dL'}{dt} = \frac{\partial R_2}{\partial l'}, \ \ldots, \ \ldots; \quad \frac{dl'}{dt} = -\frac{\partial R_2}{\partial L'}, \ \ldots, \ \ldots.$$

Further

$$T = \mu \left[\frac{E+M}{r} - \frac{\mu^2}{2L^2}(E+M)^2\right] + \mu' \left[\frac{E+M+m'}{r'} - \frac{\mu'^2}{2L'^2}(E+M+m')^2\right];$$

so that, remembering the change of meaning for L, L',

$$R_2 = F - T = -H.$$

Now any linear tangential transformation of the twelve variables to twelve others will leave the canonical form of the equations unaltered, and the new variables corresponding to l, g, h, l', g', h' will be linear combinations, with constant coefficients, of the six angular elements.

Put then $p_1, p_2, \ldots p_6$, for the six new variables corresponding to L, G, H, L', G', H', and $q_1, q_2, \ldots q_6$ for the six new angular variables; when one set has been defined, the other set is, in general, determinate. We shall suppose that such a transformation takes place that the suffixes 1, 2, 3 refer especially to the lunar and 4, 5, 6 to the solar elements, though this is unnecessary except in § (xi.) below. The particular transformation in view is that given by

$$p_1 = L, \quad p_2 = G - L, \quad p_3 = H_1 - G,$$

$$q_1 = l + g + h, \quad q_2 = g + h, \quad q_3 = h,$$

with six similar equations connecting L', G', H', l', g', h', and $p_4, p_5, p_6, q_4, q_5, q_6$.

Finally, let $$R = -R_2,$$

so that $$R = T - F = H. \tag{3}$$

The twelve canonical equations, which are now the general equations of motion of the system, become

$$\dot{p}_i = -\frac{\partial R}{\partial q_i}, \quad \dot{q}_i = \frac{\partial R}{\partial p_i} \quad (i = 1, 2, \ldots 6). \tag{4}$$

The properties of R when expressed in terms of the new variables are, first, that the time is not present explicitly in R; secondly, that R is expressible as a sum of periodic terms whose coefficients are functions of the p_i only, and whose arguments are all of the form

$$N' = j_1 q_1 + j_2 q_2 + \ldots + j_6 q_6,$$

(the j_i being positive or negative integers or zero); thirdly,

$$j_1 + j_2 + \ldots + j_6 = 0.$$

(iv.) The partial differential equation satisfied by the principal function S is

$$\frac{\partial S}{\partial t} + R = 0,$$

where R is expressed in terms of the q_i, and the derivatives of S with respect to the q_i by means of the equations

$$p_i = \frac{\partial S}{\partial q_i}. \tag{5}$$

It is known that, if a solution of the equation for S containing six arbitrary constants c_i has been obtained, all the integrals of the problem are given by (5) together with

$$l_i = \frac{\partial S}{\partial c_i},$$

where S is supposed to be expressed in terms of the q_i, c_i, t, and the l_i are six new arbitrary constants.

(v.) As no direct method of finding the form of a suitable solution of the equation is available, we shall build up a solution by assuming certain general expressions for p_i, q_i, indicated by the practical methods used for the problem.

We assume that the latter are expressible by convergent series of the form

$$\left.\begin{aligned} p_i &= (P)_i + \Sigma P_i \cos N \\ q_i &= \lambda_i \ \ + \Sigma Q_i \sin N \end{aligned}\right\}, \tag{6}$$

where $$\lambda_i = l_i + b_i t,$$

$$N = j_1\lambda_1 + j_2\lambda_2 + \ldots + j_6\lambda_6,$$

and the sign Σ denotes the sum of the periodic terms of the forms $P \cos N$, $Q \sin N$; in these expressions the time only occurs through the angles λ_i in the manner shown, and $(P)_i$, P_i, Q_i, b_i are independent of the l_i, being functions of six other arbitrary constants. As far as algebraic *form* alone is concerned, these assumptions may be readily proved by induction if we suppose the problem solved by the method which Delaunay uses in his lunar theory, for, initially, P_i, Q_i are zero, and each operation gives expressions for the variables which are always of the above forms.

From the expressions (6) we may eliminate the λ_i, and obtain the p_i in the form

$$p_i = c_i + \Sigma P'_i \cos N', \tag{7}$$

where

$$N' = j_1 q_1 + j_2 q_2 + \ldots + j_6 q_6, \tag{8}$$

the c_i, P_i being functions of six arbitrary constants which we shall take to be $c_1, c_2, \ldots c_6$, the constant terms of $p_1, p_2, \ldots p_6$ when expressed in the form (7).

A solution of the partial differential equation for S is now supposed to have been found, expressed as a function of the c_i, the q_i, and the time. The p_i being given by the equations (7), we have, from (5),

$$\frac{\partial p_i}{\partial q_{i'}} = \frac{\partial^2 S}{\partial q_i \partial q_{i'}} = \frac{\partial p_{i'}}{\partial q_i},$$

and therefore, identically,

$$\Sigma P'_i j_{i'} \cos N' = \Sigma P'_{i'} j_i \cos N.$$

As all the angles N' are different, we must have

$$\frac{P'_i}{j_i} = \frac{P'_{i'}}{j_{i'}} = s, \text{ suppose.}$$

(If j_i is zero in any term, the corresponding P'_i must be zero.) Hence

$$\frac{\partial S}{\partial q_i} = p_i = c_i + \Sigma s j_i \cos N'. \tag{9}$$

Finally, since

$$\frac{\partial S}{\partial t} = -R = -H, \quad \text{by (3)},$$

a solution of the equation for S, expressed as a function of the time, the six variables q_i, and the six arbitraries c_i, must be

$$S = -Ht + \Sigma_i c_i q_i + \Sigma s \sin N'. \tag{10}$$

The arbitrary constant additive to S is, as usual, put zero.

The other six integrals are now given by

$$l_i = \frac{\partial S}{\partial c_i} = -t\frac{\partial H}{\partial c_i} + q_i + \Sigma \frac{\partial s}{\partial c_i} \sin N'. \tag{11}$$

The complete solution is contained in (9), (11) together with the equation $R = H$. It will be noticed that for each argument N' there is but one unknown coefficient s in the form in which the solution is represented above. The unknown coefficients s and the constant H may be found in terms of the c_i by substituting the expressions (9) for p_i in the development of R, and writing down the conditions that $R - H$ is identically zero.

(vi.) The properties of the solution are now obtained very easily.

(*a*) As the reversion of the series (11) gives the q_i expressed in terms of the angles λ_i or $l_i + b_i t$, we have

$$b_i = \frac{\partial H}{\partial c_i} = -\frac{\partial T_0}{\partial c_i}, \tag{12}$$

by (2). And therefore, as H is expressible in terms of the c_i and the masses only,

$$\frac{\partial b_i}{\partial c_{i'}} = \frac{\partial b_{i'}}{\partial c_i}.$$

(*b*) Again, H is of the same order as T, that is,

$$\left[H\right] = \left[\text{mass}\right] \times \left[\text{length}\right]^2 \times \left[\text{time}\right]^{-2},$$

and

$$\left[c_i\right] = \left[\text{mass}\right] \times \left[\text{length}\right]^2 \times \left[\text{time}\right]^{-1}.$$

Also, making abstraction of the Gaussian constant,

$$\left[\text{mass}\right] = \left[\text{length}\right]^3 \times \left[\text{time}\right]^{-2}.$$

From these we easily find

$$\left[H\right] = \left[\text{mass}\right]^5 \left[c_i\right]^{-2}.$$

Thence, H being a homogeneous function of the c_i,

$$\Sigma_i c_i \frac{\partial H}{\partial c_i} = -2H;$$

and therefore, by (12) and (2),

$$\Sigma_i c_i b_i = -2H = 2T_0. \tag{13}$$

(*c*) In this last equation, let the c_i receive arbitrary independent variations δ; we find

$$\Sigma_i (c_i \delta b_i + b_i \delta c_i) = -2\delta H,$$

which, by (12), becomes

$$\Sigma_i (c_i \delta b_i) = -3\delta H.$$

Considering now that H is expressed as a function of the b_i and the masses only, we obtain

$$c_i = -3\frac{\partial H}{\partial b_i} = \frac{\partial}{\partial b_i}(T_0 + F_0), \tag{14}$$

and
$$\frac{\partial c_i}{\partial b_{i'}} = \frac{\partial c_{i'}}{\partial b_i}.$$

(*d*) Again, since the c_i and p_i are of the same order, and R and H are homogeneous and of the same order with respect to the c_i or p_i, we have

$$\Sigma_i p_i \dot{q}_i = \Sigma p_i \frac{\partial R}{\partial p_i} = -2R = -2H = \Sigma_i c_i b_i.$$

(*e*) In equations (11), s is of the same order as the c_i, and therefore

$$\Sigma_i c_i l_i = -t\Sigma_i c_i \frac{\partial H}{\partial c_i} + \Sigma_i c_i q_i + \Sigma\Sigma_i c_i \frac{\partial s}{\partial c_i} \sin N'$$

$$= -2Ht + \Sigma_i c_i q_i + \Sigma s \sin N'.$$

Hence, from (10),

$$S = \Sigma_i c_i l_i - 3Ht. \tag{15}$$

(vii.) It is now easy to obtain a proof that

$$\Sigma j_i = 0 \quad \text{or} \quad 1,$$

in all the arguments of the periodic terms which are present in the expressions for p_i, q_i, and the rectangular coordinates.

For this purpose, we first recall [§ (iii.)] that in every periodic term of the development of R in terms of p_i, q_i we have

$$\Sigma_i j_i = 0,$$

and therefore
$$\Sigma_i \frac{\partial R}{\partial q_i} = 0.$$

Hence, by (4),
$$\Sigma_i \dot{p}_i = 0,$$

giving
$$\Sigma_i p_i = \text{const.}$$

Comparing this result with (9), we see that

$$\Sigma_i p_i = \Sigma_i c_i, \quad \Sigma_i j_i = 0,$$

the latter holding for every periodic term in (9). The second result must also be true for each periodic term in (10) and therefore in (11). On reversing the last-mentioned series and substituting the values of the q_i in (9), this property will still hold. Hence, when p_i, q_i have been expressed in terms of the time, they will be of the form (6) and in every argument N we have

$$\Sigma_i j_i = 0.$$

Again, in elliptic motion, x, y, z, if referred to arbitrary fixed axes such that the inclination of the orbit to the plane of (xy) is small, can be put into the forms

$$x = \Sigma k' \cos N', \quad y = \Sigma k' \sin N', \quad z = \Sigma k'_1 \sin N',$$

where k', k'_1 are functions of p_1, p_2, p_3, and

$$N' = j_1 q_1 + j_2 q_2 + j_3 q_3,$$

$$j_1 + j_2 + j_3 = 1, \text{ in } x, y,$$

$$j_1 + j_2 + j_3 = 0, \text{ in } z,$$

the angles N' being all different. Similar results evidently hold for the solar coordinates. Since $\Sigma_i j_i = 0$ in all the arguments of the expressions for p_i, q_i in terms of the time, we see that, when the latter are substituted in the elliptic expressions for the coordinates, we shall have

$$\left.\begin{aligned} x &= \Sigma k \cos N, & x' &= \Sigma K \cos N \\ y &= \Sigma k \sin N, & y' &= \Sigma K \sin N \\ z &= \Sigma k_1 \sin N, & z' &= \Sigma K_1 \sin N \end{aligned}\right\}, \tag{16}$$

where $N = \Sigma_i j_i \lambda_i, \quad \lambda_i = l_i + b_i t,$

$$\Sigma_i j_i = 1 \quad \text{in} \quad x,\ y,\ x',\ y',$$

$$\Sigma_i j_i = 0 \quad \text{in} \quad z,\ z',$$

and k, k_1, K, K_1, b_i are functions of the c_i.

(viii.) It is now necessary to obtain expressions for deducing the c_i directly from the equations (16), one object in view being the disturbed problem of three bodies.

We have, from (14),

$$c_i = \frac{\partial}{\partial b_i}(T_0 + F_0)$$

$$= \text{const. term in } \frac{\partial T}{\partial b_i} + \frac{\partial F}{\partial b_i},$$

where T, F are expressed in terms of the time as sums of periodic terms. Remembering the composition of T, F, we therefore have

$$c_i = \text{const. term in } \left[\mu\left(\dot{x}\frac{\partial \dot{x}}{\partial b_i} + \dot{y}\frac{\partial \dot{y}}{\partial b_i} + \dot{z}\frac{\partial \dot{z}}{\partial b_i}\right) + \mu'\left(\dot{x}'\frac{\partial \dot{x}'}{\partial b_i} + \ldots + \ldots\right)\right.$$

$$\left. + \frac{\partial F}{\partial x}\frac{\partial x}{\partial b_i} + \ldots + \ldots + \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial b_i} + \ldots + \ldots\right]$$

$$= \quad \text{,,} \quad \text{,,} \quad \left[\mu\dot{x}\frac{d}{dt}\frac{\partial x}{\partial b_i} + \ldots + \mu\ddot{x}\frac{\partial x}{\partial b_i} + \ldots\right]$$

$$= \quad \text{,,} \quad \text{,,} \quad \frac{d}{dt}\left[\mu\dot{x}\frac{\partial x}{\partial b_i} + \ldots + \ldots + \mu'\dot{x}'\frac{\partial x'}{\partial b_i} + \ldots + \ldots\right].$$

Hence c_i is equal to the non-periodic part of the coefficient of t in

$$\mu\left(\dot{x}\frac{\partial x}{\partial b_i} + \dot{y}\frac{\partial y}{\partial b_i} + \dot{z}\frac{\partial z}{\partial b_i}\right) + \mu'\left(\dot{x}'\frac{\partial x'}{\partial b_i} + \dot{y}'\frac{\partial y'}{\partial b_i} + \dot{z}'\frac{\partial z'}{\partial b_i}\right).$$

But, from (16), we have, if

$$B = j_1 b_1 + \ldots + j_6 b_6,$$

$$\dot{x} = -\Sigma kB\sin N, \qquad \frac{\partial x}{\partial b_i} = \Sigma\frac{\partial k}{\partial b_i}\cos N - t\Sigma kj_i\sin N,$$

and therefore, since the angles N are all different, the part of c_i arising from the product of these is $\frac{1}{2}\mu\Sigma k^2 Bj_i$. Similar results being found for the other coordinates, we finally obtain

$$c_i = \mu\left[\Sigma k^2 Bj_i + \tfrac{1}{2}\Sigma k_1^2 Bj_i\right] + \mu'\left[\Sigma K^2 Bj_i + \tfrac{1}{2}\Sigma K_1^2 Bj_i\right]. \qquad (17)$$

A verification of this result is easily obtained. For, from the definition of B,

$$\Sigma_i c_i b_i = \mu\left[\Sigma k^2 B^2 + \tfrac{1}{2}\Sigma k_1^2 B^2\right] + \mu'\left[\Sigma K^2 B^2 + \tfrac{1}{2}\Sigma K^2 B^2\right]$$

$$= 2T_0 = -2H,$$

which is the equation (13).

(ix.) In Delaunay's method, the values of the c_i are deduced from the expressions of p_i, q_i in terms of the time.

Let arbitrary independent variations δ be given to the l_i, and let δq_i be the corresponding variation of q_i. We have

$$\Sigma_i \, p_i \delta q_i = \Sigma_i \frac{\partial S}{\partial q_i} \delta q_i = \delta S = \Sigma_i c_i \delta l_i,$$

by (15). Inserting the values of p_i, q_i from (6), and considering only the constant part of the coefficient of δl_i, we find

$$c_i = (P)_i + \Sigma P_i Q_i j_i.$$

This result enables us to obtain the c_i when the values of p_i, q_i have been found by any other method.

(x.) The constants in equations (1) may now be shown to be $-2\Sigma_i c_i$, 0, 0, respectively.

Since $\Sigma_i j_i = 1$ in the expressions (16) for x, y, x', y', and $\Sigma_i j_i = 0$ in those for z, z', it is evident that none of the terms $z\dot{y}$, $x\dot{z}$, &c., in the second and third of equations (1) can give rise to a constant term; the right-hand members of these equations must therefore be zero.

Substituting the values (16) in the first of equations (1), we have, since the result must be a constant,

$$\mu\,(y\dot{x} - x\dot{y}) + \mu'\,(y'\dot{x}' - x'\dot{y}') = -2\mu\Sigma k^2 B - 2\mu'\Sigma K^2 B.$$

But, from (17), since $\Sigma_i j_i = 1$ in the first and third terms, and $= 0$ in the second and fourth terms, we obtain

$$\Sigma c_i = \mu\Sigma k^2 B + \mu'\Sigma K^2 B.$$

Comparing this result with the previous equation, we see that the right-hand member, and therefore the "constant of areas," is $-2\Sigma_i c_i$.

(xi.) In order to complete the theory, a few further steps are necessary to find the perturbations produced by a fourth body.

The property of the system of constants introduced by Jacobi's method is well known. If a disturbing body be present, producing a new disturbing function R'_p, the corresponding variations of the arbitraries l_i, c_i are given by

$$\frac{dc_i}{dt} = \frac{\partial R'_p}{\partial l_i}, \quad \frac{dl_i}{dt} = -\frac{\partial R'_p}{\partial c_i};$$

or, if we change from l_i to $\lambda_i = l_i + b_i t$, and put

$$R_p = R'_p - H,$$

by

$$\frac{dc_i}{dt} = \frac{\partial R_p}{\partial \lambda_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial R_p}{\partial c_i}.$$

Professor Newcomb, by considering the orders of the various classes of terms, has shown ("Action of the Planets on the Moon," § 11), that the perturbations of the Sun produced by the Moon are so small that they may be neglected when we are considering the action of the Sun on the Moon. Consequently, we may solve the equations (4) for $i = 1, 2, 3$ (that is, the equations which refer more particularly to the Moon) with constant values for the solar elliptic elements. The values of c_1, c_2, c_3, in terms of a, e, γ, n', e', may therefore be taken to be those given by Delaunay's theory. This amounts to neglecting quantities of the order m^2a^2/a'^2, where m is the ratio of the mean motions, and a/a' that of the mean distances.

The same factor occurs when we are considering those parts of the perturbations produced by R'_p in $c_1, c_2, c_3, l_1, l_2, l_3$, which arise from the variable parts of $c_4, c_5, c_6, l_4, l_5, l_6$, supposed substituted in $\partial R/\partial c_1$, &c. But these variable parts are those which produce what is known as the indirect action of a planet on the Moon. Hence for the indirect action we have, to the first orders of the disturbance,

$$\delta c_1 = 0, \quad \delta c_2 = 0, \quad \delta c_3 = 0; \tag{18}$$

$$\delta l_4 = 0, \quad \delta l_2 = 0, \quad \delta l_3 = 0.$$

If, then, the coordinates of the Moon be expressed in terms of the six arbitraries c_i, l_i, and the time, the indirect action of a planet, to the first order of the disturbance, may be obtained by simply inserting the variable values of $c_4, c_5, c_6, l_4, l_5, l_6$. The application of this result to the calculation of the secular acceleration is immediate. It will be found in § 12 of Professor Newcomb's paper just referred to. We suppose c_1, c_2, c_3 expressed in terms of n, e^2, γ^2, n', e'^2, and, considering $\delta e'^2$ as known, find the values of δn, δe^2, $\delta \gamma^2$, from the equations (18). The results substituted in

$$\int \delta n\, dt, \quad \int \delta \pi_1\, dt, \quad \int \delta \theta_1\, dt,$$

where π_1, θ_1 are the mean motions of the perigee and node, will give the required secular accelerations.

On certain Properties of the Mean Motions and the Secular Accelerations of the principal Arguments used in the Lunar Theory. By Ernest W. Brown. Read November 12th, 1896. Received October 25th, 1896.

(i.) In an earlier communication to the Society* I have shown how Delaunay's lunar theory may be exhibited in a sufficiently simple manner by means of Jacobi's dynamical method. This plan of treatment, namely, the discovery by an indirect process of a solution of the partial differential equation satisfied by the principal function, I have also applied, in the preceding paper, to the general problem of three bodies, to prove the theorems obtained by Newcomb† with reference to the properties of a special series of canonical constants, and to add one or two results of minor importance.

In this paper I apply the same method: firstly, to show how we may pass directly from the elements of the ellipse to the final system of constants used by Delaunay after all his operations have been completed. Secondly, it is shown how the constant part of the parallax is connected with the motions of the perigee and the node: the equation found here is probably the most general form of this connexion. Certain previous results should be mentioned. In a "Note on a remarkable Property of the Analytical Expression for the Constant Term in the Reciprocal of the Moon's Radius Vector,"‡ Adams has proved that, if we neglect the square of the solar parallax, all the terms in the constant part of the lunar parallax which have the factors e^2, γ^2 are identically zero, and further that the coefficients of e^4, $e^2\gamma^2$, γ^4 are connected by two ratios to the four coefficients of e^2, γ^2 in the expressions which give the mean motions of the perigee and the node. I have given a more definite form to these ratios, by obtaining them in the form of four relations instead of two;§ the new factors introduced being certain functions of the coefficients of the periodic terms. All these results will now be shown to be simple deductions from a single equation numbered (13) below.

Thirdly, these properties are used in connexion with the remarkable theorem obtained by Newcomb in the latter of his two papers just referred to,* by which the secular accelerations are obtainable immediately from the final values of Delaunay's constants L, G, H. In particular, I shall show that the principal part of the acceleration of the mean motion (that independent of e, γ, a/a') is obtainable directly from the corresponding portion of the constant term in the expression for the lunar parallax; and that, when this is found, the principal parts of the accelerations of the perigee and the node may be deduced from the expressions for their mean motions.

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* *Proceedings* May, 1896, pp. 385–390.

† "General Integrals of Planetary Motion," *Smithsonian Contributions*, Vol. xxi.; "Action of the Planets on the Moon," *Amer. Eph. Papers*, Vol. v.

‡ *Monthly Notices R.A.S.*, Vol. xxxviii.

§ "Investigations in the Lunar Theory," *Amer. Jour. Math.*, Vol. xvii.

The basis which furnishes these results is a consideration of the constant parts of the various functions which naturally arise in the solution obtained by varying the arbitrary constants. In all problems of celestial mechanics there are certain arbitrary constants which must be determined from observation; and there are certain others which may also be accurately found by observation, but which depend on those previously found. The comparison of the two sets of values forms so important a test of the sufficiency of the theory that no means which will give tests of the accuracy of the theoretical calculations, or which will enable us to obtain the second class of constants without serious risk of error, will be unimportant from a practical point of view, apart from any theoretical interest which they may possess.

(ii.) The general canonical system of equations used by Delaunay is

$$\frac{dL}{dt} = \frac{\partial R_1}{\partial l}, \quad \frac{dG}{dt} = \frac{\partial R_1}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R_1}{\partial h};$$

$$\frac{dl}{dt} = -\frac{\partial R_1}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R_1}{\partial g}, \quad \frac{dh}{dt} = -\frac{\partial R_1}{\partial H};$$

where

$$R_1 = \frac{(E+M)^2}{2L^2}$$
$$+m'\left\{\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{xx'+yy'+zz'}{r'^3} - \frac{1}{r'}\right\}.$$

Here x, y, z, x', y', z' are the coordinates of the Moon and Sun referred to fixed axes through the Earth; E, M, m' are the masses of the Earth, Moon, and Sun; L, G, H, l, g, h are the well-known functions of the elliptic elements given by

$$L = a^{\frac{1}{2}}(E+M)^{\frac{1}{2}}, \quad G = L(1-e^2)^{\frac{1}{2}}, \quad H = G\cos i = G(1-2\gamma^2);$$

$l = nt+\epsilon-\pi =$ mean anomaly,

$g = \theta-\pi =$ distance from node to perigee,

$h = \theta =$ longitude of node.

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* Also proved in § (xi.) of the preceding paper.

The change necessary in order that x', y', z' may be the coordinates of the Sun referred to the centre of mass of the Earth and Moon is known to be sufficiently accounted for if we replace a/a' by $(E-M)\,a/(E+M)\,a'$ in the final results.

It will be advantageous to introduce a change of variables. Let

$$p_1 = L, \qquad p_2 = G-L, \quad p_3 = H-G;$$
$$q_1 = l+g+h, \quad q_2 = g+h, \quad q_3 = h.$$

The equations satisfied by the new variables will still be canonical, and, if we put

$$R = -R_1,$$

they will become

$$\frac{dp_i}{dt} = -\frac{\partial R}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial R}{\partial p_i}, \qquad (i = 1, 2, 3). \tag{1}$$

The new variables, while possessing all the special properties of the old ones, have this further advantage, namely, that p_2, p_3 are small quantities of the orders e^2, γ^2, respectively; also, q_1, q_2, q_3 are the longitudes of the Moon, its perigee, and node, the non-periodic parts of them being the mean values of these angles.

(iii.) The solution of the equations (1) will be carried out in a manner similar to that which has been used in the previous paper; but, as the time now occurs explicitly in R through its presence in the elliptic values of x', y', z', one or two modifications are necessary. We suppose, with Delaunay, that $z' = 0$, and that x', y' have their elliptic values.

The equations satisfied by the principal function are

$$p_i = \frac{\partial S}{\partial q_i}, \quad \frac{\partial S}{\partial t} = -R, \tag{2}$$

and it may be shown as before that p_1, p_2, p_3 are expressible by series of the form

$$p_i = c_i + \Sigma s j_i \cos N', \tag{3}$$

c_1, c_2, c_3 being arbitrary constants, the coefficients s depending only on c_1, c_2, c_3, n', e', and N' being an angle of the form

$$N' = j_1 q_1 + j_2 q_2 + j_3 q_3 + j' n' t + \alpha,$$

where α is an absolute constant and j_1, j_2, j_3, j' are positive or negative integers or zero.

As

$$\frac{\partial p_i}{\partial t} = \frac{\partial^2 S}{\partial q_i \partial t} = -\frac{\partial R}{\partial q_i},$$

all the quantities being expressed in terms of q_i, c_i, t, we have

$$R = -B - \Sigma s j' n' \cos N', \tag{4}$$

where B is a constant of the same nature as s. A solution of the partial differential equation for S, involving the q_i, the c_i, and the time, will therefore be

$$S = c_1 q_1 + c_2 q_2 + c_3 q_3 + Bt + \Sigma s \sin N'. \tag{5}$$

The remaining integrals are given by

$$l_i = \frac{\partial S}{\partial c_i} = q_i + t\frac{\partial B}{\partial c_i} + \Sigma \frac{\partial s}{\partial c_i} \sin N', \tag{6}$$

l_1, l_2, l_3 being three new arbitrary constants.

(iv.) The relation of the foregoing solution to that of Delaunay is easily seen. The three expressions

$$l_i - t\frac{\partial B}{\partial c_i}$$

are the non-periodic parts of the variables q_1, q_2, q_3, and therefore

$$-\frac{\partial B}{\partial c_1}, \quad -\frac{\partial B}{\partial c_2}, \quad -\frac{\partial B}{\partial c_3}$$

are the mean motions of the Moon, its perigee, and node respectively. The symbol B is the same as that used by Delaunay,[*] and it represents the constant remainder of his disturbing function after all the periodic terms have been eliminated. We therefore see that c_1, $c_1 + c_2$, $c_1 + c_2 + c_3$, B are the same as Delaunay's L, G, H, R [†] after the completion of the operations. We have therefore shown that the same quantities are the constant parts of Delaunay's initial variables L, G, H, and of R, when the latter are expressed in terms of the c_i, the q_i, the solar constants, and the time.

(v.) The solution contained in (iii.) gives a method of performing any one of Delaunay's operations which may possibly be useful in certain cases. Assume the values (3) of the p_i: to each periodic term there is one unknown coefficient s. We substitute these values of the p_i in the portion of R under consideration, and equate the result to the expression of R in (4). The equations of condition thus furnished will give B and all the coefficients s in terms of c_1, c_2, c_3, n', e'^2.

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[*] *Théorie de la Lune*, Vol. II., p. 234.

[†] Delaunay's R is the R_1 of this paper.

With these results we enter equations (6), and a reversion of series will give the values of q_1, q_2, q_3 in terms of the constants and the time. Substituting in (3), we then obtain p_1, p_2, p_3 in terms of the time. The six new canonical variables for the next operation are the c_i and $l_i - t\,\partial B/\partial c_i$, and the passage to it is made in the manner shown in my paper on Delaunay's method already referred to.

It is not necessary to consider only one periodic term of the disturbing function: we may take two or more at any operation, if any advantage is to be gained thereby. The method might possibly be of some advantage, if the determination of the mean motions of the perigee and node was under consideration, since no reversion of series is necessary in the latter operations. If we decide to take one periodic term at a time, the calculation of such functions as p_1^{-2}, p_1^{4}, $(p_2/p_1)^{\frac{1}{2}}$, with indeterminate values of the coefficients of the periodic series, might be done once for all, so that the coefficients s could be determined by mere numerical substitutions. The reversion of the series, necessary in order to obtain the q_i in terms of the time, could be treated in a similar way. A somewhat extended series of calculations would be necessary to see if any real advantage over Delaunay's method of procedure is gained. It may be mentioned that the introduction of e, γ at each stage will not be necessary, since c_2, c_3 are respectively of the orders e^2, γ^2.

(vi.) I now proceed to prove the property connecting the mean motions with the constant term of the parallax.

Representing by b_1, b_2, b_3 the coefficients of t in the non-periodic parts of q_1, q_2, q_3, that is, putting

$$b_i = -\frac{\partial B}{\partial c_i}, \quad (i = 1, 2, 3), \tag{7}$$

it will be shown that

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + B = \tfrac{3}{2}\left(\frac{E+M}{r}\right)_0,$$

where in all cases $(Q)_0$ represents the non-periodic part of any function Q expressed as a sum of sines or cosines in terms of the time and the constants.

First, since S is supposed to be a function of q_1, q_2, q_3, t, we have

$$\begin{aligned}\frac{dS}{dt} &= \Sigma_i \dot{q}_i \frac{\partial S}{\partial q_i} + \frac{\partial S}{\partial t} \\ &= \Sigma_i p_i \dot{q}_i - R \quad \text{by} \quad (2).\end{aligned}$$

Hence, remembering the notation just explained,

$$\left(\frac{dS}{dt}\right)_0 = (\Sigma_i p_i \dot{q}_i)_0 - (R)_0.$$

The left-hand member of this equation is the coefficient of t in the non-periodic part of S, when the latter has been expressed in terms of the time and the arbitraries. If we look at equation (5), and suppose the values of q_1, q_2, q_3 in terms of the time to be substituted therein, it is evident that the periodic terms $\Sigma s \sin N'$ will only produce periodic terms. The non-periodic part of q_i is $l_i + b_i t$, and therefore the non-periodic part of the coefficient of t in S is

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + B.$$

Hence $$(\Sigma_i p_i \dot{q}_i)_0 = b_1 c_1 + b_2 c_2 + b_3 c_3 + B + (R)_0. \tag{8}$$

Secondly, denote the ordinary disturbing function by Ω, that is, let

$$\Omega = m' \left\{ \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{xx'+yy'+zz'}{r'^3} - \frac{1}{r'} \right\};$$

so that, if $E+M = \kappa$,

$$R = -\Omega - \frac{\kappa^2}{2p_1^2}; \tag{9}$$

and the equations of motion are

$$\ddot{x} = \frac{\partial}{\partial x}\left(\frac{\kappa}{r} + \Omega\right), \quad \ddot{y} = \frac{\partial}{\partial y}\left(\frac{\kappa}{r} + \Omega\right), \quad \ddot{z} = \frac{\partial}{\partial z}\left(\frac{\kappa}{r} + \Omega\right).$$

We shall now neglect the ratio of the parallax of the Moon to that of the Sun. As the results obtained below refer only to the constant parts of the various functions, and as this ratio is well known to occur in these constant parts to an order not less than m^2a^2/a'^2, the error committed will be almost negligible. It amounts to about $2''$ in the annual motion of the perigee (that is to say, a quantity of the same order as the difference between the observed value and Hansen's theoretical value), and is less than $0''\cdot001$ in the secular accelerations per century. The accuracy of the results obtained for the latter quantities will therefore not be impaired by this limitation.

We have, on expanding Ω and neglecting the ratio of the parallaxes

$$\Omega = \frac{m'}{r'^3} \left\{ \frac{3}{2} \frac{(xx'+yy'+zz')^2}{r'^2} - \tfrac{1}{2} r^2 \right\},$$

and therefore $$x\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} + z\frac{\partial \Omega}{\partial z} = 2\Omega.$$

Multiply the equations of motion by x, y, z, respectively, and add. Using the last equation, the result may be written

$$\tfrac{1}{2}\frac{d^2}{dt^2}(r^2)-2T=-\frac{\kappa}{r}+2\Omega,$$

where $$2T=\dot{x}^2+\dot{y}^2+\dot{z}^2.$$

Since the first term cannot give rise to a constant portion, this equation gives

$$2\,(T)_0=\left(\frac{\kappa}{r}\right)_0-2\,(\Omega)_0. \tag{9'}$$

Again, from the theory of elliptic motion,

$$T=\frac{\kappa}{r}-\frac{\kappa^2}{2p_1^2}$$

and therefore, by (9),

$$T-R=\frac{\kappa}{r}+\Omega.$$

Taking the constant parts of this equation, and eliminating $(T)_0$ by means of the previous value obtained for it, we get

$$(R)_0+2\,(\Omega)_0=-\tfrac{1}{2}\left(\frac{\kappa}{r}\right)_0 \tag{10}.$$

Thirdly, the p_i are of the order $[\text{time}]^{-1}\times[\text{length}]^2$, that is, of the order $[\text{mass}]^{\frac{1}{2}}\,[\text{length}]^{\frac{1}{2}}$. As the coefficients of the elliptic values of x, y, z are expressible by means of p_1, p_2, p_3 only, and are homogeneous with respect to these quantities, x, y, z must be of the order $[p]^2$, abstraction being made of the masses. Hence Ω is of the order $[p]^4$. Euler's theorem of homogeneous functions then gives

$$p_1\frac{\partial\Omega}{\partial p_1}+p_2\frac{\partial\Omega}{\partial p_2}+p_3\frac{\partial\Omega}{\partial p_3}=4\Omega.$$

Using this result with (9), we find

$$\begin{aligned}\Sigma_i p_i\dot{q}_i=\Sigma_i p_i\frac{\partial R}{\partial p_i}&=\frac{\kappa^2}{p_1^2}-4\Omega\\ &=-6\Omega-2R.\end{aligned}$$

Taking the non-periodic terms in this equation and combining with (10), we obtain

$$(\Sigma_i p_i\dot{q}_i)_0=\tfrac{3}{2}\left(\frac{\kappa}{r}\right)_0+(R)_0,$$

whence, by (8), $$b_1c_1+b_2c_2+b_3c_3+B=\tfrac{3}{2}\left(\frac{\kappa}{r}\right)_0, \tag{11}$$

the required result.

(vii.) Before proceeding to the special applications of this equation, one or two general properties may be noted.

(α) Let arbitrary variations δc_i be given to the c_i. We have

$$\Sigma_i\,(b_i\delta c_i + c_i\delta b_i) + \delta B = \tfrac{3}{2}\delta\left(\frac{\kappa}{r}\right)_0.$$

But $b_i = -\partial B/\partial c_i$, and therefore, if we suppose $(\kappa/r)_0$ to be expressed in terms of b_1, b_2, b_3, and the solar constants,

$$c_i = \tfrac{3}{2}\frac{\partial}{\partial b_i}\left(\frac{\kappa}{r}\right)_0. \qquad (12)$$

[The result corresponding to this in the general problem of three bodies is given by the equations*

$$c_i = -3\frac{\partial H}{\partial b_i} = \tfrac{3}{2}\frac{\partial}{\partial b_i}(F)_0.]$$

(β) The equations (12), (7) show that

$$\frac{\partial c_i}{\partial b_{i'}} = \frac{\partial c_{i'}}{\partial b_i}, \quad \frac{\partial b_i}{\partial c_{i'}} = \frac{\partial b_{i'}}{\partial c_i}.$$

(γ) A formula for obtaining the c_i from the expressions for the rectangular coordinates when the latter are known can be deduced from this result.† We have

$$
\begin{aligned}
c_i &= \tfrac{3}{2}\frac{\partial}{\partial b^i}\left(\frac{\kappa}{r}\right)_0 \\
&= \frac{\partial}{\partial b_i}\left(T+\Omega+\frac{\kappa}{r}\right)_0, \quad \text{by} \quad (9'), \\
&= \text{non-periodic part of } \frac{\partial T}{\partial b_i} + \left(\frac{\partial x}{\partial b_i}\frac{\partial}{\partial x} + \frac{\partial y}{\partial b_i}\frac{\partial}{\partial y} + \frac{\partial z}{\partial b_i}\frac{\partial}{\partial z}\right)\left(\frac{\kappa}{r}+\Omega\right) \\
&= \quad ,, \qquad ,, \qquad \left(\dot{x}\frac{\partial \dot{x}}{\partial b_i} + \ldots + \ldots + \ddot{x}\frac{\partial x}{\partial bi} + \ldots + \ldots\right) \\
&= \quad ,, \qquad ,, \qquad \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial b_i} + \dot{y}\frac{\partial y}{\partial b_i} + \dot{z}\frac{\partial z}{\partial b_i}\right).
\end{aligned}
$$

Assuming

$$x = \Sigma k\cos N, \quad y = \Sigma k\sin N, \quad z = \Sigma k'\sin N,$$

where N is defined as before, we obtain from this result

$$c_i = \Sigma k^2\,(j_1b_1+j_2b_2+j_3b_3+j'n')\,j_i + \tfrac{1}{2}\Sigma k'^2\,(j_1b_1+j_2b_2+j_3b_3+j'n')\,j_i,$$

---

* See § (vi.) (c) of my paper "On the Application of Jacobi's Dynamical Method to the Problem of Three Bodies."

† *Ibid.*, § (viii.).

a result similar to that found by Newcomb (*Action of the Planets on Moon*, § 8) in the general problem.

(δ) We can also show,* as in the general problem, that, if p_i, q_i be expressed in terms of the time by series of the forms

$$p_i = (P)_i + \Sigma P_i \cos N,$$

$$q_i = l_i + b_i t + \Sigma Q_i \sin N,$$

the c_i may be found, if necessary, from the equations

$$c_i = (P)_i + \Sigma P_i Q_i j_i.$$

We suppose the l_i to receive variations δl_i, and we then have

$$\Sigma p_i \delta q_i = \delta S.$$

The constant part of the coefficient of δl_i in δS is easily seen, as before, to be c_i, and the result follows immediately.

(viii.) *Connexion between* $(\kappa/r)_0$ *and the Mean Motion of the Perigee and the Node. Adams' Theorems.*—Replace b_1, b_2, b_3, which are respectively the mean motions of the Moon, its perigee, and its node, by the letters which usually represent these quantities, namely, n, π_1, θ_1. The equation (11) is therefore written

$$nc_1 + \pi_1 c_2 + \theta_1 c_3 + B = \tfrac{3}{2}\left(\frac{\kappa}{r}\right)_0; \tag{13}$$

and the equations (7) become

$$n = -\frac{\partial B}{\partial c_1}, \quad \pi_1 = -\frac{\partial B}{\partial c_2}, \quad \theta_1 = -\frac{\partial B}{\partial c_3}. \tag{14}$$

We now consider all the quantities in these equations to be expressed as functions of n, e^2, γ^2, n', e'^2, κ, where e, γ are the eccentricity and the sine of half the inclination, defined in Delaunay's manner. These definitions are used for the sake of clearness, although it will be evident that any of the various possible modes of defining them are available. The definitions of n, π_1, θ_1 are fixed. It may be remarked, however, that the expression $b_1 c_1 + b_2 c_2 + b_3 c_3$ is an invariant for any linear tangential transformation of the variables p_i, q_i.

Differentiate (13) partially with regard to n, e^2, γ^2, successively. Since

$$\frac{\partial B}{\partial n} = \frac{\partial B}{\partial c_1}\frac{\partial c_1}{\partial n} + \frac{\partial B}{\partial c_2}\frac{\partial c_2}{\partial n} + \frac{\partial B}{\partial c_3}\frac{\partial c_3}{\partial n}$$

$$= -n\frac{\partial c_1}{\partial n} - \pi_1\frac{\partial c_2}{\partial n} - \theta_1\frac{\partial c_3}{\partial n}, \quad \text{\&c.,}$$

---

* *Ibid.*, § (ix.).

we obtain
$$\left.\begin{aligned} c_1+c_2\frac{\partial\pi_1}{\partial n}+c_3\frac{\partial\theta_1}{\partial n} &= \tfrac{3}{2}\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0, \\ c_2\frac{\partial\pi_1}{\partial e^2}+c_3\frac{\partial\theta_1}{\partial e^2} &= \tfrac{3}{2}\frac{\partial}{\partial e^2}\left(\frac{\kappa}{r}\right)_0, \\ c_2\frac{\partial\pi_1}{\partial \gamma^2}+c_3\frac{\partial\theta_1}{\partial \gamma^2} &= \tfrac{3}{2}\frac{\partial}{\partial \gamma^2}\left(\frac{\kappa}{r}\right)_0 \end{aligned}\right\}, \qquad (15)$$

where $\partial\pi_1/\partial n$, $\partial\pi_1/\partial e^2$, &c., evidently denote the partial differentials of π_1, θ_1 with respect to n, e^2, γ^2, when expressed in terms of n, e^2, γ^2, n', e'^2. It will be noticed that these three equations have the advantage of not containing B.

By means of these forms of (13) we can deduce Adams' theorems [see reference given in § (i.)].

Define a by means of the equation $\kappa = n^2a^3$, and write

$$\left(\frac{a}{r}\right)_0 = \mathrm{A}+\mathrm{B}e^2+\mathrm{C}\gamma^2+\mathrm{E}e^4+2\mathrm{F}e^2\gamma^2+\mathrm{G}\gamma^4+\ldots,$$
$$\pi_1 = n\,(\mathrm{P}+\mathrm{H}e^2+\mathrm{K}\gamma^2+\ldots),$$
$$\theta_1 = n\,(\mathrm{T}+\mathrm{M}e^2+\mathrm{N}\gamma^2+\ldots),$$

where A, B, ... P, H, ... T, M, ... are functions of m, e'^2 only. Substitute these results in the second and third of equations (15). These being identities, we can equate to zero the coefficients of the various powers of e^2, γ^2.

Equate the coefficients of e^2, γ^2 to zero. Remembering that c_2, c_3 respectively contain the factors e^2, γ^2 in all their terms, we find

$$\mathrm{B}=0, \quad \mathrm{C}=0.$$

This is Adams' first theorem.

Equate the coefficients of e^4, $e^2\gamma^2$, γ^4 to zero. We find

$$\left.\begin{aligned} c_2\mathrm{H} &= \tfrac{3}{2}na^2\mathrm{E}e^2, & c_3\mathrm{M} &= \tfrac{3}{2}na^2\mathrm{F}\gamma^2, \\ c_2\mathrm{K} &= \tfrac{3}{2}na^2\mathrm{F}e^2, & c_3\mathrm{N} &= \tfrac{3}{2}na^2\mathrm{G}\gamma^2 \end{aligned}\right\}, \qquad (16)$$

c_2/e^2, c_3/γ^2 being limited to terms of the form $na^2f(m, e'^2)$. Hence, eliminating c_2, c_3,

$$\frac{\mathrm{E}}{\mathrm{F}}=\frac{\mathrm{H}}{\mathrm{K}}, \quad \frac{\mathrm{F}}{\mathrm{G}}=\frac{\mathrm{M}}{\mathrm{N}},$$

the second theorem of Adams. The further results which he obtains for the higher powers of e^2, γ^2 follow in like manner.

It is of some interest to compare these results with those obtained by the use of rectangular coordinates.* In the paper referred to, I found†

$$HT_e = 6E, \quad MT_\gamma = 6F,$$

$$KT_e = 6F, \quad NT_\gamma = 6G,$$

where T_e, T_γ were certain functions of m, e'^2, obtained from the coefficients of the periodic terms in the expressions for the moving rectangular coordinates there used. A comparison of the two sets of results gives

$$c_2 = \tfrac{1}{4} na^2 e^2 T_e, \quad c_3 = \tfrac{1}{4} na^2 \gamma^2 T_\gamma.$$

On comparing the values obtained for T_e, T_γ in the memoir referred to, and those found for c_2, c_3 in § (vii.) above, the correctness of these two identities may be easily verified.

One or two further results of the equations (15) may be mentioned. If all powers of e^2, γ^2 be neglected, the first of them gives

$$c_1 = \tfrac{3}{2} \frac{\partial}{\partial n} \left(\frac{\kappa}{r}\right)_0. \tag{17}$$

If we consider the first powers of e^2, γ^2 only, we find from the same equation, combined with Adams' first theorem,

$$\text{coefficient of } e^2 \text{ in } c_1 = -c_2 \frac{\partial \pi_1}{\partial n},$$

$$\text{,,} \qquad \gamma^2 \text{ ,, } c_2 = -c_3 \frac{\partial \theta_1}{\partial n}.$$

When c_1, c_2, c_3, π_1, θ_1 are known in terms of n, e^2, γ^2, n', e'^2, the value of the constant term of the lunar parallax can be obtained to the same degree of accuracy as c_1, so far as its part independent of e, γ is concerned, and to two orders higher than c_2, c_3, so far as the parts depending on e^4, $e^2\gamma^2$, γ^4 are concerned. The integration of equation (17) presents no difficulty, since the form of the expression for the parallax is known, and it contains no term independent of n.

By using the value of the constant term of the parallax obtained by Adams‡ in a different manner, we have a useful verification of Delaunay's results.

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* "Investigations in the Lunar Theory," *Amer. Jour. Math.*, Vol. XVII.

† The meaning given to E, F, G is slightly different from that above.

‡ Given in his paper referred to in § (i.). The values of c_1, c_2, c_3 in terms of n, e^2, γ^2, n', e'^2 are given by Newcomb in § 11 of his memoir on "The Action of the Planets on the Moon"; the values of π_1, θ_1 are found in § 12 of the same paper. The latter were calculated by Delaunay (*Comptes Rendus*, Vol. LXXIV., p. 19), and by Cayley (*Monthly Notices R.A.S.*, Vol. XXXII., pp. 8–16, 74; *Coll. Works*, Vol. VII., pp. 532–534.

(ix.) *The Secular Accelerations.*—Professor Newcomb has shown [see references at the end of § (i.)] that the secular accelerations may be obtained by means of the equations

$$\delta c_1 = 0, \quad \delta c_2 = 0, \quad \delta c_3 = 0.$$

Here c_1, c_2, c_3 are supposed to have been expressed in terms of n, e^2, γ^2, n', e'^2, κ. A variation $\delta e'^2$, due to planetary action, induces variations δn, δe^2, $\delta\gamma^2$, which may be found by means of the above equations. The secular accelerations are then given by

$$\int \delta n\, dt, \quad \int \delta\pi_1\, dt, \quad \int \delta\theta_1\, dt.$$

Applying the variation δ to the three equations (15), we find, since δc_1, δc_2, δc_3 are all zero,

$$\left.\begin{aligned} c_2\delta\frac{\partial\pi_1}{\partial} + c_3\delta\frac{\partial\theta_1}{\partial n} &= \tfrac{3}{2}\delta\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0, \\ c_2\delta\frac{\partial\pi_1}{\partial e^2} + c_3\delta\frac{\partial\theta_1}{\partial e^2} &= \tfrac{3}{2}\delta\frac{\partial}{\partial e^2}\left(\frac{\kappa}{r}\right)_0, \\ c_2\delta\frac{\partial\pi_1}{\partial\gamma^2} + c_3\delta\frac{\partial\theta_1}{\partial\gamma^2} &= \tfrac{3}{2}\delta\frac{\partial}{\partial\gamma^2}\left(\frac{\kappa}{r}\right)_0 \end{aligned}\right\}. \qquad (18)$$

We shall use δe^2, $\delta\gamma^2$, $\delta e'^2$ to denote the variations of e^2, γ^2, e'^2, respectively.

The presence of the factor e^2 in c_2, δe^2 and of γ^2 in c_3, $\delta\gamma^2$ enables us to calculate the principal parts of δn, $\delta\pi_1$, $\delta\theta_1$ without the intervention of c_1, c_2, c_3, when π_1, θ_1, $(\kappa/r)_0$ are known.

(a) *Neglect all powers of* e^2, γ^2. The first of equations (18) gives

$$\delta\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0 = 0,$$

or

$$\delta n = \left\{\frac{\partial^2}{\partial n\,\partial e'^2}\left(\frac{\kappa}{r}\right)_0\right\}\delta e'^2 \div \left\{\frac{\partial^2}{\partial n\,\partial n}\left(\frac{\kappa}{r}\right)_0\right\},$$

giving the part of δn, independent of e^2, γ^2.

When δn has been thus found, the corresponding portions of e^2, γ^2 are given by

$$\delta\pi_1 = \frac{\partial\pi_1}{\partial n}\delta n + \frac{\partial\pi_1}{\partial e'^2}\delta e'^2,$$

$$\delta\theta_1 = \frac{\partial\theta_1}{\partial n}\delta n + \frac{\partial\theta_1}{\partial e'^2}\delta e'^2.$$

(β) *Neglect all powers of* e^2, γ^2 *above the first.* Denote by $(\delta n)_{e^i}$, $(\delta n)_{\gamma^i}$ the parts of δn which contain the factors e^i, γ^i respectively. Using Adams' first theorem, we find from (18)

$$c_2 \left\{ \frac{\partial^2 \pi_1}{\partial n \partial n} (\delta n)_{e^0} + \frac{\partial^2 \pi_1}{\partial n \partial e'^2} \delta e'^2 \right\} = \tfrac{3}{2} (\delta n)_{e^2} \frac{\partial^2}{\partial n \partial n} \left(\frac{\kappa}{r} \right)_0,$$

$$c_3 \left\{ \frac{\partial^2 \theta_1}{\partial n \partial n} (\delta n)_{e^0} + \frac{\partial^2 \theta_1}{\partial n \partial e'^2} \delta e'^2 \right\} = \tfrac{3}{2} (\delta n)_{\gamma^2} \frac{\partial^2}{\partial n \partial n} \left(\frac{\kappa}{r} \right)_0,$$

which give the terms in δn containing the factors e^2, γ^2.

To calculate the parts of $\delta\pi_1$, $\delta\theta_1$ which contain the factors e^2, γ^2 it will probably be simplest and most accurate to find δe^2, $\delta\gamma^2$ from the equations

$$\delta c_2 = 0, \quad \delta c_3 = 0,$$

using the known value of δn. We then substitute in the equations

$$\delta\pi_1 = \frac{\partial \pi_1}{\partial e^2} \delta e^2 + \frac{\partial \pi_1}{\partial \gamma^2} \delta\gamma^2 + \frac{\partial \pi_1}{\partial n} \delta n + \frac{\partial \pi_1}{\partial e'^2} \delta e'^2,$$

$$\delta\theta_1 = \frac{\partial \theta_1}{\partial e^2} \delta e^2 + \frac{\partial \theta_1}{\partial \gamma^2} \delta\gamma^2 + \frac{\partial \theta_1}{\partial n} \delta n + \frac{\partial \theta_1}{\partial e'^2} \delta e'^2.$$

The degree of accuracy obtained will, in general, correspond to that with which $\partial\pi_1/\partial e'^2$ is known. The terms of higher orders are almost insensible.

I hope shortly to publish a verification by this method of some of Delaunay's results for the secular accelerations ;* a preliminary examination of some of the terms exhibits errors which are not unimportant.

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* [*Note added March 29th*, 1897.—See *Monthly Notices*, *R.A.S.*, March, 1897, where complete numerical results are given.]

The Calculus of Equivalent Statements. (*Fifth Paper.*)

By HUGH MACCOLL, B.A. Read November 12th, 1896. Received, in revised form, March 6th, 1897.

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Introduction.

As in the following pages I have made considerable changes in the notation which I employed in my former papers on this subject, I may say a few words in explanation of my reasons.

I use α_β as synonymous with my former symbol $\alpha : \beta$ for the sake of brevity and to avoid the necessity of brackets. Thus, $\alpha_\beta \beta_\gamma : \alpha_\gamma$ is synonymous with $(\alpha : \beta)(\beta : \gamma) : (\alpha : \gamma)$. In dealing with *implications of the second order* (*i.e.*, implications whose elementary constituents are themselves implications) the advantages of the abbreviation are self-evident. The *atoms* (as it were) of two different substances combine into *molecules* of a third substance.

I use (::) as synonymous with (=) chiefly to avoid the necessity of brackets. How this is effected will appear further on.

I use the symbol ϵ instead of my former symbol 1 (unity), and η

instead of *zero*, to express *lasting and necessary truth* and *lasting and necessary falsehood* respectively; while I use θ to denote any statement which may be either true or false, and therefore belongs neither to the class ϵ nor to the class η. Thus $(2+3=5)$ belongs to the class ϵ; $(2+3=5)'$, *i.e.*, the denial of this, and $(2+3=7)$ both belong to the class η; and $(x=2)$ belongs to the class θ. The symbol ϵ' is, of course, equivalent to η, and η' is equivalent to ϵ; for, if any statement α belongs to either class (ϵ or η), its denial α' belongs to the other. These and other class symbols τ, ι, π, &c., will be treated in their due place; as will other symbolic innovations, such as α^β, $\frac{\alpha}{\beta}$, $\alpha>\beta$, $\alpha<\beta$, &c. It will be noticed that I have used the symbols α_β and ϵ in analogous, but not quite identical, senses in my Fourth Paper on this subject in the *Proc. Lond. Math. Soc.* (See Vol. XI., No. 163.)

Definitions.

DEF. 1.—All intelligible statements may be divided into *two classes*—the *true* and the *false.* The small Greek letter τ will denote any statement of the former class, and the small Greek letter ι will denote any statement of the latter.

DEF. 2.—All intelligible statements may also be divided into *three* classes, namely:

(1) *Those necessarily and always true*—these may be called *certainties*, and will be denoted by the symbol ϵ;

(2) *Those necessarily and always false*—these may be called *absurdities*, *impossibilities*, or *inconsistencies*, and will be denoted by the Greek letter η; and

(3) *Those which belong neither to the class* ϵ *nor to the class* η, and may therefore be either true or false—these may be called *variables*, and will be denoted by the Greek letter θ.

The statement $(2+2=4)$ belongs to the class ϵ; the statement $(2\times 3=7)$ belongs to the class η; and $(x=5)$ belongs to the class θ.

DEF. 3.—The symbol α^β asserts that the statement α belongs to the class of statements denoted by β. Thus α^τ asserts that α is *true*; α^ι asserts that α is *false;* α^ϵ asserts that α is *always true;* α^η asserts that α is *always false;* and α^θ asserts that α is *neither always true nor always false.*

But (especially when we are considering statements of the classes τ and ι *only*) the symbol τ is usually left understood, and the symbol ι will be replaced by an acute accent. Thus $\alpha=\alpha^\tau$, and $\alpha'=\alpha^\iota$.

Def. 4.—The symbol $\alpha^x : \beta^y$ asserts that, whenever α belongs to the class x, and so long as α belongs to the class x, then, and so long, will β belong to the class y. Thus, $\alpha^\tau : \beta^\tau$ (or briefly $\alpha : \beta$) asserts that, whenever α is true, β is true; and $\alpha^\tau : \beta^\iota$ (or briefly $\alpha : \beta'$) asserts that, whenever α is true, β is false.*

Def. 5.—The symbol $\alpha^x\beta^y$ makes *two* assertions: it asserts that α belongs to the class x, and that β belongs to the class y. If no indices be expressed, the index τ is understood. Thus $\alpha\beta$ asserts that α and β *both* belong to the class τ; that is, that they are *both true*. Similarly, $\alpha\beta'$ asserts that α is true and β false. By the same convention $\alpha^x\beta^y\gamma^z$ makes *three* statements, and so on.

Def. 6.—The symbol $\alpha^x + \beta^y$ is an *alternative*, and makes only *one* assertion, namely, that *one at least* (and possibly both) of the statements α^x and β^y is true. The symbol $\alpha + \beta'$, on this convention, is, of course, equivalent to $\alpha^\tau + \beta^\iota$.

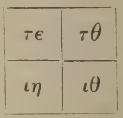
Def. 7.—The symbol $\phi^x(\alpha, \beta)$ asserts that the complex statement denoted by $\phi(\alpha, \beta)$ belongs to the class x. For example, the symbol $(\alpha+\beta)^\theta : \alpha^\theta\beta^{o\epsilon} + \beta^\theta\alpha^{o\epsilon}$ asserts (see Defs. 10, 11) that, when the symbol $\alpha+\beta$ (which is short for $\alpha^\tau + \beta^\tau$) belongs to the class θ (that is, when it denotes a *variable*), then α is a *variable* and β an *uncertainty*; or else β is a *variable* and α an *uncertainty*. (Proved on p. 169.)

Note.—$\beta^{o\epsilon} = (\beta^\epsilon)^\iota$, and $\alpha^{o\epsilon} = (\alpha^\epsilon)^\iota$.

Def. 8.—When any symbol α denotes *one statement only*, it is called a *singular statement*, or simply a *singular*; when it is applicable to *more than one statement*, it is called a *plural* or *class statement*. Thus $(2 \times 3 = 6)$ and $(2 \times 3 = 7)$ are two *singular* statements belonging respectively to the *plurals* or *class statements* ϵ and η.

Def. 9.—The symbol α^β, when α and β are *both* class statements, asserts that *every* statement of the class α belongs also to the class β. Hence we have $\tau^\epsilon = \eta$, $\theta^\epsilon = \eta$, $\theta^\theta = \epsilon$, $\theta^\tau = \eta$, $\theta^\iota = \eta$, $\epsilon^\tau = \epsilon$, $\eta^\iota = \epsilon$. (See Def. 3.)

Note.—The symbol $(\alpha^\beta = x)$ here is understood to mean $(\alpha^\beta)^x$. The accompanying diagram will illustrate the mutual relations of the five class statements τ, ι, ϵ, η, θ. The two upper squares constitute the class τ (or *true* statements); and the two lower squares the class ι (or *false* statements). The two squares to the right constitute the class θ (or *variable* statements); and the two squares to the left constitute



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[*] Observe that $\alpha : \beta$ makes no assertion as to whether β is true when α is not true. (See Def. 15.)

the *non-variables* or *constants* (see next Def.). The ϵ statements are restricted to the square $\tau\epsilon$; and the η statements to the square $\iota\eta$.

DEF. 10.—Statements which do *not* belong to the class ϵ are called *uncertainties;* those which do *not* belong the class η are called *possibilities;* and those which do *not* belong to the class θ are called *constants*, or *non-variables.* Thus $\alpha^{o\epsilon}\beta^{o\eta}\gamma^{o\theta}$, or its equivalent $\alpha^{\epsilon\iota}\beta^{\eta\iota}\gamma^{\theta\iota}$ (see Defs. 11 and 12), asserts that α is an *uncertainty*, that β is a *possibility*, and that γ is a *constant*—neither always true nor always false.

DEF. 11.—The symbol $\alpha^{o\beta}$ is the *denial* of α^{β}, and $\alpha_{o\beta}$ is the *denial* of α_{β}; so that $\alpha_{o\beta} = (\alpha_{\beta})'$, and $\alpha^{o\beta} = (\alpha^{\beta})'$.

DEF. 12.—The symbol α^{xy} means $(\alpha^{x})^{y}$, and asserts that α^{x} belongs to the class y. Similarly, α^{xyz} means $(\alpha^{xy})^{z}$; and so on.

Note.—This law does not hold for *subscripts.* Thus, α_{xy} means $\alpha_{(xy)}$, and *not* $(\alpha_{x})_{y}$. The symbol of denial o affects the whole expression following it. Thus α_{oxy} means $\alpha_{o(xy)}$, and α^{oxy} means $\alpha^{xy\iota}$; that is, $(\alpha^{xy})^{\iota}$. It is clear that α^{xy} is not always synonymous with α^{yx}. For example, when α is a *singular* (*i.e.*, when it denotes *one* statement only), we have $\alpha^{\iota\epsilon} = (\alpha^{\iota})^{\epsilon} = \alpha^{\eta}$; but $\alpha^{\epsilon\iota} = (\alpha^{\epsilon})^{\iota} = \alpha^{o\epsilon}$; so that $\alpha^{\iota\epsilon}$ asserts that α is *always false;* while $\alpha^{\epsilon\iota}$ merely asserts that α is *not always true.*

DEF. 13.—To remove all possible ambiguity as to the meaning of the symbol $\alpha^{\tau} : \beta^{\tau}$ and of its synonyms $\alpha : \beta$ and α_{β}, we may say (as an alternative to Def. 4) that α_{β} means $(\alpha^{\tau}\beta^{o\tau})^{\eta}$; that is to say, it asserts that the combined statement that α *is true and* β *false* is a statement *which is always false.* Briefly, this definition asserts $\alpha : \beta = (\alpha\beta^{\iota})^{\eta}$.

DEF. 14.—The symbol $\alpha :: \beta$ means $(\alpha : \beta)(\beta : \alpha)$; the symbol $\alpha \,!\, \beta$ means $\beta : \alpha$; the symbol $\alpha > \beta$ means $\alpha_{\beta}\beta_{o\alpha}$; the symbol $\alpha < \beta$ means $\beta > \alpha$; the symbol $\alpha : \beta : \gamma$ means $(\alpha : \beta)(\beta : \gamma)$; the symbol $\alpha \,!\, \beta \,!\, \gamma$ means $(\alpha \,!\, \beta)(\beta \,!\, \gamma)$; the symbol $\alpha > \beta > \gamma$ (with its equivalent $\gamma < \beta < \alpha$) means $(\alpha > \beta)(\beta > \gamma)$.

DEF. 15.—The symbol $\frac{\alpha}{\beta}$ (called a *causal implication*, to distinguish it from the *general implication* α_{β}) means $\alpha_{\beta}\alpha'_{o\beta}$ or its equivalent $\alpha_{\beta}\epsilon_{o\beta}$; in which $\epsilon_{o\beta}$ is the denial of β^{ϵ}, and is therefore synonymous with $\beta^{o\epsilon}$.

DEF. 16.—The symbols $(\alpha, \beta : \gamma)$ and $(\alpha, \beta)_{\gamma}$ are each synonymous with $\alpha_{\gamma}+\beta_{\gamma}$; and $(\alpha : \beta, \gamma)$ and $\alpha_{\beta,\gamma}$ are each synonymous with $\alpha_{\beta}+\alpha_{\gamma}$.

Also $\frac{\alpha, \beta}{\gamma}$ means $\frac{\alpha}{\gamma}+\frac{\beta}{\gamma}$, and $\frac{\alpha}{\beta, \gamma}$ means $\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}$. Also $\alpha^{u, v}$ means $\alpha^u+\alpha^v$, and $(\alpha, \beta)^u$ means $\alpha^u+\beta^u$.

Def. 17.—The symbol $(\alpha, \beta=u)$ means $(\alpha=u)(\beta=u)$; and $(u=\alpha, \beta)$ means $(u=\alpha)+(u=\beta)$.

When α and β are *singulars*, and u is a *plural* (or *class statement*), the symbol $(\alpha=u)$ means α^u, and $(\alpha, \beta=u)$ means $\alpha^u\beta^u$. But, when α and β are *plurals* (or class statements), and u is a *singular*, then $(u=\alpha, \beta)$ means $u^\alpha+u^\beta$.

Def. 19.—The symbol u sometimes denotes an *indefinite* class statement. Thus $\phi(\alpha, \beta)=u$ or $\phi^u(\alpha, \beta)$ means that nothing is asserted as to the class to which $\phi(\alpha, \beta)$ belongs; it may be of the class ϵ or η or θ. But u is often used in other senses also, like any other letter.

Def. 20.—The symbol π denotes any or all of the premisses of a proposition to be proved; ω denotes the conclusion, or any of the conclusions, to be proved; and ρ denotes a proposition proved in the course of the argument.

ρ may, or may not, be equivalent to $\pi : \omega$, the proposition to be proved.

π throughout the argument is understood to have the force of ϵ, and provisionally to represent ϵ, though (unlike ϵ) the statement or statements which it represents may conceivably and without inconsistency be false.

The way in which π, ω, and ρ are employed will be seen in such statements as

$$\rho=u : v=\alpha : \beta=\pi\omega' : \eta=\pi : \omega,$$

which may be read thus:—"Now it has been proved that u implies v; which is equivalent to proving that α implies β; which is the same as proving that $\pi\omega'$ contains an inconsistent or impossible factor (*i.e.*, one of the class η); which is the same as proving $\pi : \omega$, the proposition required to be proved.

Def. 21.—The following conventions will enable us to shorten and simplify expressions by avoiding the necessity of repetitions and a confusing multiplicity of brackets:—

(1) Let it be understood that the symbol $(=)$, though synonymous with $(::)$, has a *longer reach*; so that $\alpha :: \beta=\gamma$ means $(\alpha :: \beta)=\gamma$, and *not* $\alpha :: (\beta=\gamma)$, which would be expressed by $\alpha=\beta :: \gamma$.

(2) Let it be understood that the symbol $(\|)$ is synonymous with $(:)$ but of *longer reach*; so that $\alpha \| \beta : \gamma$ means $\alpha : (\beta : \gamma)$ and *not*

$(\alpha \parallel \beta) : \gamma$, or its synonym $(\alpha : \beta) : \gamma$, which would be expressed by $\alpha : \beta \parallel \gamma$.

(3) Let $(\parallel)$ be of *equal reach* with $(=)$; let $(!)$, $(::)$, $(>)$, $(<)$ be all of *equal reach* with $(:)$, and of *longer reach* than $(+)$.

Any of these symbols of relation is understood to affect the whole expression to the right or left of it *till it meets another symbol of equal or superior reach;* then its power in that direction ceases. Thus,

$$\alpha+\beta : \gamma+\delta > m :: n = r : u+v \parallel \alpha : \beta$$

means $(A = B)(B : C)$; in which $C = (\alpha : \beta)$, $B = \{r : (u+v)\}$ while $A = (D : E)(E > m)(m :: n)$; in which $D = \alpha+\beta$, $E = \gamma+\delta$.

Prop. 1.—Let x, y be *class statements*, and α, β *singular statements*, of which α asserts that a certain singular statement A belongs to the class x, and β asserts that A belongs to the class y. Then it is clear (see Def. 9) that $x^y : \alpha_\beta$.* And if A (while remaining *singular*) be a *representative* statement, representing *any* (*i.e.*, *every*) statement of the class or classes considered—just as an arbitrary illustrative triangle drawn on a blackboard represents *any* triangle of the kind discussed—then we have $x^y = \alpha_\beta$, and therefore $\phi(x^y) = \phi(\alpha_\beta)$.

And if we have a *series* of *class* statements, x, y, z, &c., thus corresponding respectively to a series of *singular representative* statements, α, β, γ, &c., we shall have $\phi(x^y, y^z, x^z, \text{\&c.}) = \phi(\alpha_\beta, \beta_\gamma, \alpha_\gamma, \text{\&c.})$; in which α, β, γ, &c., respectively take the places of x, y, z, &c., except that, when the latter are *indices*, the former are *subscripts*.

Hence, the standard equivalent syllogisms

$$(\alpha^\beta \beta^\gamma : \alpha^\gamma), \quad (\alpha^\beta \alpha^{0\gamma} : \beta^{0\gamma}), \quad (\alpha^{0\gamma} \beta^\gamma : \alpha^{0\beta}),$$

in which α, β, γ denote *classes*, remain valid when we change indices into subscripts, provided we interpret the class symbols as now denoting not classes but *singular representative statements*, and consider α as short for "A belongs to the class α," β as short for "A belongs to the class β," and γ as short for "A belongs to the class γ." (See my Second Paper on the "Calculus of Equivalent Statements.")

Prop. 2.—To prove the following four formulæ:—

$$(1) \quad (\alpha+\beta)_u :: \alpha_u + \beta_u = \alpha_u :: \beta_u,$$

$$(2) \quad \frac{\alpha+\beta}{u} :: \frac{\alpha}{u} + \frac{\beta}{u} = \frac{\alpha}{u} :: \frac{\beta}{u},$$

$$(3) \quad u_\alpha u_\beta :: u_\alpha + u_\beta = u_\alpha :: u_\beta,$$

$$(4) \quad \frac{u}{\alpha}\,\frac{u}{\beta} :: \frac{u}{\alpha} + \frac{u}{\beta} = \frac{u}{\alpha} :: \frac{u}{\beta}.$$

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* More fully, $x^y : (A^x : A^y) : \alpha_\beta$.

Defs.—The *capitals* A, B, C, &c., in the following proof of Form (1) denote *formulæ assumed* (*i.e.*, statements of the class ϵ), which will be given afterwards. The *numbers* in the proof represent each the equational statement of its own line. The symbol 1^A asserts that the statement 1 is deducible from the formula A; 2^B asserts that 2 is deducible from the formula B; and so on. The letter P denotes the left side of the identity to be proved; and Q its right side. The symbols ϵ_1, ϵ_2, ϵ_3, &c., denote *particular* statements of the class ϵ.

$$\begin{aligned} P &= (\alpha+\beta)_u :: \alpha_u+\beta_u = \alpha_u\beta_u :: \sigma_u+\beta_u && (1^A)\\ &= (\sigma_u\beta_u : \alpha_u+\beta_u)(\alpha_u+\beta_u : \alpha_u\beta_u) && (2^B)\\ &= \epsilon_1\,(\alpha_u+\beta_u : \alpha_u\beta_u) && (3^C)\\ &= (\alpha_u : \alpha_u\beta_u)(\beta_u : \alpha_u\beta_u) && (4^A)\\ &= (\alpha_u : \alpha_u)(\alpha_u : \beta_u)(\beta_u : \alpha_u)(\beta_u : \beta_u) && (5^D)\\ &= \epsilon_2\,(\alpha_u : \beta_u)(\beta_u : \alpha_u)\,\epsilon_3 && (6)\\ &= (\alpha_u : \beta_u)(\beta_u : \alpha_u)\\ &= \alpha_u :: \beta_u\\ &= Q. \end{aligned}$$

Here

$$\begin{aligned} A &= (\alpha+\beta)_u :: \alpha_u\beta_u,\\ B &= (\alpha :: \beta) :: (\alpha : \beta)(\beta : \alpha),\\ C &= \alpha : \alpha+\beta;\ \text{and, } \textit{a fortiori},\ \alpha\beta : \alpha+\beta,\\ D &= (\alpha : \beta\gamma) :: (\alpha : \beta)(\alpha : \gamma). \end{aligned}$$

Next, to prove formula (2). This is simply effected by changing the *general* implications α_u, β_u, $(\alpha+\beta)_u$, &c., into *causal* implications $\frac{\alpha}{u}$, $\frac{\beta}{u}$, $\frac{\alpha+\beta}{u}$, &c., *throughout*—both in the proof and in the assumed formula A, which is valid for causal as well as for general implications. Formulæ (3) and (4) may be proved similarly.

Of course, the preceding proof is much longer than is necessary; but I have given every step to show the working of the notation.

Prop. 3.—The formula

$$A_\alpha B_\beta C_\gamma : (A+B+C : \alpha+\beta+\gamma)$$

(which holds good for any number of terms) will enable us to resolve any expression of the form $amx+by+cz$, &c., into its elementary factors, by taking $A = amx$, $B = by$, $C = cz$; while α successively represents a, m, x; and β successively b, y; and γ successively c, z, &c.

As an example, take $\alpha\beta'+\beta\gamma'+\alpha\gamma$. This resolves into eight elementary factors which

$$= (\alpha+\beta+\alpha)(\alpha+\beta+\gamma)(\alpha+\gamma'+\alpha)(\alpha+\gamma'+\gamma)(\beta'+\beta+\alpha)$$
$$\times(\beta'+\beta+\gamma)(\beta'+\gamma'+\alpha)(\beta'+\gamma'+\gamma)$$

$$= (\alpha+\beta)\underline{(\alpha+\beta+\gamma)}(\alpha+\gamma')\,\epsilon_1\epsilon_2\epsilon_3\,\underline{(\beta'+\gamma'+\alpha)}\,\epsilon_4$$

$$= (\alpha+\beta)(\alpha+\gamma');$$

the two factors underlined and the four ϵ-factors being omitted, since they are implied in their co-factors; for $\alpha+\beta$ implies $\alpha+\beta+\gamma$, and $\alpha+\gamma'$ implies $\beta'+\gamma'+\alpha$; while *every* statement implies ϵ. (See next proposition.)

Note.—The same result may be obtained more briefly by first reducing to its *primitive form* the expression to be resolved. (See my Third Paper on the "Calculus of Equivalent Statements" in the *Proc. Lond. Math. Soc.*) Putting P for the given expression, we have $P=\alpha\beta'+\beta\gamma'+\alpha\gamma$; therefore $P'=(\alpha'+\beta)(\beta'+\gamma)(\alpha'+\gamma')=(\alpha'+\beta\gamma')(\beta'+\gamma)=\alpha'(\beta'+\gamma)$. Hence $P=\alpha+\beta\gamma'=(\alpha+\beta)(\alpha+\gamma')$.

Prop. 4.—To prove the formulæ (1) $\alpha:\epsilon=\epsilon$; (2) $\eta:\alpha=\epsilon$; (3) $\theta:\eta=\eta$; (4) $\epsilon:\theta=\eta$; (5) $\theta':\eta=\eta$; (6) $\epsilon:\theta'=\eta$; (7) $\epsilon:\alpha=\epsilon::\alpha$; (8) $\alpha:\eta=\alpha::\eta$.

Defs.—The symbols ϵ_1, ϵ_2, &c.; η_1, η_2, &c.; and θ_1, θ_2, &c., denote *particular* statements of their respective classes; and (to avoid accents) the symbol η_r, for every value of the number r, represents ϵ'_r, the denial of ϵ_r, and *vice versa.*

For (1) we have, by Def. 13,

$$\alpha:\epsilon_1=(\alpha\eta_1)^\eta=\eta^\eta=\epsilon;\ \text{for }\alpha\eta_1=\eta_2.^*$$

Similarly, we get $\alpha:\epsilon_2=\epsilon$, and so on, for every possible *singular* or *particular* statement of the class ϵ. Hence, the formula $(\alpha:\epsilon=\epsilon)$ holds good universally.

Similarly we may prove (2). To prove (3) we have

$$\theta_1:\eta_1=(\theta_1\epsilon_1)^\eta=\theta_2^\eta=\eta;\ \text{for }\theta_1\epsilon_1=\theta_2,$$

and, by definition of a variable, θ_2 is *not* always false; so that θ_2^η is impossible. Similarly, we show that $\theta_2 : \eta_2 = \eta$; that $\theta_3 : \eta_3 = \eta$; and so on. Hence $\theta : \eta = \eta$ universally.

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* Put into words, the statement $(\alpha\eta_1=\eta_2)$ asserts that $\alpha\eta_1$ is another or *second statement* of the class η.

To prove (5) we get in like manner $\theta'_1 : \eta_1 = (\theta'_1)^{\eta} = \eta$; for, since θ_1, by definition, is neither always true nor always false, its denial θ'_1 cannot always be false. Similarly for θ'_2, θ'_3, &c.; so that $\theta' : \eta = 0$ is true universally.

Formulæ (4) and (6) are proved in the same way.

From this we see that $\eta : \eta$ is equivalent to ϵ, even when the η of the antecedent and the η of the consequent represent *different* statements. But this does not hold good in the case of $\theta : \theta$; for the implication $\theta_1 : \theta_2$ is not necessarily, nor even generally, of the class ϵ.

For all values of α we have also

$$(7)\ \epsilon : \alpha = \epsilon :: \alpha = \alpha^{\epsilon}; \quad (8)\ \alpha : \eta = \alpha :: \eta = \alpha^{\eta}.$$

This requires no formal proof.

PROP. 5.—To find the weakest premiss that must be added to any given insufficient premiss or premisses π to justify the conclusion ω.

DEF.—α is said to be *stronger* and β the *weaker* statement when we have $\alpha > \beta$. (See Def. 14.)

Let s be the *strongest* inference obtainable from $\pi\omega'$. Then, since (by hypothesis) $\pi\omega' : s$, we get $\pi s' : \omega$, in which s' is the weakest obtainable co-factor of π to justify the conclusion ω. Now from $\pi s' : \omega$ we get $s' : \pi' + \omega$. But, the antecedent of an implication being generally stronger, and *never weaker*, than the consequent, it follows that no values of s' can be weaker than $\pi' + \omega$; so that $\pi' + \omega$ is a *limit of weakness* to the supplementary premiss required.

Note.—Since $\alpha : \beta = \beta' : \alpha'$, it follows that $\alpha > \beta = \alpha' < \beta'$; so that the stronger a statement the weaker its denial, and *vice versa.*

PROP. 6.—To find the weakest premiss that must be added to the usual premisses of the defective syllogism *Darapti* to justify the conclusion.

Interpreting the representative singular statements as in the concluding portion of Prop. 1, and as in my Second Paper on the "Calculus of Equivalent Statements" (see *Proc. Lond. Math. Soc.*), *Darapti* may be expressed as

$$\alpha_{\beta}\alpha_{\gamma} : \beta_{o\gamma'}.$$

Putting π for the two combined premisses $\alpha_\beta \alpha_\gamma$, and ω for the conclusion $\beta_{o\gamma'}$, we have

$$\begin{aligned}
\pi\omega' &= \alpha_{\beta\gamma}\beta_{\gamma'} \\
&= \{\alpha(\beta\gamma)'\}_\eta(\beta\gamma)_\eta \quad \text{(see Prop. 2, Defs.)} && (1^A) \\
&= \alpha(\beta\gamma)' + \beta\gamma : \eta && (2^B) \\
&= \alpha + \beta\gamma : \eta && (3^C) \\
&= \alpha_\eta(\beta\gamma)_\eta && (4^B) \\
&= \alpha_\eta\beta_{\gamma'} && (5^A) \\
&= \alpha_\eta\omega' = \alpha^\eta\omega'.
\end{aligned}$$

Representing a *proved* proposition by the symbol ρ (Def. 20), we have, therefore,

$$\begin{aligned}
\rho &= \pi\omega' :: \alpha^\eta\omega' \\
&= (\pi\omega' : \alpha^\eta\omega')(\alpha^\eta\omega' : \pi\omega') \\
&= (\pi\omega' : \alpha^\eta)(\alpha^\eta\omega' : \pi) \\
&= \pi\omega' : \alpha^\eta. && (7^D)
\end{aligned}$$

Hence, by the preceding proposition, the weakest premiss required is $\alpha^{o\eta}$, and, to justify the conclusion ω, *Darapti* should be written

$$\pi\alpha^{o\eta} : \omega;$$

that is, $$\alpha_\beta\alpha_\gamma\alpha_{o\eta} : \omega.$$

The formulæ assumed in this proof are

$$A = \text{Def. } 13; \quad B = \alpha_u\beta_u :: (\alpha+\beta)_u; \quad C = \alpha\beta' + \beta :: \alpha + \beta;$$

and $D =$ Def. 20, which implies that the second factor $(\alpha^\eta\omega' : \pi)$ is equivalent to ϵ, and therefore *unnecessary*. And, even if it were otherwise, we should still have $\rho : (\pi\omega' : \alpha^\eta)$, which would give the same result.

Prop. 7.—Now let us suppose *Barbara* to be defective (though we know the contrary), and ascertain the weakest premiss that must be added as a factor to its usual premisses π to justify the conclusion ω.

Here $\pi = \alpha_\beta\beta_\gamma$, and $\omega = \alpha_\gamma$; so that we have

$$\begin{aligned}
\pi\omega' &= \alpha_\beta\beta_\gamma\alpha_{o\gamma} \\
&= (\alpha\beta')_\eta(\beta\gamma')_\eta\alpha_{o\gamma} \\
&= (\alpha\beta' + \beta\gamma' : \eta)\alpha_{o\gamma} \\
&= \{(\alpha+\beta)(\alpha+\gamma')(\beta'+\gamma') : \eta\}\alpha_{o\gamma} \quad \text{(see Prop. 3)} \\
&= \{(\alpha+\beta\gamma')(\beta'+\gamma') : \eta\}\alpha_{o\gamma} \\
&= (\alpha\beta' + \beta\gamma' + \alpha\gamma' : \eta)\alpha_{o\gamma} \\
&= \alpha_\beta\beta_\gamma\alpha_\gamma\alpha_{o\gamma} \\
&= \eta \,(\text{for } \alpha_\gamma\alpha_{o\gamma} = \eta).
\end{aligned}$$

Hence, the weakest premiss required is the denial of η; that is, ϵ. Now ϵ is the weakest factor that exists, since (like unity in mathematics) its presence or absence as a factor of any statement does not alter the value of that statement. In other words, *Barbara* is a valid syllogism, requiring no additional premiss to justify its conclusion.

Note.—It is worthy of notice that the assumption $\pi\omega'$ is the one made by Euclid in *reductio ad absurdum* proofs. He always finds $\pi\omega' : \eta$, or $\pi\omega' : \pi'$; either of which is equivalent to $\pi : \omega$. Were he to find $\pi\omega' : s$, the statement s being consistent with π, it would show that his premisses must be increased by the premiss s' (the denial of s) to justify his conclusion ω.

PROP. 8.—To find the weakest factor that must be added to the premisses of the defective syllogism *Bramantip*, to justify the conclusion.

Here $\pi = \alpha_\beta \beta_\gamma$, and $\omega = \alpha_{o\gamma'}$; so that we have

$$\pi\omega' = \alpha_\beta \beta_\gamma \alpha_{\gamma'} = \alpha\beta' + \beta\gamma' + \alpha\gamma : \eta$$

$$= \alpha + \beta\gamma' : \eta \quad \text{(see note to Prop. 3)}$$

$$= \alpha_\eta (\beta\gamma')_\eta = \alpha_\eta \beta_\gamma = \alpha_\eta \pi ;$$

therefore

$$\rho = \pi\omega' :: \alpha_\eta \pi = (\pi\omega' : \alpha_\eta \pi)(\alpha_\eta \pi : \pi\omega')$$

$$= (\pi\omega' : \alpha_\eta)(\alpha_\eta : \omega') = (\pi\omega' : \alpha_\eta)(\alpha_\eta : \alpha_{\gamma'})$$

$$= \pi\omega' : \alpha_\eta ;$$

for the formula $\alpha_\eta : \alpha_u$ is of the class ϵ for all values of α and u. Hence we get $\pi\omega' : \alpha^\eta$; so that (as in the case of *Darapti*) the weakest additional premiss required is $\alpha^{o\eta}$.

PROP. 9.—To prove the equivalence of $\alpha_\beta \epsilon_{o\beta}$ and $\alpha_\beta \alpha'_{o\beta}$, the two definitions given of the *causal* implication $\dfrac{\alpha}{\beta}$ (see Def. 15),

$$\alpha_\beta \epsilon_{o\beta} = \alpha_\beta (\alpha' + \alpha)_{o\beta} \quad \text{(see Prop. 2, Defs.)} \qquad (1^A)$$

$$= \alpha_\beta (\alpha'_{o\beta} + \alpha_{o\beta}) \qquad (2^B)$$

$$= \alpha_\beta \alpha'_{o\beta} + \alpha_\beta \alpha_{o\beta}$$

$$= \alpha_\beta \alpha'_{o\beta} + \eta \qquad (3^C)$$

$$= \alpha_\beta \alpha'_{o\beta}.$$

Here
$$A = \alpha' + \alpha :: \epsilon,$$
$$B = (\alpha + \beta)_{ou} :: \alpha_{ou} + \beta_{ou},$$
$$C = \alpha\alpha' :: \eta.$$

Note.—It is evident that, when β is a variable, $\frac{\alpha}{\beta} = \alpha_\beta$; for in this case $\epsilon_{o\beta} = \epsilon$. It is also clear that, in all cases, $\frac{\alpha}{\beta} : \epsilon_{o\alpha}$; so that we have
$$\frac{\alpha}{\beta} = \alpha_\beta \alpha'_{o\beta} \epsilon_{o\beta} \epsilon_{o\alpha},$$
which we can strike out as redundant either $\epsilon_{o\beta}\epsilon_{o\alpha}$ or $\alpha'_{o\beta}\epsilon_{o\alpha}$.

It is worthy of remark that *Barbara* is valid in the form $\frac{\alpha}{\beta} . \frac{\beta}{\gamma} : \frac{\alpha}{\gamma}$, as well as in the form $\alpha_\beta \beta_\gamma : \alpha_\gamma$. The proof is easy.

The Limits of Statements.

Prop. 10.—To find the nearest limits of A^u, where A is some functional statement involving simpler statements α, β, γ, &c.; and u is one or other of the class statements ϵ, η, θ.

Def.—In any implication $\alpha : \beta$, the antecedent α is said to be a *superior* limit to the consequent β; and β is said to be an *inferior* limit to α. When we have $\alpha : \beta : x$, α and β are *both* superior limits of x, but β is the *nearest* superior limit; and, when we have $\alpha ! \beta ! x$ (or its equivalent $x : \beta : \alpha$), α and β are both *inferior* limits of x, and β is the *nearest* inferior limit. Any implication $\phi : \psi$, *constituting a formula*, is understood to mean $(\phi : \psi)^\epsilon (\psi : \phi)^{o\epsilon}$. Similarly for $\phi ! \psi$. Since $\eta : x : \epsilon$ is always true whatever x may be, it follows that η is a *superior* limit, and ϵ an *inferior* limit, to *any* statement. It is evident that $\alpha : x : \beta : \alpha$ is equivalent to $\alpha :: x :: \beta$, which implies both $\alpha : x : \beta$ and $\beta : x : \alpha$. In other words, when α, β, and x are all three equivalent, α and β may each be considered either a superior or inferior limit to x; just as in mathematics $\alpha - \beta$ may, without invalidating our argument, be considered positive or negative when $\alpha = \beta$.

We seem here to be entering upon an important development of the subject, analogous to my method for discovering the limits of multiple integrals, as explained in my First Paper on the "Calculus of Equivalent Statements." This development may possibly form the subject of a future communication to the Society; here I shall only give two or three simple cases, just to show the working of the principles and notation already explained.

The problem now before us is to find W and S in the double implication $W : A^u : S$; where A is short for $A(\alpha, \beta, \gamma, \&c.)$, and denotes some function of α, β, γ, &c.; u denotes one or other of the class statements ϵ, η, θ; W denotes the *weakest* obtainable antecedent or premiss from which we can draw the conclusion A^u; and S denotes the *strongest* conclusion which can be inferred from the premiss A^u. In the result W and S are to be both expressed in terms of $\alpha^\epsilon, \alpha^\eta, \alpha^\theta, \beta^\epsilon, \beta^\eta, \beta^\theta$, &c.

Two or three simple examples will illustrate the nature of the problem. Let A^u or $A^u(\alpha, \beta)$ denote $(\alpha\beta)^\theta$. We get

$$\begin{aligned} A^u = (\alpha\beta)^\theta &= \alpha^{\epsilon,\eta,\theta}(\alpha\beta)^\theta \quad [\text{for } \alpha^{\epsilon,\eta,\theta} = \epsilon \text{ (Def. 16)}] \\ &= \alpha^\epsilon(\alpha\beta)^\theta + \alpha^{\eta,\theta}(\alpha\beta)^\theta \\ &= \alpha^\epsilon\beta^\theta + \alpha^{\eta,\theta}(\alpha\beta)^\theta \\ &= \alpha^\epsilon\beta^\theta + \alpha^\theta(\alpha\beta)^\theta \quad [\text{for } \alpha^\eta(\alpha\beta)^\theta = \eta] \\ &= \alpha^\epsilon\beta^\theta + \alpha^\theta\beta^{\epsilon,\eta,\theta}(\alpha\beta)^\theta \\ &= \alpha^\epsilon\beta^\theta + \alpha^\theta\beta^\epsilon + \alpha^\theta\beta^{\eta,\theta}(\alpha\beta)^\theta \\ &= \alpha^\epsilon\beta^\theta + \alpha^\theta\beta^\epsilon + \alpha^\theta\beta^\theta(\alpha\beta)^\theta \\ &= W + \alpha^\theta\beta^\theta(\alpha\beta)^\theta. \end{aligned}$$

In this case, therefore, $W = \alpha^\epsilon\beta^\theta + \alpha^\theta\beta^\epsilon$,

and

$$\begin{aligned} S &= W + \alpha^\theta\beta^\theta = W + \alpha^\theta\beta^\theta + \alpha^\theta\beta^\theta \\ &= \alpha^\theta(\beta^\theta + \beta^\epsilon) + \beta^\theta(\alpha^\theta + \alpha^\epsilon) \\ &= \alpha^\theta\beta^{\theta,\epsilon} + \beta^\theta\alpha^{\theta,\epsilon} \\ &= \alpha^\theta\beta^{o\eta} + \beta^\theta\alpha^{o\eta}. \end{aligned}$$

Resolving W and S into their simplest factors, and then employing the abridged notation of Def. 16, we get

$$W = (\alpha, \beta)_\epsilon(\alpha, \beta)^\theta, \quad S = \alpha^{o\eta}\beta^{o\eta}(\alpha, \beta)^\theta,$$

so that $W : (\alpha\beta)^\theta : S$ may be read :—

"If we know that one of the factors of $\alpha\beta$ is a *certainty*, and one a *variable*, we may conclude that $\alpha\beta$ is a *variable*.

"And, if we know that $\alpha\beta$ is a *variable*, we may conclude that *both* its factors are *possibilities* $(\alpha^{o\eta}\beta^{o\eta})$, and *one at least* a *variable* $(\alpha, \beta)^\theta$."

Similarly, for $W : (\alpha\beta)^{\eta} : S$, we get

$$(\alpha,\ \beta)^{\eta} : (\alpha\beta)^{\eta} : (\alpha,\ \beta)^{\eta} + \alpha^{\theta}\beta^{\theta};$$

and for $W : (\alpha\beta)^{\epsilon} : S$, by inspection, we get

$$\alpha^{\epsilon}\beta^{\epsilon} : (\alpha\beta)^{\epsilon} : \alpha^{\epsilon}\beta^{\epsilon}.$$

Here $$W = S = (\alpha\beta)^{\epsilon}.$$

The values of W and S in the case of $W : (\alpha+\beta)^{\theta} : S$ may be found in the same way independently; or it may be derived from the values of W and S found before in the solution of $W : (\alpha\beta)^{\theta} : S$ as follows:—

From the evident formula $\alpha^{\theta} = (\alpha')^{\theta}$,

we get $$(\alpha+\beta)^{\theta} = (\alpha'\beta')^{\theta}.$$

Hence, changing α and β into their denials in the results already obtained for $(\alpha\beta)^{\theta}$, we get

$$W = (\alpha',\ \beta')^{\epsilon}\,(\alpha',\ \beta')^{\theta} = (\alpha,\ \beta)^{\eta}\,(\alpha,\ \beta)^{\theta} = \alpha^{\theta}\beta^{\eta} + \alpha^{\eta}\beta^{\theta},$$

$$S = (\alpha')^{o\eta}\,(\beta')^{o\eta}\,(\alpha',\beta')^{\theta} = \alpha^{o\epsilon}\beta^{o\epsilon}\,(\alpha,\ \beta)^{\theta} = \alpha^{\theta}\beta^{o\epsilon} + \alpha^{o\epsilon}\beta^{\theta}.$$

For $W : (\alpha+\beta)^{\eta} : S$ we get, in the same way, either independently or from a proved formula,

$$W = \alpha^{\eta}\beta^{\eta} = S,$$

For $W : (\alpha+\beta)^{\epsilon} : S$, we get

$$W = (\alpha,\ \beta)^{\epsilon} = \alpha^{\epsilon} + \beta^{\epsilon},$$

$$S = (\alpha,\ \beta)^{\epsilon} + \alpha^{\theta}\beta^{\theta} = \alpha^{\epsilon} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}.$$

We will now solve $W : A^{u}\,(\alpha,\ \beta) : S$ when $A\,(\alpha,\ \beta)$ is an *implication*, beginning with

$$W : (\alpha : \beta)^{\theta} : S.$$

Here
$$\begin{aligned}(\alpha : \beta)^{\theta} &= \alpha^{\epsilon,\eta,\theta}\,\beta^{\epsilon,\eta,\theta}\,(\alpha : \beta)^{\theta}\\ &= (\alpha^{\epsilon}\beta^{\epsilon} + \alpha^{\epsilon}\beta^{\eta} + \alpha^{\epsilon}\beta^{\theta}\\ &\quad + \alpha^{\eta}\beta^{\epsilon} + \alpha^{\eta}\beta^{\eta} + \alpha^{\eta}\beta^{\theta}\\ &\quad + \alpha^{\theta}\beta^{\epsilon} + \alpha^{\theta}\beta^{\eta} + \alpha^{\theta}\beta^{\theta})(\alpha : \beta)^{\theta}\\ &= \alpha^{\theta}\beta^{\theta}\,(\alpha : \beta)^{\theta};\end{aligned}$$

for all the nine terms in the bracket *except the last* are inconsistent with $(\alpha : \beta)^{\theta}$.

Our final result therefore is

$$(\alpha : \beta)^{\theta} : \alpha^{\theta}\beta^{\theta};$$

so that in this case $W = \eta$, and $S = \alpha^{\theta}\beta^{\theta}$.

The result $\eta : (\alpha : \beta)^{\theta} : \alpha^{\theta}\beta^{\theta}$ may be read:—

"From *no* data *expressible in terms of* α^{ϵ}, α^{η}, α^{θ}, β^{ϵ}, β^{η}, β^{θ}, *or their combinations*, can we infer that $(\alpha : \beta)$ is a *variable*.

"But, if we know that $(\alpha : \beta)$ is a *variable*, we may infer that α and β are both *variables*."

Next, take $W : (\alpha : \beta)^{\epsilon} : S$.

This may be briefly obtained from a former result as follows:—

$$(\alpha : \beta)^{\epsilon} = (\alpha\beta')^{\eta\epsilon} = (\alpha\beta')^{\eta}$$

(see Defs. 12, 13). We have therefore only to change β into β' in the result obtained already for $(\alpha\beta)^{\eta}$, thus getting

$$W = (\alpha, \beta')^{\eta}, \quad S = W + \alpha^{\theta}\beta^{\theta};$$

that is, $W = \alpha^{\eta} + (\beta')^{\eta} = \alpha^{\eta} + \beta^{\epsilon}, \quad S = \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}$.

But, it will be instructive to work out this case independently, as follows:—

$$\begin{aligned}(\alpha : \beta)^{\epsilon} &= \alpha^{\epsilon, \eta, \theta}\beta^{\epsilon, \eta, \theta}(\alpha : \beta)^{\epsilon}\\ &= \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\epsilon, \theta}\beta^{\eta \cdot \theta}(\alpha : \beta)^{\epsilon} \quad [\text{for } \alpha^{\eta} + \beta^{\epsilon} \text{ implies } (\alpha : \beta)^{\epsilon}]\\ &= \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\eta, \theta}(\alpha : \beta)^{\epsilon} \quad [\text{for } \alpha^{\epsilon}\beta^{\eta, \theta}(\alpha : \beta)^{\epsilon} = \eta]\\ &= \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}(\alpha : \beta)^{\epsilon} \quad [\text{for } \alpha^{\theta}\beta^{\eta}(\alpha : \beta) = \eta].\end{aligned}$$

Thus we get, as before,

$$W = \alpha^{\eta} + \beta^{\epsilon}, \quad S = \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}.$$

Lastly, take $W : (\alpha : \beta)^{\eta} : S$,

$$\begin{aligned}(\alpha : \beta)^{\eta} &= \alpha^{\epsilon, \eta, \theta}\beta^{\epsilon, \eta, \theta}(\alpha : \beta)^{\eta}\\ &= \alpha^{\epsilon}\beta^{\eta, \theta} + \alpha^{\theta}\beta^{\eta} + \alpha^{\theta}\beta^{\theta}(\alpha : \beta)^{\eta}\\ &= \alpha^{\epsilon}(\beta^{\eta} + \beta^{\theta}) + \alpha^{\theta}\beta^{\eta} + \alpha^{\theta}\beta^{\theta}(\alpha : \beta)^{\eta}\\ &= \alpha^{\epsilon}\beta^{\eta} + \alpha^{\epsilon}\beta^{\theta} + \alpha^{\theta}\beta^{\eta} + \alpha^{\theta}\beta^{\theta}(\alpha : \beta)^{\eta}.\end{aligned}$$

So we get

$$\begin{aligned}W &= \alpha^{\epsilon}\beta^{\eta} + \alpha^{\epsilon}\beta^{\theta} + \alpha^{\theta}\beta^{\eta}\\ &= \alpha^{\epsilon}\beta^{o\epsilon} + \alpha^{\theta}\beta^{\eta}\\ &= \alpha^{o\eta}\beta^{\eta} + \alpha^{\epsilon}\beta^{o\epsilon},\end{aligned}$$

and

$$\begin{aligned} S &= W + \alpha^{\theta}\beta^{\theta} \\ &= \alpha^{\epsilon}(\beta^{\eta} + \beta^{\theta}) + \alpha^{\theta}(\beta^{\eta} + \beta^{\theta}) \\ &= (\alpha^{\epsilon} + \alpha^{\theta})(\beta^{\eta} + \beta^{\theta}) \\ &= \alpha^{0\eta}\beta^{0\epsilon}. \end{aligned}$$

The following method of solving $W : (\alpha : \beta)^{\eta} : S$ deserves notice, as it involves an important principle not appealed to in our previous solutions of $W : A^{u}(\alpha, \beta) : S$,

$$\begin{aligned} (\alpha : \beta)^{\eta} &= (\alpha\beta')^{\eta\eta} \text{ (see Defs. 12, 13)} \\ &= (\alpha\beta')^{\eta\iota\epsilon} \text{ (see Note to Def. 12 and Formula 42)} \\ &= (\alpha\beta')^{\eta\iota}. \end{aligned}$$

Now (*Definition*) let W placed immediately before a function lose its force as an independent statement—like the symbol d in the differential calculus—so that $W\phi^{u}(\alpha, \beta)$ denotes, *not* the product of the factors W and $\phi^{u}(\alpha, \beta)$, but *the weakest premiss* (or antecedent), *expressed in terms of* α^{ϵ}, α^{η}, α^{θ}, β^{ϵ}, β^{η}, β^{θ}, *from which we can infer the conclusion* $\phi^{u}(\alpha, \beta)$; and let the symbol $S\phi^{u}(\alpha, \beta)$ in like manner denote the *strongest conclusion* (expressed in terms of α^{ϵ}, α^{η}, &c.) *inferable from the premiss* $\phi^{u}(\alpha, \beta)$. Also let $W'v$, $S'v$ [where v denotes $\phi^{u}(\alpha, \beta)$] be abbreviations for $(Wv)'$, $(Sv)'$. Assuming this definition, and having just shown that

$$(\alpha : \beta)^{\eta} = (\alpha\beta')^{\eta\iota},$$

we get

$$\begin{aligned} W(\alpha : \beta)^{\eta} &= W(\alpha\beta')^{\eta\iota} \\ &= S'(\alpha\beta')^{\eta} && (1^{A}) \\ &= \alpha^{\epsilon}\beta^{0\epsilon} + \alpha^{0\eta}\beta^{\eta}, && (2^{B}) \end{aligned}$$

$$\begin{aligned} S(\alpha : \beta)^{\eta} &= S(\alpha\beta')^{\eta\iota} \\ &= W'(\alpha\beta')^{\eta} && (3^{A}) \\ &= \{(\alpha, \beta')^{\eta}\}' && (4^{C}) \\ &= \{\alpha^{\eta} + (\beta')^{\eta}\}' && (5^{C}) \\ &= (\alpha^{\eta} + \beta^{\epsilon})' \\ &= \alpha^{0\eta}\beta^{0\epsilon}, \end{aligned}$$

We thus get the same results for $W(\alpha : \beta)^{\eta}$ and $S(\alpha : \beta)^{\eta}$ as we obtained before by another method. The assumptions A, B, C are

$A =$ the *nearest superior* limit of any statement α is the *denial* of the *nearest inferior* limit of its denial α', and *vice versa.*

B is our Def. of Sv and $S'v$, combined with the result obtained for $(\alpha\beta)^{\eta}$.

C is our definition of $(\alpha, \beta)^{u}$ (Def. 16).

As an instance of the application of this method to an implication of a higher degree, let it be required to ascertain for what values of α, β, u the implication $(u_{\alpha} : u_{\beta})$ is false.

Proceeding by a chain of what may be called *inductive sorites*, we get

$$(u_{\alpha} : u_{\beta})^{\eta} \; ! \; (u_{\alpha})^{\epsilon} (u_{\beta})^{\eta} \; ! \; W(u_{\alpha})^{\epsilon} W(u_{\beta})^{\eta}$$

$$! \; (u^{\eta} + \alpha^{\epsilon})(u^{\epsilon}\beta^{\epsilon\iota} + u^{\theta}\beta^{\eta}) \; ! \; \alpha^{\epsilon} (u^{\epsilon}\beta^{\epsilon\iota} + u^{\theta}\beta^{\eta}).$$

Hence, $(u_{\alpha} : u_{\beta})$ is false in at least *three cases*, namely, $\alpha^{\epsilon}u^{\epsilon}\beta^{\eta}$, $\alpha^{\epsilon}u^{\epsilon}\beta^{\theta}$, $\alpha^{\epsilon}u^{\theta}\beta^{\eta}$.

This inductive process is useful in detecting fallacies.

The preceding solutions are given as an introduction to a general method of dealing with the limits of statements, and thence with the limits of chances, when these last limits can be deduced from the data, while the *exact values* of the chances cannot be found. To this class belongs Boole's "Challenge Problem" (as I believe it is called), which I have discussed in my Fourth Paper on the "Calculus of Equivalent Statements" (see Appendix to Fourth Paper). Boole's result (on which he staked the validity of his "General Method") is therein shown to be wrong.

Implications of Constants.

When α and β are *constants*, we get

$$\epsilon : \alpha = \alpha, \quad \alpha : \beta = \alpha' + \beta, \quad (\alpha : \beta)' = \alpha\beta',$$

and universally, in the case of *constants*, the implication

$$\phi(a, b, c) : \psi(a, b, c)$$

will be equivalent to the *alternative*

$$\phi'(a, b, c) + \psi(a, b, c);$$

the symbol $\phi'(a, b, c)$ denoting the denial of $\phi(a, b, c)$.

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* For a more complicated example, see Appendix, Note 1.

This affords a simple mechanical method of testing the validity of formulæ which only contain *constant constituents.* Suppose, for example, the constituents of *Darapti* to be the *constants* a, b, c; so that we write it $a_{bc} : b_{oc'}$. This, by the preceding principle, is now equivalent to $a_{obc}+b_{oc'}$, which (on the same principle) is equivalent to $a\,(bc)'+bc$, that is, to $a+bc$; a result which may be true or false. It is true, for instance, in the two cases α^{ϵ} and $(bc)^{\epsilon}$; but it is false in the case $\alpha^{\eta}\,(bc)^{\eta}$.

Let us next suppose the constituents of *Barbara* to be the *constants* a, b, c; so that *Barbara* may be written $a_b b_c : a_c$. Applying the same test, this finally reduces to the form $(b'+b)+a'+c$, which is evidently equivalent to $\epsilon+a'+c$; that is, to ϵ. This, of course, only proves *Barbara* in the case of *constants;* a general proof, applicable to constants and variables alike, has been already given.

Other Meanings of the Symbol α^{β}.

Hitherto we have used the symbol α^{β} to assert that the statement α belongs to the class of statements denoted by β; the index β in such cases generally representing one or other of the three mutually exclusive classes ϵ, η, θ, to one or other of which every intelligible and unambiguous statement must belong. But we obtain some remarkable analogies to mathematical formulæ by departing from this restriction. For example, let α^{β} first be used as a synonym of β_{α} (just as $\alpha\,!\,\beta$ is a synonym of $\beta : \alpha$), and assert that the statement α *is implied* in the statement β. Thus interpreted, α^{ϵ} will still remain synonymous with ϵ_{α}, but α^{η} will change its meaning and be no longer (as hitherto) synonymous with α_{η}, but with η_{α}, which $=\epsilon$. With this new definition of the symbol α^{β}, we get the following three analogies to mathematical formulæ :—

$$(1)\ \alpha^{u}\beta^{u}=(\alpha\beta)^{n};\quad (2)\ u^{\alpha}u^{\beta}=u^{\alpha+\beta};\quad (3)\ \alpha^{\eta}=\epsilon,$$

the last of which corresponds to $\alpha^{\circ}=1$ in mathematics; and we also get, as before (see Prop. 1),

$$(\alpha^{\beta}\beta^{\gamma} : \alpha^{\gamma}),\quad (\alpha^{\beta}\alpha^{o\gamma} : \beta^{o\gamma}),\quad (\alpha^{o\gamma}\beta^{\gamma} : \alpha^{o\beta});$$

but with a different interpretation of the symbols; the indices now representing *singular* statements, and *not classes.*

Let us now give a *third* interpretation to the symbol α^{β}, and make it this time synonymous with the alternative $\alpha+\beta'$, so that we shall have

$$\beta_{\alpha}=\epsilon : \alpha^{\beta}.$$

Thus interpreted, we shall still have the three previous mathematical analogies and three syllogistic formulæ, and, in addition, we shall have the mathematical analogies

$$(a^u)^v = (a^v)^u = a^{uv}.$$

But, on the other hand, we shall now have also

$$a^{uv} = a^u + a^v,$$

which does *not* hold good in mathematics, nor yet when the symbol a^β is used as a synonym of β_a.

The difference between $\frac{a}{\beta}$, a_β, and β^a (when the last is used as a synonym of $\beta + a'$) may be shown geometrically by the accompanying three diagrams.

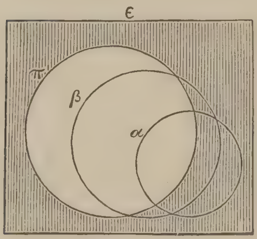

FIG. 1.

$$\left(\frac{\alpha}{\beta}\right)^\epsilon (\alpha_\beta)^\epsilon (\beta^\alpha)^\epsilon$$

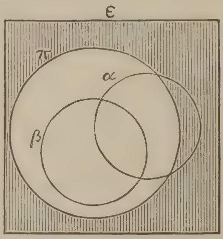

FIG. 2.

$$\left(\frac{\alpha}{\beta}\right)^\eta (\alpha_\beta)^\eta (\beta^\alpha)^\theta$$

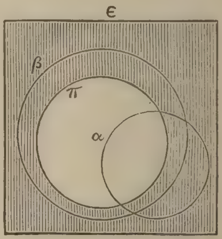

FIG. 3.

$$\left(\frac{\alpha}{\beta}\right)^\eta (\alpha_\beta)^\epsilon (\beta^\alpha)^\epsilon$$

Explanatory Note.

To fix our ideas, let us suppose first that the classes π, a, β are the three infinite collections of points enclosed in three circumferences, drawn each through three random points in the square ϵ—circles which do not fall wholly within this square not being counted—and that the figures above are the results of actual trials. Next let π, a, β be three *statements*, respectively asserting that a certain random point P will fall in the circle π, that it will fall in a, that it will fall in β. *Till the three random circles* π, a, β *are formed*, the implications $\frac{a}{\beta}$ and a_β are *variables*, since they may turn out true or false in the future result. But when the trial has taken place, and *the three circles are formed*, then we choose one of them π as our *universe of discourse*, and in this universe the point P is taken at random. The implications $\frac{\alpha}{\beta}$ and α_β are now *constants* (since the circles are now formed and fixed) belonging, according to the result, either to the class ϵ or to the class η; while α, β, and β^α (meaning $\beta+\alpha'$) are all three variables in Fig 2; α and β alone are variables in Fig. 1; and α alone a variable in Fig. 3.

The *statement* α asserts that P will be one of the points in the *class* α, and β asserts that P will be one of the points in the *class* β. Inspection of the diagrams will show—

(1) That all three statements $\frac{\alpha}{\beta}$, α_β, β^α (meaning $\beta+\alpha'$) are always true of Fig. 1.

(2) That $\frac{\alpha}{\beta}$ and α_β are always false of Fig. 2, since they each assert the evident untruth that, whenever P falls within the boundary α, it also falls within the boundary β, while β^α (meaning the alternative $\beta+\alpha'$) is a *variable*, being neither always true, nor always false. The alternative $\beta+\alpha'$ is true every time P falls within β, and it is also true every time P falls *outside* α; but it is false when its *denial* $\alpha\beta'$ is true, that is, *when P falls within A and outside β.*

(3) In Fig. 3, α_β and β^α are both true; but $\frac{\alpha}{\beta}$ is false, since it contains the false factor $\epsilon_{o\beta}$ $\left(\text{see definition of } \frac{\alpha}{\beta}\right)$, the denial of the true statement ϵ_β (or β^ϵ) *which in* Fig. 3 *is necessarily true, since the boundary β includes the whole universe of possibilities π.*

Note 1.—Observe that in the case of Fig. 3 the statement β is a *constant*, since there it is *always true*; whereas in Figs. 1 and 2 α and β are both *variables.*

Note 2.—It is evident that the statement β^α (when synonymous with $\beta+\alpha'$) refers to a *single instance only*; whereas $\frac{\alpha}{\beta}$ and α_β, *after the classes π, α, β have been formed*, refer to a *general law* and many cases. Hence, the equation

$$(\alpha^u)^v = \alpha^{uv}$$

is an *identity* when the symbol α^β is understood to be a synonym of

$\alpha+\beta'$; for then the equation

$$(\alpha^u)^v = \alpha^{uv}$$

becomes the equivalent of

$$(\alpha+u')+v' = \alpha+(uv)',$$

which is evidently an identity. But, when α^β is synonymous with β_α, the equation

$$(\alpha^u)^v = \alpha^{uv}$$

becomes synonymous with $v : u_\alpha = (uv)_\alpha$,

which is *not* always true, as the accompanying diagram (Fig. 4) will show. With reference to a point P taken at random within the boundary ϵ, let the statements u, v, α respectively assert the falling of this random point within the *fixed* boundaries which the letters indicate in the figure. It is evident, by reference to the figure, that u (like v and α) is a *variable*; that u_α is *always false*, since the boundaries u and α are *fixed* (the *always true* statement being $u_{o\alpha}$); and that $(uv)_\alpha$ is *always true*, so that the equation

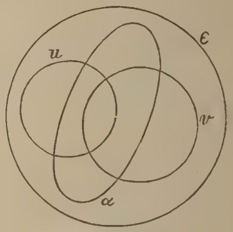

FIG. 4.

$$v : u_\alpha = (uv)_\alpha$$

becomes $$\theta_1 : \eta_1 = \epsilon_1 ;$$

or its equivalent $$\theta_1^\eta = \epsilon_1,$$

which is an impossibility, since the left side θ_1^η asserts the impossibility that a certain statement of the class θ belongs to the class η, and the equation asserts that this impossibility θ_1^η is a particular case of the class of certainties ϵ.

APPENDIX.

Formulæ and Notes. (See Def. p. 167.)

The following formulæ are *identities*, and therefore *true for all values of their constituents*, that is, they are all statements of the class ϵ. Thus $F_1 = \epsilon_1$, $F_2 = \epsilon_2$, $F_3 = \epsilon_3$, &c., which means that the first formula F_1 is one instance of the class ϵ; the second formula F_2 a second instance of the class ϵ, &c. The *numerical* symbol m^n asserts that F_m is deducible from F_n; the symbol m_n asserts that F_m implies F_n; the symbol $m^{n.r}$ asserts that F_m is deducible from $F_n F_r$;

the symbol $m_{n.r}$ asserts that F_m implies $F_n F_r$; and the symbol m^{n+r} is equivalent to $m^n m^r$, which asserts that F_m is deducible from F_n and also from F_r.

1. $\alpha\alpha' = \eta;\quad \alpha\alpha' : \eta;\quad (\alpha\alpha')^{\eta}.$

2. $\alpha+\alpha' = \epsilon;\quad \epsilon : \alpha+\alpha';\quad (\alpha+\alpha')^{\epsilon}.$

3. $(\alpha=\beta) = (\alpha :: \beta) = (\alpha:\beta)(\beta:\alpha) = \alpha_{\beta}\beta_{\alpha}.$

4. $\begin{cases} \alpha:\epsilon = \alpha_{\epsilon} = \epsilon;\quad \eta:\alpha = \eta_{\alpha} = \epsilon;\quad \alpha:\alpha = \alpha_{\alpha} = \epsilon; \\ \eta:\epsilon = \epsilon;\quad \eta:\eta = \epsilon;\quad \epsilon:\eta = \eta. \end{cases}$

5. $\begin{cases} \alpha_{o\epsilon} = \eta;\quad \eta_{o\alpha} = \eta;\quad \alpha_{o\alpha} = \eta; \\ \eta_{o\epsilon} = \eta;\quad \eta_{o\eta} = \eta;\quad \epsilon_{o\eta} = \epsilon. \end{cases}$

6. $\alpha:\beta = \epsilon:\alpha'+\beta = \alpha\beta':\eta = (\alpha'+\beta)^{\epsilon} = (\alpha\beta')^{\eta}.$

$7^{3.4}$. $\epsilon:\alpha = (\epsilon=\alpha);\quad \alpha:\eta = (\alpha=\eta) = \alpha_{\eta} = \alpha^{\eta}.$

8. $\alpha:\beta = \beta':\alpha' = \alpha\beta':\eta = (\alpha\beta')^{\eta}.$

9. $\alpha_u : (\alpha\beta)_u : (\alpha\beta\gamma)_u :$ &c.

10. $\begin{cases} \alpha : \alpha+\beta : \alpha+\beta+\gamma : \text{\&c.}; \\ (\alpha\beta)' = \alpha'+\beta';\quad (\alpha+\beta)' = \alpha'\beta'. \end{cases}$

11. $\begin{cases} \alpha+\beta = \alpha+\alpha'\beta; \\ \alpha+\beta+\gamma = \alpha+\alpha'\beta+\alpha'\beta'\gamma, \text{ \&c.} \end{cases}$

12. $\alpha_{\beta}\beta_{\gamma} : \alpha_{\gamma}.$

13^{12}. $\alpha_{\beta}\beta_{\gamma}\alpha_{o\gamma} : \eta.$

14^{13}. $\alpha_{\beta}\alpha_{o\gamma} : \beta_{o\gamma};\quad \alpha_{o\gamma}\beta_{\gamma} : \alpha_{o\beta}.$

15. $\begin{cases} \alpha_{\eta} : \alpha_{\beta};\quad \alpha_{o\beta} : \alpha_{o\eta} : \alpha^{o\eta}, \\ \epsilon_{\alpha} : \beta_{\alpha};\quad \alpha_{o\beta} : \epsilon_{o\beta} : \beta^{o\epsilon}. \end{cases}$

$16^{10.12}$. $\alpha_{\beta} : \alpha_{\beta+\gamma}.$

17. $\begin{cases} \alpha_{\beta}\alpha_{\gamma} = \alpha_{\beta\gamma}, \\ \alpha_{o\beta\gamma} = \alpha_{o\beta}+\alpha_{o\gamma}. \end{cases}$

$18^{8.17}$. $(\alpha+\beta)_{\gamma} = \alpha_{\gamma}\beta_{\gamma}.$

19^{18}. $\alpha_{o\gamma}+\beta_{o\gamma} = (\alpha+\beta)_{o\gamma}.$

$20^{3\cdot17\cdot18}$. $\alpha : \beta = (\alpha = \alpha\beta) = (\alpha + \beta = \beta).$

21^{20}. $$\begin{cases} \alpha + \alpha\beta = \alpha\,; & \epsilon + \alpha = \epsilon\,; \quad \eta + \alpha = \alpha\,; \\ \alpha + \alpha \ \ = \alpha\,; & \alpha\alpha = \alpha\,; \quad \ \ \eta\alpha = \eta. \end{cases}$$

22. $$\begin{cases} (\epsilon = \alpha) : \{\phi(\alpha) = \phi(\epsilon)\}, \\ (\alpha = \eta) : \{\phi(\alpha) = \phi(\eta)\}. \end{cases}$$

23. $$(\alpha = \epsilon) : \{\phi(\alpha') = \phi(\eta)\},$$
$$(\alpha = \eta) : \{\phi(\alpha') = \phi(\epsilon)\}.$$

$24^{22\cdot23}$. We may assume

$$\pi = \epsilon, \quad \pi' = \eta, \quad \phi(\pi) = \phi(\epsilon), \quad \phi(\pi') = \phi(\eta),$$

in order to prove the proposition $\pi : \omega$; but *not* in order to prove the conclusion ω. (See Def. 20.)

25. $$\begin{cases} (\theta_1)_\eta = \ \ \theta_1^\eta \ \ = \eta\,; \ (\theta_1)_{o\eta} = \ \ \theta_1^{o\eta} \ \ = \epsilon, \\ (\theta_1')_\eta = (\theta_1')^\eta = \eta\,; \ (\theta_1')_{o\eta} = (\theta_1')^{o\eta} = \epsilon. \end{cases}$$

26. $\theta_1 : \theta_1' = \theta_1' : \theta_1 = \eta$; though $\eta_1 : \eta_1' = \eta_1 : \epsilon_1 = \epsilon.$

$27^{9\cdot10\cdot16}$. $A_\alpha B_\beta C_\gamma : (A + B + C : \alpha + \beta + \gamma).$

$28^{9\cdot18}$. $\alpha_\gamma + \beta_\gamma : (\alpha\beta)_\gamma.$

29. $$\frac{\theta_1}{\theta_2} = \theta_1 : \theta_2.$$

30. $$\frac{\alpha}{\epsilon} = \frac{\epsilon}{\alpha} = \eta\,; \ \frac{\eta}{\eta} = \epsilon.$$

31. $$\frac{\alpha}{u} : \frac{\alpha\beta}{u} : \frac{\alpha\beta\gamma}{u}, \text{ \&c.}$$

32. $$\frac{\alpha}{u} \cdot \frac{\beta}{u} = \frac{\alpha + \beta}{u}.$$

33. $$\frac{u}{\alpha\beta} = u_{\alpha\beta}\,(u_\alpha + u_\beta).$$

34. $\alpha : \beta = \alpha^\tau : \beta^\tau.$ (Defs. 1, 2, 3, 4.)

35. $$\left\{ \begin{array}{l} \alpha_x + \beta_x = \alpha, \beta : x = (\alpha, \beta)_x \\ x_\alpha + x_\beta = x : \alpha, \beta = x_{\alpha, \beta} \\ \dfrac{\alpha}{x} + \dfrac{\beta}{x} = \dfrac{\alpha, \beta}{x} \\ \dfrac{x}{\alpha} + \dfrac{x}{\beta} = \dfrac{x}{\alpha, \beta} \end{array} \right\}. \quad \text{(Def. 16.)}$$

36. $$\left\{\begin{array}{l} \alpha^{u} + \alpha^{v} = \alpha^{u,\,v} \\ \alpha^{u} + \alpha^{v\iota} = \alpha^{u,\,v\iota} \end{array}\right\}. \quad \text{(Defs. 1, 16, 12.)}$$

37. $$\left\{\begin{array}{l} (\alpha,\,\beta = u) = (\alpha = u)(\beta = u) = (\alpha + \beta)_{u} u_{\alpha\beta}, \\ (u = \alpha,\,\beta) = (u = \alpha) + (u = \beta) = u_{\alpha}\alpha_{u} + u_{\beta}\beta_{u}. \end{array}\right. \quad \text{(Def. 17.)}$$

38. $$\left\{\begin{array}{l} (\alpha : \beta) : (u_{\alpha} : u_{\beta}), \\ (\alpha : \beta) : (\beta_{u} : \alpha_{u}). \end{array}\right.$$

39. $\alpha^{\eta\iota} = \alpha^{o\eta}$; $\alpha^{\iota\eta} = \alpha^{\epsilon}$. (Defs. 1, 12.)

40. $\alpha^{\epsilon\iota} = \alpha^{o\epsilon}$; $\alpha^{\iota\epsilon} = \alpha^{\eta}$. (Defs. 1, 12.)

41. $\alpha^{\epsilon\epsilon} = \alpha^{\epsilon}$; $\alpha^{\epsilon\eta} = \alpha^{\epsilon\iota} = \alpha^{o\epsilon}$.

42. $\alpha^{\eta\epsilon} = \alpha^{\eta}$; $\alpha^{\eta\eta} = \alpha^{\eta\iota} = \alpha^{o\eta}$.

43. $$\left\{\begin{array}{l} W(\alpha\beta)^{\theta} = \alpha^{\epsilon}\beta^{\theta} + \beta^{\epsilon}\alpha^{\theta}, \quad \text{(See Prop. 10 and Def. p. 171.)} \\ S(\alpha\beta)^{\theta} = \alpha^{\theta}\beta^{o\eta} + \beta^{\theta}\alpha^{o\eta}. \end{array}\right.$$

44. $W(\alpha\beta)^{\epsilon} = \alpha^{\epsilon}\beta^{\epsilon} = S(\alpha\beta)^{\epsilon}$.

45. $$\left\{\begin{array}{l} W(\alpha\beta)^{\eta} = (\alpha,\,\beta)^{\eta} = \alpha^{\eta} + \beta^{\eta}, \\ S(\alpha\beta)^{\eta} = (\alpha,\,\beta)^{\eta} + \alpha^{\theta}\beta^{\theta} = \alpha^{\eta} + \beta^{\eta} + \alpha^{\theta}\beta^{\theta}. \end{array}\right.$$

46^{43}. $$\left\{\begin{array}{l} W(\alpha + \beta)^{\theta} = (\alpha,\,\beta)^{\eta}\,(\alpha,\,\beta)^{\theta} = \alpha^{\theta}\beta^{\eta} + \alpha^{\eta}\beta^{\theta}, \\ S(\alpha + \beta)^{\theta} = \alpha^{o\epsilon}\beta^{o\epsilon}\,(\alpha,\,\beta)^{\theta} = \alpha^{\theta}\beta^{o\epsilon} + \alpha^{o\epsilon}\beta^{\theta}. \end{array}\right.$$

47. $$\left\{\begin{array}{l} W(\alpha + \beta)^{\epsilon} = (\alpha,\,\beta)^{\epsilon} = \alpha^{\epsilon} + \beta^{\epsilon}, \\ S(\alpha + \beta)^{\epsilon} = (\alpha,\,\beta)^{\epsilon} + \alpha^{\theta}\beta^{\theta} = \alpha^{\epsilon} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}. \end{array}\right.$$

48. $W(\alpha + \beta)^{\eta} = \alpha^{\eta}\beta^{\eta} = S(\alpha + \beta)^{\eta}$.

49. $$\left\{\begin{array}{l} W(\alpha : \beta)^{\theta} = \eta \\ S(\alpha : \beta)^{\theta} = \alpha^{\theta}\beta^{\theta} \end{array}\right\}. \quad \text{(See page 169.)}$$

50. $$\left\{\begin{array}{l} W(\alpha : \beta)^{\epsilon} = \alpha^{\eta} + \beta^{\epsilon}, \\ S(\alpha : \beta)^{\epsilon} = \alpha^{\eta} + \beta^{\epsilon} + \alpha^{\theta}\beta^{\theta}. \end{array}\right.$$

51. $$\left\{\begin{array}{l} W(\alpha : \beta)^{\eta} = \alpha^{\epsilon}\beta^{o\epsilon} + \alpha^{\theta}\beta^{\eta} = \alpha^{o\eta}\beta^{\eta} + \alpha^{\epsilon}\beta^{\theta}, \\ S(\alpha : \beta)^{\eta} = \alpha^{o\eta}\beta^{o\epsilon}. \end{array}\right.$$

52. $$\left\{\begin{array}{l} Wu = S'u', \\ Su = W'u' \end{array}\right\}, \text{ in which } u \text{ denotes } \varphi(\alpha,\,\beta) \text{ or } \psi(\alpha,\,\beta,\,\gamma), \text{ \&c.}$$

53. $$\begin{cases} uv \,!\, W(uv) \,!\, Wu\,Wv, \\ uv : \; S(uv) : Su\,Sv. \end{cases}$$

54. $$\begin{cases} u+v \,!\, W(u+v) \,!\, Wu+Wv, \\ u+v : \; S(u+v) : Su+Sv. \end{cases}$$

55. $$\left.\begin{cases} W\alpha^u = \alpha^u \\ S\alpha^u = \alpha^u \end{cases}\right\},$$ in which α^u denotes any of the six statements
$$\alpha^\epsilon,\ \alpha^\eta,\ \alpha^\theta,\ \alpha^{o\epsilon},\ \alpha^{o\eta},\ \alpha^{o\theta}.$$

Note 1.—As a good example of the application of the seven formulæ F_{49} to F_{55}, and of the principles on which they are founded, let us take the following:—

The formula $(\alpha : \beta) : (u_\alpha : u_\beta)$ is *always* true; but *not always* its converse, the implication

$$(u_\alpha : u_\beta) : (\alpha : \beta).$$

Let it be required to ascertain for what values of u, α, β this last implication must be true, and for what values false.

Let $\phi(u, \alpha, \beta)$, or simply ϕ, denote the implication

$$(u_\alpha : u_\beta) : (\alpha : \beta).$$

We get by *inductive sorites*, using $F_{50.51.49}$,

$$\begin{aligned} \phi^\epsilon \,!\, W\phi^\epsilon \,&!\, (u_\alpha : u_\beta)^\eta + (\alpha : \beta)^\epsilon \\ &!\, (u_\alpha)^\epsilon (u_\beta)^{\epsilon\iota} + (u_\alpha)^\theta (u_\beta)^\eta + (\alpha^\eta + \beta^\epsilon) \\ &!\, (u_\alpha)^\epsilon (u_\beta)^\eta + \alpha^\eta + \beta^\epsilon \\ &!\, (u^\eta + \alpha^\epsilon)(u^\epsilon \beta^{\epsilon\iota} + u^\theta \beta^\eta) + \alpha^\eta + \beta^\epsilon \\ &!\, \alpha^\epsilon (u^\epsilon \beta^{\epsilon\iota} + u^\theta \beta^\eta) + \alpha^\eta + \beta^\epsilon \\ &!\, \alpha^\epsilon (u^\epsilon \beta^\eta + u^\epsilon \beta^\theta + u^\theta \beta^\eta) + \alpha^\eta + \beta^\epsilon \\ &!\, \alpha^\epsilon (u^\epsilon \beta^{\epsilon\iota} + u^{\eta\iota} \beta^\eta) + \alpha^\eta + \beta^\epsilon. \end{aligned}$$

Similarly, we get

$$\phi^\eta \,!\, u^\eta \alpha^\epsilon \beta^{o\epsilon} + u^\epsilon \alpha^\theta \beta^\eta + u^\eta \alpha^{o\eta} \beta^\eta.$$

The results obtained are easily verified. For example, the term α^η in the first result informs us that $\phi(u, \eta, \beta)$ is *always true*, which is evident; for

$$\begin{aligned} \phi(u, \eta, \beta) &= (u_\eta : u_\beta) : (\eta : \beta) \\ &= (u_\eta : u_\beta) : \epsilon = \epsilon; \quad \text{for } A : \epsilon = \epsilon, \end{aligned}$$

whatever A may be.

Again, the term $u^{\epsilon}\alpha^{\theta}\beta^{\eta}$ in the second result informs us that $\phi(\epsilon, \theta, \eta)$ is *always false.* This also is evident, for

$$\begin{aligned}\phi(\epsilon, \theta, \eta) &= (\epsilon_{\theta} : \epsilon_{\eta}) : (\theta : \eta) \\ &= (\eta : \eta) : \eta \\ &= \epsilon : \eta \\ &= \eta.\end{aligned}$$

Note 2.—The leading principle which runs through this paper is that symbols of abbreviation should be employed—not as invariable substitutes, but as synonyms—for all expressions (verbal or symbolic) that are found by experience to recur frequently. None of the symbols adopted, not even the symbols $(+)$ and $(=)$, need always be used in the same sense; though it is advisable to be as conservative as possible in regard to the meanings attached to symbols of very frequent recurrence; and, of course, every change of signification should be accompanied by a fresh definition. The following mathematical example will illustrate this principle, and also the meanings of the important symbols W and S.

Let $\phi(x)$ be any function of x in the *mathematical* sense, and let the symbols $\phi^{u}(x)$, $\phi^{v}(x)$, $\phi^{w}(x)$ be *logical statements*, asserting respectively—that $\phi(x)$ is *real and positive;* that it is *real and negative;* that it is *imaginary*—any function of x not belonging to one or other of the three classes u, v, w, not entering into our universe of discourse. Let x^{u}, $\phi^{w}(x, y)$, &c., be similarly interpreted as *statements* when the indices are u, v, or w; while they retain their customary mathematical meaning with other indices. As a special and simple case take

$$\phi(x, y) = x+y;$$

and let it be required to find $W(x+y)^{u}$ and $S(x+y)^{u}$ in terms of x^{u}, x^{v}, x^{w}, y^{u}, y^{v}, y^{w}. We have

$$\begin{aligned}(x+y)^{u} &= (x^{u}+x^{v}+x^{w})(y^{u}+y^{v}+y^{w})(x+y)^{u} \\ &= x^{u,v,w}\, y^{u,v,w}(x+y)^{u} \qquad \text{(by Def. 16)} \\ &= (x^{u}y^{u}+x^{u}y^{v}+x^{v}y^{u}+x^{w}y^{w})(x+y)^{u}\end{aligned}$$

[omitting terms inconsistent with $(x+y)^{u}$]

$$= x^{u}y^{u}+(x^{u}y^{v}+x^{v}y^{u}+x^{w}y^{w})(x+y)^{u}$$

[for $x^{u}y^{u}$ implies $(x+y)^{u}$].

Hence, we get $W(x+y)^u = x^u y^u$

$$S(x+y)^u = x^u y^u + (x^u y^v + x^v y^u + x^w y^w)$$
$$= x^w y^w + x^{ow} y^{ow} (x^u + y^u).$$

Thus, the *weakest premiss* from which we can infer $(x+y)^u$ (*i.e.*, that the number or fraction $x+y$ is *real and positive*) is $x^u y^u$ (*i.e.*, that x is *real and positive* and that y also is *real and positive*). And the *strongest conclusion* we can infer from $(x+y)^u$ alone as our data is

$$x^w y^w + x^{ow} y^{ow} (x^u + y^u);$$

that is, that either x and y are *both imaginary*, or else that *neither of them is imaginary* while *one at least of them is real and positive.*

Observe that in the expression $(x+y)^u$ the symbol $(+)$ has its usual mathematical meaning of *addition;* whereas in $(x^u + y^u)$ and other places the symbol $(+)$ has its *logical* sense of the conjunction *or.* Yet there is no ambiguity, for, since x and y are numbers (or fractions), $x+y$ must also be a number or fraction; and, since x^u and y^u are *statements* (or *propositions*), $x^u + y^u$ must also be a *statement.*

[*Postscript.*

To show the utility of this logic of *three dimensions* (ϵ, η, θ), I may mention that (since I wrote the preceding) it has enabled me to discover a well-concealed fallacy in the arguments of two of my critics, Dr. Venn and Dr. Schröder. The latter, on pp. 258–260 of the second volume of his *Algebra der Logik*, says (if the translation given me be accurate) that formula 13 of my Third Paper in the *Proceedings* should be an *equation*, and not merely an *implication.* Translated into the notation of this Fifth Paper, Dr. Schröder asserts that $(\alpha\beta)_u : \alpha_u + \beta_u$, the converse of formula 13, is also *certain.* Denoting this statement by ϕ, we readily get (see p. 172) the result $\phi^\eta ! \, \alpha^\theta \beta^\theta u^\eta (\alpha\beta)^\eta$, which shows that Dr. Schröder's assertion ϕ is wrong when α and β are *contradictory variables* and u an *impossibility.*

Dr. Venn, on p. 377 of his *Symbolic Logic* (first edition), says that my compound implication $(x : y) : (z : w)$ is synonymous with Boole's equation

$$(1 - x\bar{y}) z\bar{w} = 0;$$

an assertion founded on the assumption that Boole's equation is also synonymous with my implication $zw' : xy'$. Dr. Venn's assertion therefore implies that ϕ is *certain* when it denotes $(x_y : z_w) : (zw' : xy')$. Now, by the process of p. 172, we get $\phi^\eta ! \, (x^\theta y^\eta + x^\epsilon y^\theta) z^\epsilon w^\eta$; so that there are two cases at least, $x^\theta y^\eta z^\epsilon w^\eta$ and $x^\epsilon y^\theta z^\epsilon w^\eta$, in which ϕ is an *impossibility*. This may be verified by substituting for x, y, z, w in the statement ϕ their *class values* or *indices*, as on pp. 180 and 181.

Dr. Venn and Dr. Schröder appear to have both stumbled over the same fallacy, and one which, in a logic of only *two dimensions* (*unity* and *zero*), is extremely difficult to avoid. They confounded *truth* with *certainty*, and *falsehood* with *impossibility;* labelling the first two indiscriminately with the symbol 1, and the last two with the symbol 0. When Dr. Venn's *Symbolic Logic* first appeared, I did not discover his error; but, in the light of this new logic of *three dimensions*, its detection becomes a mere matter of accuracy in working out an easy algebraic process.—*April 25th.*]

The Differentiation of Spherical Harmonics. By E. G. Gallop.

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Let $P(a, b, c)$ denote the surface spherical harmonic

$$\frac{(-1)^n}{a!\,b!\,c!}\,r^{n+1}\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r},$$

where $n = a+b+c$, and $r^2 = x^2+y^2+z^2$. This definition only applies when a, b, c are positive integers or zero; in the case when any one of these is a negative integer, $P(a, b, c)$ is defined to be zero.

If the solid harmonic $r^nP(a, b, c)$ is differentiated, with respect to x, y, z, m times in all, there results a solid harmonic of degree $n-m$, which it must be possible to express as a sum of harmonics of the type $r^{n-m}P(a', b', c')$. In this communication a general formula is obtained expressing symbolically the result in this form, and application made to zonal and tesseral harmonics. It is also shown that a precisely similar general formula holds for the functions obtained by differentiating $\frac{1}{r^s}$ instead of $\frac{1}{r}$, and for hyper-spherical harmonics in any number of variables.

1. To prove that

$$\begin{aligned}\frac{\partial}{\partial x}r^nP(a, b, c) = -r^{n-1}\big[&(a+1)\,P(a+1, b-2, c)\\ &+(a+1)\,P(a+1, b, c-2)-(2n-a)\,P(a-1, b, c)\big]. \qquad (1)\end{aligned}$$

We have

$$\frac{\partial}{\partial x} r^{2n+1}\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}=r^{2n-1}\Big[(x^2+y^2+z^2)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$$
$$+(2n+1)\,x\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}\Big]. \qquad (2)$$

Now

$$\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{x^2+y^2+z^2}{r}=\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}r$$
$$=\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{x}{r}.$$

Therefore

$$(x^2+y^2+z^2)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}+2\,(a+1)\,x\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$$
$$+2by\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-1}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}+2cz\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c-1}\frac{1}{r}$$
$$+(a+1)\,a\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}+b(b-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-2}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$$
$$+c\,(c-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c-2}\frac{1}{r}$$
$$=x\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}+a\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}.$$

Eliminating $(x^2+y^2+z^2)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$

from (2) by help of this equation, we find

$$r^{-(2n-1)}\frac{\partial}{\partial x}r^{2n+1}\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$$
$$=-b\,(b-1)\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^{b-2}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}-c\,(c-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c-2}\frac{1}{r}$$
$$-a^2\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}+2\,(b+c)\,x\left(\frac{\partial}{\partial x}\right)^{a}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}$$
$$-2by\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-1}\left(\frac{\partial}{\partial z}\right)^{c}\frac{1}{r}-2cz\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b}\left(\frac{\partial}{\partial z}\right)^{c-1}\frac{1}{r}. \qquad (3)$$

Now
$$\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\frac{1}{r}=0.$$

Operating with $\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^{b-1}\left(\frac{\partial}{\partial z}\right)^c$ on this equation, we obtain

$$x\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}-y\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-1}\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}$$
$$+a\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}-(b-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-2}\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}=0.$$

Similarly

$$x\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}-z\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^{c-1}\frac{1}{r}$$
$$+a\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}-(c-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^{c-2}\frac{1}{r}=0.$$

Multiplying the last two equations by $-2b$ and $-2c$, and adding the left-hand sides to the right-hand of (3), we have

$$r^{-(2n-1)}\frac{\partial}{\partial x}r^{2n+1}\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}$$
$$=b(b-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^{b-2}\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}+c(c-1)\left(\frac{\partial}{\partial x}\right)^{a+1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^{c-2}\frac{1}{r}$$
$$-a(2n-a)\left(\frac{\partial}{\partial x}\right)^{a-1}\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r}.$$

This result is equivalent to the formula enunciated. It is to be particularly noticed that the formula holds even when one or more of the indices of the P-functions involved is a negative integer. Thus, if b is zero or unity, $b-2$ is negative, and in accordance with the definition $P(a+1, b-2, c)$ is zero.

2. In order to obtain the result of any number of differentiations it is convenient to introduce a symbolical notation.

Let λ be an operator which changes the a in $P(a, b, c)$ into $a-1$, so that
$$\lambda^m P(a, b, c) = P(a-m, b, c);$$
let μ change b into $b-1$, and let ν change c into $c-1$.

We may therefore write

$$P(a, b, c) = \lambda^{-a}\mu^{-b}\nu^{-c} P(0, 0, 0);$$

or, since $P(0, 0, 0) = 1$, we shall write more simply

$$P(a, b, c) = \lambda^{-a}\mu^{-b}\nu^{-c}.$$

With this notation we therefore have, if $n = a+b+c$,

$$r^{n+1}\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r} = (-1)^n a!\, b!\, c!\, \lambda^{-a}\mu^{-b}\nu^{-c}$$

$$= \lambda\mu\nu\left(\frac{\partial}{\partial\lambda}\right)^a\left(\frac{\partial}{\partial\mu}\right)^b\left(\frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu};$$

and, if $f_n(xyz)$ denotes a homogeneous integral function of degree n in x, y, z,

$$r^{n+1} f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = \lambda\mu\nu f_n\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}.$$

Any homogeneous integral function of x, y, z which satisfies Laplace's equation may be expressed symbolically. For, if $H_n(xyz)$ denotes such a function of degree n,

$$H_n(xyz) = (-1)^n\frac{2^n n!}{(2n)!} r^{2n+1} H_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= (-1)^n\frac{2^n n!}{(2n)!} r^n \lambda\mu\nu H_n\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}.$$

The harmonics $P(a, b, c)$ are not all independent; for, in consequence of Laplace's equation,

$$(a+1)(a+2) P(a+2, b, c) + (b+1)(b+2) P(a, b+2, c)$$
$$+ (c+1)(c+2) P(a, b, c+2) = 0;$$

or in the symbolical notation

$$\lambda\mu\nu\left(\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\frac{1}{\lambda^{a+1}\mu^{b+1}\nu^{c+1}} = 0.$$

Therefore, if f denotes any homogeneous integral function,

$$\lambda\mu\nu\left(\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) f(\lambda^{-1}, \mu^{-1}, \nu^{-1}) = 0. \tag{4}$$

This is the full equivalent of Laplace's equation as applied to the functions here considered.

3. The formula (1) of § 1 may now be written symbolically,

$$\frac{\partial}{\partial x} r^n (\lambda^{-a} \mu^{-b} \nu^{-c}) = - r^{n-1} \Big[(a+1) \lambda^{-(a+1)} \mu^{-(b-2)} \nu^{-c}$$

$$+ (a+1) \lambda^{-(a+1)} \mu^{-b} \nu^{-(c-2)} - (2n-a) \lambda^{-(a-1)} \mu^{-b} \nu^{-c} \Big]$$

$$= - r^{n-1} \big[(a+1) \lambda^{-1} (\lambda^2+\mu^2+\nu^2) - (2n+1) \lambda \big] \lambda^{-a} \mu^{-b} \nu^{-c}$$

$$= \quad r^{n-1} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{-(n-\frac{1}{2})} \frac{\partial}{\partial\lambda} \frac{(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}}{\lambda^{a+1} \mu^{b+1} \nu^{c+1}};$$

and generally

$$\frac{\partial}{\partial x} r^n \psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})$$

$$= r^{n-1} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{-(n-\frac{1}{2})} \frac{\partial}{\partial\lambda} \left[(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}} \frac{\psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})}{\lambda\mu\nu} \right].$$

Here, as elsewhere in this paper, the general functional symbol represents an integral homogeneous function of degree indicated by the suffix. To differentiate again with respect to x, we repeat the operations, writing $n-1$ for n, and thus obtain

$$\frac{\partial^2}{\partial x^2} r^n \psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})$$

$$= r^{n-2} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{-(n-\frac{3}{2})} \frac{\partial^2}{\partial\lambda^2} \frac{(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}} \psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})}{\lambda\mu\nu}.$$

Differentiations with respect to y and z follow the same rule, and therefore, if $f+g+h=m$,

$$\left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h r^n \psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})$$

$$= r^{n-m} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{m-n-\frac{1}{2}} \left(\frac{\partial}{\partial\lambda}\right)^f \left(\frac{\partial}{\partial\mu}\right)^g \left(\frac{\partial}{\partial\nu}\right)^h \frac{(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}} \psi_n(\lambda^{-1}, \mu^{-1}, \nu^{-1})}{\lambda\mu\nu};$$

and more generally

$$\phi_m \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) r^n \psi \, (\lambda^{-1}, \mu^{-1}, \nu^{-1})$$

$$= r^{n-m} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{m-n-\frac{1}{2}} \phi_m \left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right) \frac{(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}} \psi_n (\lambda^{-1}, \mu^{-1}, \nu^{-1})}{\lambda\mu\nu}. \tag{5}$$

The last result may also be written in the equivalent form

$$r^{m-n}\phi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) r^{2n+1}\psi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)^{m-n-\frac{1}{2}}\phi_m\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}\psi_n\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}. \tag{6}$$

4. It is a good illustration of the formulæ to verify that the values of the differential coefficients just found are such as to make

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) r^n\,(\lambda^{-a}\mu^{-b}\nu^{-c})$$

vanish. This expression, by formula (5), is equal to

$$r^{n-2}\lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)^{-(n-\frac{3}{2})}\left(\frac{\partial^2}{\partial\lambda^2}+\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\nu^2}\right)\frac{(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}}{\lambda^{a+1}\mu^{b+1}\nu^{c+1}}.$$

After a few transformations this reduces to

$$r^{n-2}\Big[-4\,(2n+1)\,\frac{\lambda^2+\mu^2+\nu^2}{\lambda^a\mu^b\nu^c}$$

$$+\lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)^2\left(\frac{\partial^2}{\partial\lambda^2}+\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\nu^2}\right)\frac{1}{\lambda^{a+1}\mu^{b+1}\nu^{c+1}}\Big]$$

$$= r^{n-2}\lambda\mu\nu\left(\frac{\partial^2}{\partial\lambda^2}+\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\nu^2}\right)\frac{(\lambda^2+\mu^2+\nu^2)^2}{\lambda^{a+1}\mu^{b+1}\nu^{c+1}};$$

and this expression vanishes in virtue of equation (4).

5. The following lemma sometimes facilitates the application of the formulæ of § 3.

If $$f_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi_n(x, y, z) = \psi(x, y, z),$$

then $$xyzf_m(x, y, z)\,\phi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{xyz}$$

differs from $$(-1)^m\,xyz\,\psi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{xyz}$$

by terms each of which contains a positive power of x, y, or z; and consequently in the symbolical notation

$$\lambda\mu\nu f_m(\lambda, \mu, \nu)\, \phi_n\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}$$

$$= (-1)^m\, \lambda\mu\nu\psi_n\left(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}.$$

To prove this lemma it is sufficient to examine the case in which

$$f_m(x, y, z) = x^a y^b z^c, \quad \text{and} \quad \phi_n(x, y, z) = x^f y^g z^h,$$

where $$a+b+c = m, \quad \text{and} \quad f+g+h = n.$$

In this case, if we first suppose $a \leqq f$, $b \leqq g$, $c \leqq h$,

$$\psi(x, y, z) = \frac{f!\, g!\, h!}{(f-a)!\, (g-b)!\, (h-c)!}\, x^{f-a} y^{g-b} z^{h-c}.$$

Therefore

$$xyz\, \psi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{xyz} = (-1)^{n-m}\, \frac{f!\, g!\, h!}{x^{f-a} y^{g-b} z^{h-c}}$$

$$= (-1)^m\, xyz\, x^a y^b z^c \left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h \frac{1}{xyz}$$

$$= (-1)^m\, xyz f_m(x, y, z)\, \phi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{xyz}.$$

Next suppose $a > f$; then $\psi = 0$. And

$$xyz f_m(x, y, z)\, \phi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{xyz} = (-1)^n\, xyz\, x^a y^b z^c\, \frac{f!\, g!\, h!}{x^{f+1} y^{g+1} z^{h+1}},$$

and therefore contains a power of x with positive exponent.

6. A transformation of the formula (6) of § 3 may be obtained in the following manner.

Let $$\phi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = \frac{H_m(x, y, z)}{r^{2m+1}}.$$

Then, by Dr. Hobson's "Theorem in Differentiation," *Proceedings*, Vol. XXIV., we see that, for operations on a function which satisfies Laplace's equation,

$$\phi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = (-1)^m \frac{2^m m!}{(2m)!} H_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Now, by the formula (6),

$$r^{m-n} H_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) r^{2n+1} \psi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r}$$

$$= \lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)^{m-n-\frac{1}{2}} H_m\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}\right)(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}$$

$$\times \psi_n\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}\right)\frac{1}{\lambda\mu\nu}.$$

To expand this expression write

$$\frac{\partial}{\partial\lambda} = \frac{\partial}{\partial\lambda_1} + \frac{\partial}{\partial\lambda_2}, \quad \frac{\partial}{\partial\mu} = \frac{\partial}{\partial\mu_1} + \frac{\partial}{\partial\mu_2}, \quad \frac{\partial}{\partial\nu} = \frac{\partial}{\partial\nu_1} + \frac{\partial}{\partial\nu_2},$$

where suffix 1 indicates differentiation of $(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}$, and suffix 2 indicates differentiation of ψ_n.

Then, by the extension of Taylor's theorem, if $H_m^{(\lambda)}(\lambda, \mu, \nu)$ denotes

$$\frac{\partial}{\partial\lambda} H_m(\lambda, \mu, \nu), \text{ \&c.,}$$

we have
$$H_m\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}\right)$$

$$= H_m\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}\right) + H_m^{(\lambda)}\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}\right)\frac{\partial}{\partial\lambda_2}$$

$$+ H_m^{(\mu)}\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}\right)\frac{\partial}{\partial\mu_2} + H_m^{(\nu)}\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}\right)\frac{\partial}{\partial\nu_2}$$

$$+ \frac{1}{2!} H_m^{(\lambda\lambda)}\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \nu_1}\right)\left(\frac{\partial}{\partial\lambda_2}\right)^2 + \dots .$$

But, by § 7, of Dr. Hobson's paper, p. 65, if $\chi_m(x, y, z)$ satisfies Laplace's equation,

$$\chi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) r^s = s\,(s-2)\,\dots\,(s-2m+2)\,\chi_m(x, y, z)\, r^{s-2m}.$$

Hence

$$H_m\left(\frac{\partial}{\partial\lambda},\ \frac{\partial}{\partial\mu},\ \frac{\partial}{\partial\nu}\right)(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}$$

$$=(2n+1)(2n-1)\dots(2n-2m+3)(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}-m}H_m(\lambda,\mu,\nu),$$

$$H_m^{(\lambda)}\left(\frac{\partial}{\partial\lambda},\ \frac{\partial}{\partial\mu},\ \frac{\partial}{\partial\nu}\right)(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}$$

$$=(2n+1)\dots(2n-2m+5)(\lambda^2+\mu^2+\nu^2)^{n+\frac{3}{2}-m}\frac{\partial}{\partial\lambda}H_m(\lambda,\mu,\nu).$$

Therefore, writing $\rho^2=\lambda^2+\mu^2+\nu^2,$

and $$\Omega=2\left(\frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\lambda_2}+\frac{\partial}{\partial\mu_1}\frac{\partial}{\partial\mu_2}+\frac{\partial}{\partial\nu_1}\frac{\partial}{\partial\nu_2}\right),$$

$$r^{m-n}H_m\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)r^{2n+1}\psi_n\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$=(2n+1)(2n-1)\dots(2n-2m+3)\,\lambda\mu\nu$$

$$\times\left[1+\frac{\rho^2\Omega}{2\,.\,2n-2m+3}+\frac{\rho^4\Omega^2}{2.4.2n-2m+3.2n-2m+5}+\dots\right]$$

$$H_m(\lambda,\mu,\nu)\,\psi_n\left(\frac{\partial}{\partial\lambda},\ \frac{\partial}{\partial\mu},\ \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu},$$

where suffix 1 indicates differentiation of H_m, and suffix 2 indicates differentiation of ψ_n.

If $$\psi_n\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{r}=\frac{K_n(x,y,z)}{r^{2n+1}},$$

we have

$$(-1)^{m+n}\frac{(2m)!}{2^m m!}\frac{(2n)!}{2^n n!}r^{m-n}\phi_m\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)r^{2n+1}\psi_n\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$=r^{m-n}H_m\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)r^{2n+1}K_n\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$=(2n+1)(2n-1)\dots(2n-2m+3)\,\lambda\mu\nu$$

$$\times\left[1+\frac{\rho^2\Omega}{2.2n-2m+3}+\frac{\rho^4\Omega^2}{2.4.2n-2m+3.2n-2m+5}+\dots\right]$$

$$H_m(\lambda,\mu,\nu)\,K_n\left(\frac{\partial}{\partial\lambda},\ \frac{\partial}{\partial\mu},\ \frac{\partial}{\partial\nu}\right)\frac{1}{\lambda\mu\nu}. \qquad (7)$$

With this result we may compare the following formula for the differentiation of a harmonic K_n of degree $-(n+1)$. By Dr. Hobson's theorem

$$r^{2m+2n+1} H_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) K_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r}$$

$$= (-1)^{m+n} \frac{(2m+2n)!}{2^{m+n}\,(m+n)!}$$
$$\times\left[1-\frac{r^2\nabla^2}{2\,(2n+2m-1)}+\frac{r^4\nabla^4}{2.4\,(2n+2m-1)(2n+2m-3)}-\dots\right]$$
$$H_m(x, y, z)\, K_n(x, y, z)$$

$$= (-1)^{m+n} \frac{(2m+2n)!}{2^{m+n}\,(m+n)!}$$
$$\times\left[1-\frac{r^2O^n}{2\,(2n+2m-1)}+\frac{r^4O^2}{2.4\,(2n+2m-1)(2n+2m-3)}-\dots\right]$$
$$H_m(x, y, z)\, K_n(x, y, z), \tag{8}$$

where $$O = 2\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2}+\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right),$$

and suffix 1 indicates differentiation of H_m, suffix 2 indicates differentiation of K_n.

The operators enclosed in square brackets in (7) and (8) differ only in form by the interchange of $-(n+1)$ and n.

7. With a view to applying the formulæ of § 3 to the differentiation of a solid zonal harmonic it is desirable to introduce some additional considerations.

It is obvious that the result of the formula (5) would not be affected if the expressions

$$(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}, \quad (\lambda^2+\mu^2+\nu^2)^{m-n-\frac{1}{2}}$$

were expanded in ascending powers of λ, μ, and ν^{-1}. We will therefore consider a harmonic $S(a, b, c)$ defined by the equation

$$S(a, b, c) = \lambda^{-a}\mu^{-b}\nu^{-c}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{-(n+\frac{1}{2})},$$

where $$a+b+c = n,$$

and where the right-hand side is to be expanded in ascending powers of λ, μ, and ν^{-1}. When a and b are positive integers this harmonic S consists of a finite number of P-harmonics; for after some term the exponents of λ and μ are positive, and the succeeding terms represent P-harmonics of zero value. If a or b is a negative integer, the S-harmonic is zero; but, if c is negative, S is not necessarily zero. If $a+b+c=0$, $S(a, b, c)$ reduces to a single P-harmonic of order zero, that is, to a constant. For in this case

$$S(a, b, c) = \lambda^{-a}\mu^{-b}\nu^{-c}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{-\frac{1}{2}}$$

$$= \Sigma \frac{(-1)^{p+q}(2p+2q)!}{2^{2p+2q}(p+q)!\,p!\,q!} P(a-2p, b-2q, c+2p+2q),$$

where the summation extends to all values of p and q which make $a-2p$, $b-2q$, $c+2p+2q$ positive integers or zero. But, since the sum of these three quantities is zero, the only values of p and q applicable are such as make each of the three quantities zero; and therefore $p=\frac{1}{2}a$, $q=\frac{1}{2}b$. Hence, if either a or b is odd, $S(a, b, c)$ vanishes, when

$$a+b+c=0;$$

but, if a and b are both even,

$$S(a, b, c) = \frac{(-1)^{\frac{1}{2}(a+b)}(a+b)!}{2^{a+b}\,\frac{a+b}{2}!\,\frac{a}{2}!\,\frac{b}{2}!}.$$

Or we may write this last result more simply. If

$$c = a+b,$$

$$S(a, b, -c) = \frac{(-1)^{\frac{1}{2}c}}{2^c}\,\frac{c!}{\frac{a}{2}!\,\frac{b}{2}!\,\frac{c}{2}!}. \qquad (9)$$

8. The differential coefficients of the S-harmonic are easily found by formula (5). We have, putting

$$f(\lambda^{-1}, \mu^{-1}, \nu^{-1}) = \lambda^{-a}\mu^{-b}\nu^{-c}\left[1+\nu^{-2}(\lambda^2+\mu^2)\right]^{-(n+\frac{1}{2})},$$

$$\frac{\partial}{\partial x} r^n S(a, b, c) = r^{n-1}\lambda\mu\nu^{-(2n-2)}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{-(n-\frac{1}{2})}\frac{\partial}{\partial\lambda}\frac{\nu^{2n-c}}{\lambda^{a+1}\mu^{b+1}}$$

$$= -(a+1)\,r^{n-1}S(a+1, b, c-2).$$

Similarly,

$$\frac{\partial}{\partial y} r^n S(a, b, c) = -(b+1)\,r^{n-1}S(a, b+1, c-2),$$

$$\frac{\partial}{\partial z} r^n S(a, b, c) = (2n-c)\,r^{n-1}S(a, b, c-1);$$

and therefore

$$\left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h r^n S(a, b, c)$$

$$= (-1)^{f+g} \frac{(a+f)!\,(b+g)!\,(2n-c)!}{a!\,b!\,(2n-c-h)!} r^{n-f-g-h}$$

$$\times S(a+f, b+g, c-2f-2g-h). \quad (10)$$

The expression for $r^n S(a, b, c)$ in terms of x, y, z follows at once. For, if

$$f+g+h = n = a+b+c,$$

the sum of the indices of the S-harmonic last written is zero, and the value of this harmonic can be found from formula (9). Hence the coefficient of $x^f y^g z^h$ in $r^n S(a, b, c)$ is

$$\frac{(-1)^{f+g}}{f!\,g!\,h!} \frac{(a+f)!\,(b+g)!\,(2n-c)!}{a!\,b!\,(2n-c-h)!} \frac{(-1)^{n-\frac{1}{2}(c+h)}}{2^{2n-c-h}} \frac{(2n-c-h)!}{\left(n-\frac{c+h}{2}\right)!\,\frac{a+f}{2}!\,\frac{b+g}{2}!}$$

$$= \frac{(-1)^{\frac{1}{2}(c-h)}}{2^{2n-c-h}} \frac{(a+f)!\,(b+g)!\,(2n-c)!}{a!\,b!\,f!\,g!\,h!\,\frac{a+f}{2}!\,\frac{b+g}{2}!\left(n-\frac{c+h}{2}\right)!}. \quad (12)$$

9. The differential coefficients of the P-harmonics may now be found by expressing the P-harmonics in terms of S-harmonics. Such expression is always possible. For

$$P(a, b, c)$$

$$= \lambda^{-a}\mu^{-b}\nu^{-c}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{n+\frac{1}{2}}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{-(n+\frac{1}{2})}$$

$$= S(a, b, c) + (n+\tfrac{1}{2})\left\{S(a-2, b, c+2) + S(a, b-2, c+2)\right\} + \ldots. \quad (13)$$

The number of S-harmonics in this expression will be finite, for after some term of the series one of the first two indices of the S-harmonic will be negative, and the succeeding terms will be S-harmonics of zero value.

The expansion of $r^n P(a, b, c)$ in powers of x, y, z may be obtained from the formula (12) with the help of (13).

In particular the zonal harmonic P_n is equal to

$$P(0, 0, n) = \nu^{-n} = S(0, 0, n).$$

Therefore

$$\left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h r^n P_n$$

$$= (-1)^{f+g} f!\, g!\, \frac{n!}{(n-h)!}\, r^{n-f-g-h}\, S(f,\, g,\, n-2f-2g-h). \quad (14)$$

For convenience, write $f+g+h = m.$

Then $S(f,\, g,\, n-2f-2g-h)$

$$= \lambda^{-f} \mu^{-g} \nu^{-n+2m-h} \left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{-(n-m+\frac{1}{2})}$$

$$= \lambda^{-f} \mu^{-g} \nu^{-n+2m-h} \Sigma\, (-1)^{p+q} \frac{\Pi\,(n-m+p+q-\frac{1}{2})}{p!\; q!\; \Pi\,(n-m-\frac{1}{2})} \lambda^{2p} \mu^{2q} \nu^{-2p-2q}.$$

Hence

$$\left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h r^n P_n = (-1)^{f+g} f!\, g!\, \frac{n!}{(n-h)!}\, r^{n-m}$$

$$\times\, \Sigma\, (-1)^{p+q} \frac{\Pi\,(n-m+p+q-\frac{1}{2})}{p!\; q!\; \Pi\,(n-m-\frac{1}{2})} P(f-2p,\, g-2q, n-2m+h+2p+2q), \quad (15)$$

where the summation extends to all positive integral values of p and q which make $f-2p$, $g-2q$, $n-2m+h+2p+2q$ all positive or zero.

When $f=0$ and $g=0$, we therefore require $p=0$ and $q=0$; and therefore

$$\left(\frac{\partial}{\partial z}\right)^h r^n P_n = \frac{n!}{(n-h)!}\, r^{n-h} P\,(0,\, 0,\, n-h)$$

$$= \frac{n!}{(n-h)!}\, r^{n-h} P_{n-h}.$$

Again, by formula (12), the coefficient of $x^{2f} y^{2g} z^{n-2f-2g}$ in $r^n P_n$ or $r^n S\,(0,\, 0,\, n)$ is

$$\frac{(-1)^{f+g}}{2^{2f+2g}} \frac{(2f)!\; (2g)!\; n!}{(2f)!\; (2g)!\; (n-2f-2g)!\; f!\; g!\; (f+g)!}$$

$$= \frac{(-1)^{f+g}}{2^{2f+2g}} \frac{n!}{(n-2f-2g)!\; \{(f+g)!\}^2} \frac{(f+g)!}{f!\; g!}.$$

Hence writing m for $f+g$, and gathering together terms for which the sum of the exponents of x and y is $2m$, we find

$$r^n P_n = \Sigma \frac{(-1)^m}{2^{2m}} \frac{n!}{(n-2m)!\,(m!)^2} (x^2+y^2)^m z^{n-2m};$$

whence we deduce the known expansion

$$P_n(\cos\theta) = \cos^n\theta - \frac{n(n-1)}{2^2}\sin^2\theta\cos^{n-2}\theta + \dots$$

$$+(-1)^m \frac{n(n-1)\dots(n-2m+1)}{(2.4\dots 2m)^2}\sin^{2m}\theta\cos^{n-2m}\theta + \dots .$$

This result may be regarded as a verification of the formulæ obtained.

We are now in a position to determine what the S-harmonic really represents. In equation (14) write a for f, b for g, c for h, so that

$$m = a+b+c,$$

and take

$$n = 2m.$$

We have, therefore,

$$r^m S(a, b, c) = (-1)^{a+b} \frac{(2m-c)!}{a!\,b!\,(2m)!} \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial y}\right)^b \left(\frac{\partial}{\partial z}\right)^c r^{2m} P_{2m}. \qquad (16)$$

Equation (13) therefore shows that every P-harmonic may be obtained by differentiating a solid zonal harmonic; and the result may be thus stated. The expression for

$$\frac{(-1)^n}{a!\,b!\,c!} r^{2n+1} \left(\frac{\partial}{\partial x}\right)^a \left(\frac{\partial}{\partial y}\right)^b \left(\frac{\partial}{\partial z}\right)^c \frac{1}{r},$$

where $a+b+c=n$, may be obtained by expanding

$$\lambda^{-a}\mu^{-b}\nu^{-c}\left\{1+\nu^{-2}(\lambda^2+\mu^2)\right\}^{n+\frac{1}{2}},$$

in ascending powers of λ, μ and ν^{-1}, leaving out all terms which have any exponent positive, and replacing any term $\lambda^{-f}\mu^{-g}\nu^{-h}$ which occurs with negative or zero exponents by

$$(-1)^{f+g} \frac{(2n-h)!}{f!\,g!\,(2n)!} \left(\frac{\partial}{\partial x}\right)^f \left(\frac{\partial}{\partial y}\right)^g \left(\frac{\partial}{\partial z}\right)^h r^{2n} P_{2n}.$$

11. We will next consider the tesseral harmonics. Let $Y(a, b, c)$ be a harmonic defined by the equation

$$Y(a, b, c) = (-1)^n r^{n+1} \left(\frac{\partial}{\partial\xi}\right)^a \left(\frac{\partial}{\partial\eta}\right)^b \left(\frac{\partial}{\partial z}\right)^c \frac{1}{r},$$

where $n = a+b+c$, $\xi = x+iy$, $\eta = x-iy$, and $r^2 = z^2+\xi\eta$.

These harmonics are not all independent, but, in virtue of Laplace's equation,

$$4Y(a+1, b+1, c-2) = -Y(a, b, c);$$

and therefore the harmonics $Y(a, b, c)$ and $Y(a', b', c')$ will differ only by a numerical factor if

$$2a+c = 2a'+c' \text{ and } 2b+c = 2b'+c'.$$

The real and imaginary parts of $Y(a, b, c)$ are tesseral harmonics.

Thus, in the notation of Ferrers' *Spherical Harmonics*, where

$$T_n^\sigma(\mu) = (1-\mu^2)^{\frac{1}{2}\sigma} \frac{d^\sigma P_n}{d\mu^\sigma},$$

$$Y(\sigma, 0, n-\sigma) = \frac{(n-\sigma)!}{2^\sigma} T_n^\sigma(\mu)(\cos\sigma\phi + i\sin\sigma\phi),$$

$$Y(0, \sigma, n-\sigma) = \frac{(n-\sigma)!}{2^\sigma} T_n^\sigma(\mu)(\cos\sigma\phi - i\sin\sigma\phi).$$

In the symbolical notation,

$$Y(a, b, c) = \frac{(-1)^n}{2^{a+b}} r^{n+1} \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^a \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^b \left(\frac{\partial}{\partial z}\right)^c \frac{1}{r}$$

$$= \frac{(-1)^n}{2^{a+b}} \lambda\mu\nu \left(\frac{\partial}{\partial\lambda} - i\frac{\partial}{\partial\mu}\right)^a \left(\frac{\partial}{\partial\lambda} + i\frac{\partial}{\partial\mu}\right)^b \left(\frac{\partial}{\partial\nu}\right)^c \frac{1}{\lambda\mu\nu}.$$

12. We have therefore, by (6) of § 3,

$$r^{-(n-1)} \frac{\partial}{\partial z} r^n Y(a, b, c)$$

$$= \frac{(-1)^n}{2^{a+b}} \lambda\mu\nu (\lambda^2+\mu^2+\nu^2)^{-(n-\frac{1}{2})}$$

$$\frac{\partial}{\partial\nu}\left[(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}} \left(\frac{\partial}{\partial\lambda} - i\frac{\partial}{\partial\mu}\right)^a \left(\frac{\partial}{\partial\lambda} + i\frac{\partial}{\partial\mu}\right)^b \left(\frac{\partial}{\partial\nu}\right)^c \frac{1}{\lambda\mu\nu}\right]$$

$$= \frac{(-1)^n}{2^{a+b}} \lambda\mu\nu \left[(2n+1)\nu + (\lambda^2+\mu^2+\nu^2)\frac{\partial}{\partial\nu}\right]$$

$$\left(\frac{\partial}{\partial\lambda} - i\frac{\partial}{\partial\mu}\right)^a \left(\frac{\partial}{\partial\lambda} + i\frac{\partial}{\partial\mu}\right)^b \left(\frac{\partial}{\partial\nu}\right)^c \frac{1}{\lambda\mu\nu}.$$

Now $\nu\left(\frac{\partial}{\partial\nu}\right)^{c}\frac{1}{\lambda\mu\nu}=-c\left(\frac{\partial}{\partial\nu}\right)^{c-1}\frac{1}{\lambda\mu\nu}$, and therefore

$$\lambda\mu\nu\,(2n+1)\,\nu\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b}\left(\frac{\partial}{\partial\nu}\right)^{c}\frac{1}{\lambda\mu\nu}$$

$$=(-1)^{n}\,2^{a+b}(2n+1)\,cY(a,\,b,\,c-1).$$

Also $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x-iy)^{a}\,(x+iy)^{b}\,z^{c+1}$

$$=4ab\,(x-iy)^{a-1}\,(x+iy)^{b-1}\,z^{c+1}+(c+1)\,c\,(x-iy)^{a}\,(x+iy)^{b}\,z^{c-1};$$

therefore, by the lemma of § 5,

$$\lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b}\left(\frac{\partial}{\partial\nu}\right)^{c+1}\frac{1}{\lambda\mu\nu}$$

$$=\lambda\mu\nu\Bigg[4ab\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a-1}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b-1}\left(\frac{\partial}{\partial\nu}\right)^{c+1}$$

$$+c\,(c+1)\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b}\left(\frac{\partial}{\partial\nu}\right)^{c-1}\Bigg]\frac{1}{\lambda\mu\nu}$$

$$=(-1)^{n-1}2^{a+b}\big[ab\,Y(a-1,\,b-1,\,c+1)+c\,(c+1)\,Y(a,\,b,\,c-1)\big]$$

$$=(-1)^{n}\,2^{a+b}\big[4ab-c\,(c+1)\big]\,Y(a,\,b,\,c-1).$$

Hence

$$r^{-(n-1)}\frac{\partial}{\partial z}r^{n}\,Y(a,\,b,\,c)=\big[(2n+1)\,c+4ab-c\,(c+1)\big]\,Y(a,\,b,\,c-1)$$

$$=(2a+c)(2b+c)\,Y(a,\,b,\,c-1) \tag{17}$$

$$=-\tfrac{1}{4}(2a+c)(2b+c)\,Y(a-1,\,b-1,\,c+1). \tag{17a}$$

Again,

$$r^{-(n-1)}\frac{\partial}{\partial\xi}r^{n}\,Y(a,\,b,\,c)=r^{-(n-1)}\tfrac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)r^{n}\,Y(a,\,b,\,c)$$

$$=\frac{(-1)^{n}\lambda\mu\nu}{2^{a+b+1}}(\lambda^2+\mu^2+\nu^2)^{-(n-\frac{1}{2})}\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)$$

$$\left[(\lambda^2+\mu^2+\nu^2)^{n+\frac{1}{2}}\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b}\left(\frac{\partial}{\partial\nu}\right)^{c}\frac{1}{\lambda\mu\nu}\right]$$

$$=\frac{(-1)^{n}}{2^{a+b+1}}\lambda\mu\nu\left[(2n+1)(\lambda-i\mu)+(\lambda^2+\mu^2+\nu^2)\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)\right]$$

$$\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b}\left(\frac{\partial}{\partial\nu}\right)^{c}\frac{1}{\lambda\mu\nu}.$$

Now, $\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(x-iy)^a(x+iy)^b z^c = 2b\,(x-iy)^a(x+iy)^{b-1}z^c$;

therefore, by the lemma of § 5,

$$\lambda\mu\nu\,(\lambda-i\mu)\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^a\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^b\left(\frac{\partial}{\partial\nu}\right)^c\frac{1}{\lambda\mu\nu}$$

$$=-2b\lambda\mu\nu\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^a\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b-1}\left(\frac{\partial}{\partial\nu}\right)^c\frac{1}{\lambda\mu\nu}$$

$$=(-1)^n 2^{a+b}\,bY(a,\,b-1,\,c).$$

Also
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x-iy)^{a+1}(x+iy)^b z^c$$

$$=4b\,(a+1)(x-iy)^a(x+iy)^{b-1}z^c+c\,(c-1)(x-iy)^{a+1}(x+iy)^b z^{c-2};$$

therefore, by the lemma,

$$\lambda\mu\nu\,(\lambda^2+\mu^2+\nu^2)\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a+1}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^b\left(\frac{\partial}{\partial\nu}\right)^c\frac{1}{\lambda\mu\nu}$$

$$=4b\,(a+1)\,\lambda\mu\nu\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^a\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^{b-1}\left(\frac{\partial}{\partial\nu}\right)^c\frac{1}{\lambda\mu\nu}$$

$$+c\,(c-1)\,\lambda\mu\nu\left(\frac{\partial}{\partial\lambda}-i\frac{\partial}{\partial\mu}\right)^{a+1}\left(\frac{\partial}{\partial\lambda}+i\frac{\partial}{\partial\mu}\right)^b\left(\frac{\partial}{\partial\nu}\right)^{c-2}\frac{1}{\lambda\mu\nu}$$

$$=(-1)^{n-1}2^{a+b+1}\left[(a+1)\,bY(a,\,b-1,\,c)+c\,(c-1)\,Y(a+1,\,b,\,c-2)\right]$$

$$=(-1)^{n-1}2^{a+b+1}\left[(a+1)\,b-\tfrac{1}{4}c\,(c-1)\right]Y(a,\,b-1,\,c).$$

Hence
$$r^{-(n-1)}\frac{\partial}{\partial\xi}r^n Y(a,\,b,\,c)$$

$$=\left[\tfrac{1}{2}(2n+1)\,b-(a+1)\,b+\tfrac{1}{4}c\,(c-1)\right]Y(a,\,b-1,\,c)$$

$$=\tfrac{1}{4}\left[2b\,(2n-2a-1)+c\,(c-1)\right]Y(a,\,b-1,\,c)$$

$$=\tfrac{1}{4}\,(2b+c)(2b+c-1)\,Y(a,\,b-1,\,c) \qquad (18)$$

$$=-(2b+c)(2b+c-1)\,Y(a+1,\,b,\,c-2). \qquad (18a)$$

Similarly

$$r^{-(n-1)}\frac{\partial}{\partial\eta}r^n Y(a,\,b,\,c)=\tfrac{1}{4}\,(2a+c)(2a+c-1)\,Y(a-1,\,b,\,c) \qquad (19)$$

$$=-\,(2a+c)(2a+c-1)\,Y(a,\,b+1,\,c-2). \qquad (19a)$$

Hence, if $m = f+g+h$,

$$r^{m-n}\left(\frac{\partial}{\partial\xi}\right)^f\left(\frac{\partial}{\partial\eta}\right)^g\left(\frac{\partial}{\partial z}\right)^h r^n Y(a,\, b,\, c)$$

$$= \frac{(2a+c)!\ (2b+c)!}{2^{2f+2g}(2a+c-2g-h)!\ (2b+c-2f-h)!}\, Y(a-g,\, b-f,\, c-h)\,; \quad (20)$$

but the result is zero if either $2a+c < 2g+h$ or $2b+c < 2f+h$. If $a-g$, $b-f$, or $c-h$ is negative, $Y(a-g,\, b-f,\, c-h)$ should be replaced by a form obtained by use of the forms (17a), (18a), (19a). This will be equivalent to transforming $Y(a-g,\, b-f,\, c-h)$ directly by the formula

$$4Y(a+1,\, b+1,\, c-2) = -Y(a,\, b,\, c).$$

The expansion of $r^n Y(a,\, b,\, c)$ in powers of ξ, η, z may be at once obtained from (20). For the coefficient of $\xi^f\eta^g z^h$, where $f+g+h = n$, is equal to

$$\frac{1}{f!\ g!\ h!}\left(\frac{\partial}{\partial\xi}\right)^f\left(\frac{\partial}{\partial\eta}\right)^g\left(\frac{\partial}{\partial z}\right)^h r^n Y(a,\, b,\, c).$$

The coefficient will be zero unless both

$$2a+c \geqq 2g+h \quad \text{and} \quad 2b+c \geqq 2f+h.$$

Neither inequality can hold, since

$$f+g+h = a+b+c.$$

Hence the only case in which the result is not zero is when

$$2a+c = 2g+h \quad \text{and} \quad 2b+c = 2f+h.$$

If these conditions are satisfied, the coefficient is

$$\frac{1}{f!\ g!\ h!}(-1)^{a-g}\frac{(2a+c)!\ (2b+c)!}{2^{2a+2f}}.$$

Therefore

$$r^n Y(\sigma,\, 0,\, n-\sigma) = \frac{(n+\sigma)!\ (n-\sigma)!}{2^{2\sigma}}\Sigma\frac{(-1)^f}{2^{2f}}\frac{\xi^f\eta^{f+\sigma}z^{n-\sigma-2f}}{f!\ (f+\sigma)!\ (n-\sigma-2f)!}\,;$$

$$r^n Y(0,\, \sigma,\, n-\sigma) = \frac{(n+\sigma)!\ (n-\sigma)!}{2^{2\sigma}}\Sigma\frac{(-1)^f}{2^{2f}}\frac{\xi^{f+\sigma}\eta^f z^{n-\sigma-2f}}{(f+\sigma)!\, f!\ (n-\sigma-2f)!}\,;$$

whence the expressions for tesseral harmonics involving powers of $\cos\theta$ and $\sin\theta$ follow directly.

13. The preceding results are applicable with very slight modification to the functions obtained by differentiating $\frac{1}{r^s}$ instead of $\frac{1}{r}$.

If $P_s(a, b, c)$ is defined to be equal to

$$\frac{(-1)^n}{a!\,b!\,c!}\,r^{n+s}\left(\frac{\partial}{\partial x}\right)^a\left(\frac{\partial}{\partial y}\right)^b\left(\frac{\partial}{\partial z}\right)^c\frac{1}{r^s},$$

where $$a+b+c=n,$$

it will be found, exactly in the same way as in § 1, that

$$\frac{\partial}{\partial x}\,r^n P_s(a, b, c) = -r^{n-1}\big[(a+1)P_s(a+1, b-2, c)$$
$$+(a+1)P_s(a+1, b, c-2)-(2n+s-1-a)P_s(a-1, b, c)\big].$$

The right-hand side only differs from the result of § 1 in having $n+\frac{1}{2}(s-1)$ instead of n. Hence, with the symbolical notation

$$P_s(a, b, c) = \lambda_s^{-1}\mu_s^{-1}\nu_s^{-1},$$

the results of § 3 will hold if we make the necessary modification.

Therefore, if $$f+g+h=m,$$

$$r^{m-n}\left(\frac{\partial}{\partial x}\right)^f\left(\frac{\partial}{\partial y}\right)^g\left(\frac{\partial}{\partial z}\right)^h r^n\left(\lambda_s^{-a}\mu_s^{-b}\nu_s^{-c}\right)$$
$$=\lambda_s\mu_s\nu_s\left(\lambda_s^2+\mu_s^2+\nu_s^2\right)^{m-n-\frac{1}{2}s}\left(\frac{\partial}{\partial\lambda_s}\right)^f\left(\frac{\partial}{\partial\mu_s}\right)^g\left(\frac{\partial}{\partial\nu_s}\right)^h\frac{\left(\lambda_s^2+\mu_s^2+\nu_s^2\right)^{n+\frac{1}{2}s}}{\lambda_s^{a+1}\mu_s^{b+1}\nu_s^{c+1}}; \tag{21}$$

and $$r^{m-n}\phi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)r^{2n+s}\psi_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r^s}$$

$$=\lambda_s\mu_s\nu_s\left(\lambda_s^2+\mu_s^2+\nu_s^2\right)^{m-n-\frac{1}{2}s}\phi_m\left(\frac{\partial}{\partial\lambda_s}, \frac{\partial}{\partial\mu_s}, \frac{\partial}{\partial\nu_s}\right)$$
$$\left(\lambda_s^2+\mu_s^2+\nu_s^2\right)^{n+\frac{1}{2}s}\psi_n\left(\frac{\partial}{\partial\lambda_s}, \frac{\partial}{\partial\mu_s}, \frac{\partial}{\partial\nu_s}\right)\frac{1}{\lambda_s\mu_s\nu_s}. \tag{22}$$

A formula of the same kind holds when there are p variables

$$x_1, x_2, \ldots x_p \quad\text{and}\quad r^2 = x_1^2+x_2^2+\ldots+x_p^2.$$

If we write

$$\frac{(-1)^n}{a_1!\,a_2!\,\ldots\,a_p!}\,r^{n+s}\left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\ldots\left(\frac{\partial}{\partial x_p}\right)^{a_p}\frac{1}{r^s} = \lambda_1^{-a_1}\lambda_2^{-a_2}\ldots\lambda_p^{-a_p},$$

where $$n = a_1 + a_2 + \dots + a_p,$$

we shall find

$$r^{m-n}\phi_m\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right) r^{2n+s}\psi_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right)\frac{1}{r^s}$$

$$= \lambda_1\lambda_2 \dots \lambda_p\,(\lambda_1^2+\lambda_2^2+\dots+\lambda_p^2)^{m-n-\frac{1}{2}s}\,\phi_m\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots \frac{\partial}{\partial\lambda_p}\right)$$

$$(\lambda_1^2+\lambda_2^2+\dots+\lambda_p^2)^{n+\frac{1}{2}s}\,\psi_n\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots \frac{\partial}{\partial\lambda_p}\right)\frac{1}{\lambda_1\lambda_2\dots\lambda_p}. \quad (23)$$

If $$s = p-2,$$

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2}\right)\frac{1}{r^s} = 0,$$

and the relations between the different functions are given by

$$\lambda_1\lambda_2 \dots \lambda_p\left(\frac{\partial^2}{\partial\lambda_1^2} + \frac{\partial^2}{\partial\lambda_2^2} + \dots + \frac{\partial^2}{\partial\lambda_p^2}\right)\lambda_1^{-a_1}\lambda_2^{-a_2}\dots\lambda_p^{-a_p} = 0. \quad (24)$$

14. When there are $p+1$ variables $x_1, x_2, \dots x_{p+1}$, we write

$$\begin{aligned}
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
\dots &\quad \dots \quad \dots \quad \dots\\
x_p &= r\sin\theta_1\sin\theta_2\dots\sin\theta_{p-1}\cos\theta_p,\\
x_{p+1} &= r\sin\theta_1\sin\theta_2\dots\sin\theta_{p-1}\sin\theta_p,
\end{aligned}$$

and the standard hyperspherical harmonic is of the form

$$r^n P^n_{n_1}(p, \cos\theta_1)\,P^{n_1}_{n_2}(p-1, \cos\theta_2)\,P^{n_2}_{n_3}(p-2, \cos\theta_3)\dots$$

$$\dots P^{n_{p-2}}_{n_{p-1}}(2, \cos\theta_{p-1})\,\frac{\cos}{\sin}(n_{p-1}\theta_p), \quad (25)$$

where

$$P^n_m(p, \cos\theta) = \sin^m\theta\left[\cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{2.2n+p-3}\cos^{n-m-2}\theta + \dots\right].$$

In order to show how this harmonic may be obtained by differentiation of $1/r^{p-1}$, let $f(x_2, x_3, \dots x_{p+1})$ be a homogeneous function of degree n_1, which satisfies the equation

$$\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\dots+\frac{\partial^2}{\partial x_{p+1}^2}\right)f=0.$$

Then, by Dr. Hobson's "Theorem in Differentiation," *Proceedings*, Vol. XXIV.,

$$r^{n+p-1}\frac{\partial^{n-n_1}}{\partial x_1^{n-n_1}}f\left(\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\dots\frac{\partial}{\partial x_{p+1}}\right)\frac{1}{r^{p-1}}$$

$$=\frac{1}{r^n}C_1\left[1-\frac{r^2\nabla^2}{2.2n+p-3}+\frac{r^4\nabla^4}{2.4(2n+p-3)(2n+p-5)}-\dots\right]$$
$$x_1^{n-n_1}f(x_2, x_3, \dots x_{p+1}),$$

where C_1 is a definite numerical constant, and

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_{p+1}^2}.$$

Performing the differentiations, we obtain

$$\frac{1}{r^n}C_1\left[1-\frac{r^2}{2.2n+p-3}\frac{\partial^2}{\partial x_1^2}+\dots\right]x_1^{n-n_1}f(x_2, \dots x_{p+1})$$

$$=C_1\frac{1}{r^n}\left[x_1^{n-n_1}-\frac{(n-n_1)(n-n_1-1)}{2.2n+p-3}r^2x_1^{n-n_1-2}+\dots\right]f(x_2, \dots x_{p+1})$$

$$=C_1\left[\cos^{n-n_1}\theta_1-\frac{(n-n_1)(n-n_1-1)}{2.2n+p-3}\cos^{n-n_1-2}\theta_1+\dots\right]\frac{\sin^{n_1}\theta_1}{r_1^{n_1}}f(x_2, \dots x_{p+1}),$$

where $\qquad r_1=r\sin\theta_1=(x_2^2+x_3^2+\dots+x_{p+1}^2)^{\frac{1}{2}}$

$$=C_1P_{n_1}^n(p, \cos\theta_1)\frac{1}{r_1^{n_1}}f(x_2, \dots x_{p+1}).$$

We may now take

$$f(x_2, \dots x_{p+1})=r_1^{2n_1+p-2}\frac{\partial^{n_1-n_2}}{\partial x_2^{n_1-n_2}}\phi_{n_2}\left(\frac{\partial}{\partial x_3}\dots\frac{\partial}{\partial x_{p+1}}\right)\frac{1}{r_1^{p-2}},$$

where $\qquad \nabla^2\phi_{n_2}(x_3, \dots x_{p+1})=0,$

and therefore

$$f(x_2, \dots x_{p+1})=r_1^{n-n_1}P_{n_2}^{n_1}(p-1, \cos\theta_2)\,\phi_{n_2}(x_3, \dots x_{p+1}).$$

Proceeding in this way, we shall finally obtain the harmonic (25) by continued differentiation.

15. We can now illustrate the application of formula (23) to the differentiation of hyperspherical harmonics. Let the harmonic be

$$r^n Y(n, n_1, n_2, \dots n_{p-1})$$

$$= C r^n P^{n}_{n_1}(p, \cos\theta_1) \dots P^{n_{p-2}}_{n_{p-1}}(2, \cos\theta_{p-1}) \begin{matrix}\cos\\ \sin\end{matrix} (n_{p-1}\theta_p)$$

$$= (-1)^n r^{2n+p-1} \frac{\partial^{n-n_1}}{\partial x_1^{n-n_1}} f_{n_1}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \dots \frac{\partial}{\partial x_{p+1}}\right) \frac{1}{r^{p-1}}.$$

Then $$r^{-(n-1)} \frac{\partial}{\partial x_1} r^n Y(n, n_1, n_2, \dots n_{p-1})$$

$$= (-1)^n \lambda_1 \dots \lambda_{p+1} (\lambda_1^2 + \dots + \lambda_{p+1}^2)^{-n-\frac{1}{2}(p-3)} \frac{\partial}{\partial \lambda_1}$$

$$\left[(\lambda_1^2 + \dots + \lambda_{p+1}^2)^{n+\frac{1}{2}(p-1)} \left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1} f_{n_1}\left(\frac{\partial}{\partial \lambda_2}, \dots \frac{\partial}{\partial \lambda_{p+1}}\right) \frac{1}{\lambda_1 \dots \lambda_{p+1}}\right]$$

$$= (-1)^n \lambda_1 \dots \lambda_{p+1} \left[(2n+p-1)\lambda_1 + (\lambda_1^2 + \dots + \lambda_{p+1}^2) \frac{\partial}{\partial \lambda_1}\right]$$

$$\left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1} f_{n_1}\left(\frac{\partial}{\partial \lambda_2}, \dots \frac{\partial}{\partial \lambda_{p+1}}\right) \frac{1}{\lambda_1 \dots \lambda_{p+1}}.$$

Now $$\lambda_1 \left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1} \frac{1}{\lambda_1} = -(n-n_1)\left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1-1} \frac{1}{\lambda_1}.$$

Also $$\left(\frac{\partial^2}{\partial x_1^2} + \dots \frac{\partial}{\partial x_{p+1}^2}\right) x_1^{n-n_1+1} f_{n_1}(x_2, \dots x_{p+1})$$

$$= (n-n_1+1)(n-n_1)\, x_1^{n-n_1-1} f_{n_1}(x_2, \dots x_{p+1}).$$

Hence, by an obvious extension of the lemma of § 5 to $p+1$ variables,

$$\lambda_1 \dots \lambda_{p+1} (\lambda_1^2 + \dots + \lambda_{p+1}^2) \left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1+1} f_{n_1}\left(\frac{\partial}{\partial \lambda_2}, \dots \frac{\partial}{\partial \lambda_{p+1}}\right) \frac{1}{\lambda_1 \dots \lambda_{p+1}}$$

$$= (n-n_1)(n-n_1+1)\lambda_1 \dots \lambda_{p+1} \left(\frac{\partial}{\partial \lambda_1}\right)^{n-n_1-1} f_n\left(\frac{\partial}{\partial \lambda_2}, \dots \frac{\partial}{\partial \lambda_{p+1}}\right) \frac{1}{\lambda_1 \dots \lambda_{p+1}}$$

$$= (-1)^{n-1} (n-n_1)(n-n_1+1)\, Y(n-1, n_1, n_2, \dots n_{p-1}).$$

Hence $$r^{-(n-1)}\frac{\partial}{\partial x_1}r^nY(n,\, n_1,\, n_2,\, \dots\, n_{p-1})$$

$$= (n-n_1)\left\{2n+p-1-(n-n_1+1)\right\}Y(n-1,\, n_1,\, n_2,\, \dots\, n_{p-1})$$

$$= (n-n_1)(n+n_1+p-2)\,Y(n-1,\, n_1,\, n_2,\, \dots\, n_{p-1}).$$

The differentiations with respect to $x_2, x_3, \dots$ may be similarly effected, but the results are not so simple.

Note on the Electric Capacity of a Conductor in the form of Two Intersecting Spheres. *By* W. D. NIVEN.

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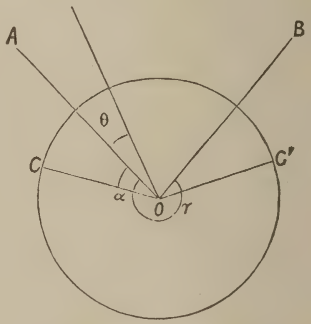
1. In the course of an interesting solution of the problem of "The Electrical Distribution on a Conductor bounded by two Spherical Surfaces cutting at any angle" (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 156), the author, Mr. H. M. Macdonald, has alluded to a paper of mine contained in Vol. XII., and has called attention to the discrepancies between his results and mine. He says: "In the paper by W. D. Niven mentioned above, an attempt is made to deduce the capacity of such a conductor from the solution of a functional equation for a particular value of one of the variables, but the result obtained does not seem in the case of the spherical bowl to agree with Lord Kelvin's. The results obtained hereafter also differ from those given by Niven."

In answer to this, I have to say (1) that the capacity of a hemispherical bowl as given by me agrees with Mr. Macdonald's own result, but (2) that in the other case where the results come into comparison, viz., when the conductor is bounded by a hemisphere and its base, I have to plead guilty to numerical mistakes in reducing this particular case from the general formulæ. In the latter case the accurate expression for the capacity is that given by Mr. Macdonald, viz., $2a\left(1-\frac{1}{\sqrt{3}}\right)$, where a is the radius.

That the difference between the general expressions obtained by us for the capacity is only one of form is, in fact, shown by Mr. Macdonald in a note which follows this paper (see p. 214).

2. On a revision of my previous paper, I find that the work was presented in a needlessly difficult form, and that the proof of the main proposition only applied for an inclination of the two planes enclosing the electrified point less than two right angles. The results themselves may also be expressed more simply and compactly. I therefore take the opportunity of supplementing that paper by a new proof which is applicable to either case. I shall consider specially the case when the inclination of the planes (γ) in the region containing the electrified point C is greater than two right angles.



Let the plane of the paper pass through the inducing point C, and cut the two conducting planes in OA, OB.

Let $COA = \alpha$, $OC = R$,

and let the inducing charge be $-R$.

The potential at any point P due to the electricity induced in OA must be a function of P's coordinates z, r, θ, where z is the ordinate perpendicular to the plane of the paper, and r, θ are the polar coordinates in that plane, with O as pole; θ being measured from OA in the direction of the hands of a watch.

Denoting this potential by $f(z, r, \theta)$, and in like manner the potential due to the induced charge on OB by $g(z, r, \theta')$, where θ' denotes the same point P's angular distance measured in the same direction from OB, so that

$$\theta' = \gamma + \theta,$$

we shall have, throughout the region $\theta = 0$ to $2\pi - \gamma$,

$$f(z, r, \theta) + g(z, r, \gamma + \theta) = R/CP. \qquad (1)$$

If, instead of being placed at C, the charge be placed at C', so situated that its angular distance from OB is equal to that of C from OA, the effect will be to produce the same distributions on OB, OA, respectively, as formerly existed on OA, OB. The potentials at z, r, θ will then be connected by the equation

$$g(z, r, \theta) + f(z, r, \gamma + \theta) = R/C'P. \qquad (2)$$

3. If we denote $e^{\partial/\partial\theta}$ by E, so that

$$f(\gamma+\theta) = E^{\gamma} f(\theta),$$

then, by the addition of (1) and (2), we obtain

$$f(z, r, \theta) + g(z, r, \theta) = \frac{R}{1+E^{\gamma}}\left(\frac{1}{CP} + \frac{1}{C'P}\right), \tag{3}$$

and, by the subtraction of (2) from (1),

$$f(z, r, \theta) - g(z, r, \theta) = \frac{R}{1-E^{\gamma}}\left(\frac{1}{CP} - \frac{1}{C'P}\right). \tag{4}$$

Since E is a differential operator with regard to θ, the expressions on the right hand of (3) and (4) will, when substituted for V, satisfy Laplace's equation

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0;$$

and therefore so also with $f(z, r, \theta)$ and $g(z, r, \theta)$.

4. In preparation for the application of the operators in (3) and (4), it becomes necessary to find expressions for $1/CP$ and $1/C'P$ which will lead to finite results. In my former paper this was effected for the particular values $z = 0$, $r = R$, but Mehler has expressed $1/CP$ in a form which is as easy for the operations in the general as in the particular case [Heine, II., § 75 (32)]. The required expression may be established as follows:—

Writing $$t^2 = z^2 + R^2 + r^2 - 2Rr\cos\theta$$

$$= 2Rr(\cosh\sigma - \cos\theta),$$

where $$2Rr\cosh\sigma = z^2 + R^2 + r^2,$$

we have $$\frac{R}{t} = \sqrt{\frac{R}{2r}}\,\frac{1}{\sqrt{\cosh\sigma - \cos\theta}}$$

$$= \tfrac{1}{2}\sqrt{\frac{R}{r}}\,\frac{1}{\sqrt{\sinh^2\tfrac{1}{2}\sigma + \sin^2\tfrac{1}{2}\theta}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{R}{r}}\int_0^{\pi}\frac{d\phi}{\sin\tfrac{1}{2}\theta + i\sinh\tfrac{1}{2}\sigma\cos\phi}$$

$$= \frac{1}{\pi}\sqrt{\frac{R}{r}}\int_0^{\frac{1}{2}\pi}\frac{\sin\tfrac{1}{2}\theta\, d\phi}{\sin^2\tfrac{1}{2}\theta + \sinh^2\tfrac{1}{2}\sigma\cos^2\phi}.$$

Now, putting $$\sinh \tfrac{1}{2}\sigma \cos\phi = \sinh \tfrac{1}{2}v,$$

$$d\phi = \frac{1}{\sqrt{2}} \frac{\cosh \frac{1}{2}v}{\sqrt{\cosh\sigma - \cosh v}}\, dv,$$

and observing that

$$\sin^2 \tfrac{1}{2}\theta + \sinh^2 \tfrac{1}{2}v = \sin \tfrac{1}{2}(\theta + iv) \sin \tfrac{1}{2}(\theta - iv),$$

we obtain

$$\frac{R}{t} = \frac{1}{2\sqrt{2}\,\pi} \sqrt{\frac{R}{r}} \int_0^\sigma \left[\operatorname{cosec} \tfrac{1}{2}(\theta + iv) + \operatorname{cosec} \tfrac{1}{2}(\theta - iv)\right] \times \frac{dv}{\sqrt{\cosh\sigma - \cosh v}}.$$

But, if z be a complex number, then (Hobson's *Trigonometry*, § 295)

$$\pi \operatorname{cosec} \pi z = \frac{1}{z} - \frac{1}{z+1} - \frac{1}{z-1} + \frac{1}{z+2} + \frac{1}{z-2} - \&c.$$

$$= \int_0^1 \frac{x^{z-1} + x^{-z}}{1+x}\, dx,$$

where, it is clear, the real part of z must be less than unity.

On writing $$x = e^{-2\pi\xi},$$

this becomes $$\operatorname{cosec} \pi z = 2 \int_0^\infty \frac{\cosh (1-2z)\pi\xi}{\cosh \pi\xi}\, d\xi.$$

Hence $$\operatorname{cosec} \tfrac{1}{2}(\theta + iv) + \operatorname{cosec} \tfrac{1}{2}(\theta - iv) = 4 \int_0^\infty \frac{\cosh (\pi - \theta)\xi \cos v\xi}{\cosh \pi\xi}\, d\xi,$$

and $$\frac{R}{t} = \frac{\sqrt{2}}{\pi} \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh (\pi - \theta)\xi}{\cosh \pi\xi} \int_0^\sigma \frac{\cos \xi v}{\sqrt{\cosh\sigma - \cosh v}}\, dv\, d\xi$$

$$= \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh (\pi - \theta)\xi}{\cosh \pi\xi} k_\xi (\cosh\sigma)\, d\xi, \tag{5}$$

where $$\frac{\pi}{\sqrt{2}} k_\xi (\cosh\sigma) = \int_0^\sigma \frac{\cos \xi v}{\sqrt{\cosh\sigma - \cosh v}}\, dv.$$

5. Heine, by the employment of Cauchy's method, has found two other forms of the integral expressing $\frac{\pi}{\sqrt{2}} k_\xi (\cosh\sigma)$, or, more briefly, $\frac{\pi}{\sqrt{2}} k_\xi$, one of which, viz.,

$$\coth \xi \int_\sigma^\infty \frac{\sin v\xi}{\sqrt{\cosh v - \cosh\sigma}}\, dv,$$

may be established by the method of the preceding article.

6. The longitudinal angles of any point P in the region not containing the charge relatively to C, C' being $\theta+\alpha$, $\theta+\gamma-\alpha$, we have thus

$$R\left(\frac{1}{CP}+\frac{1}{C'P}\right)=\quad 2\sqrt{\frac{R}{r}}\int_0^\infty \frac{\cosh\left(\theta+\tfrac{1}{2}\gamma-\pi\right)\xi\cosh\left(\tfrac{1}{2}\gamma-\alpha\right)\xi}{\cosh\pi\xi}\,k_\xi d\xi,$$

$$R\left(\frac{1}{CP}-\frac{1}{C'P}\right)=-2\sqrt{\frac{R}{r}}\int_0^\infty \frac{\sinh\left(\theta+\tfrac{1}{2}\gamma-\alpha\right)\xi\sinh\left(\tfrac{1}{2}\gamma-\alpha\right)\xi}{\cosh\pi\xi}\,k_\xi d\xi.$$

Now, observing that

$$\frac{1}{1+E^\gamma}\cosh\theta\xi=\frac{\cosh\left(\theta-\tfrac{1}{2}\gamma\right)\xi}{2\cosh\tfrac{1}{2}\gamma\xi},$$

and

$$\frac{1}{1-E^\gamma}\sinh\theta\xi=-\frac{\cosh\left(\theta-\tfrac{1}{2}\gamma\right)\xi}{2\sinh\tfrac{1}{2}\gamma\xi},$$

on referring back to (3) and (4), we see that

$$f(z,r,\theta)+g(z,r,\theta)=\sqrt{\frac{R}{r}}\int_0^\infty \frac{\cosh\left(\theta-\pi\right)\xi\cosh\left(\tfrac{1}{2}\gamma-\alpha\right)\xi}{\cosh\pi\xi\cosh\tfrac{1}{2}\gamma\xi}\,k_\xi d\xi,$$

and $f(z,r,\theta)-g(z,r,\theta)=\sqrt{\frac{R}{r}}\int_0^\infty \frac{\cosh\left(\theta-\pi\right)\xi\sinh\left(\tfrac{1}{2}\gamma-\alpha\right)\xi}{\cosh\pi\xi\sinh\tfrac{1}{2}\gamma\xi}\,k_\xi d\xi.$

From these, by addition and subtraction, we obtain

$$f(z,r,\theta)=\sqrt{\frac{R}{r}}\int_0^\infty \frac{\cosh\left(\pi-\theta\right)\xi\sinh\left(\gamma-\alpha\right)\xi}{\cosh\pi\xi\sinh\gamma\xi}\,k_\xi d\xi, \tag{6}$$

$$g(z,r,\theta)=\sqrt{\frac{R}{r}}\int_0^\infty \frac{\cosh\left(\pi-\theta\right)\xi\sinh\alpha\xi}{\cosh\pi\xi\sinh\gamma\xi}\,k_\xi d\xi. \tag{7}$$

Since $\sinh(\gamma-\alpha)\xi$ and $\sin\alpha\xi$ are each of them less than $\sinh\gamma\xi$, the expressions just found are more convergent than the original integrals from which they have been derived.

7. It will be observed that f and g are unchanged when $2\pi-\theta$ is put for θ, so that, as was to be expected, the potentials at points similarly situated with respect to the two sides of either plane are equal as regards the charge on that plane.

The potential U due to the induced charges at any point z, r, θ in the region containing the electrified point is therefore given by

$$U=f(z,r,\theta)+g(z,r,\theta-2\pi+\gamma),$$

where θ may have any value between $2\pi-\gamma$ and 2π. This gives

$$U=\sqrt{\frac{R}{r}}\int_0^\infty \frac{N}{\cosh\pi\xi\sinh\gamma x}\,k_\xi d\xi, \tag{8}$$

where $N = \cosh(\pi-\theta)\,\xi \sinh(\gamma-\alpha)\,\xi + \cosh(3\pi-\theta-\gamma)\sinh \alpha\xi$

$$= \tfrac{1}{2}\left\{\sinh(\pi+\gamma-\alpha-\theta)\,\xi + \sinh(-\pi+\gamma-\alpha+\theta)\,\xi \right.$$
$$\left. + \sinh(3\pi-\gamma+\alpha-\theta)\,\xi - \sinh(3\pi-\gamma-\alpha-\theta)\,\xi\right\}$$
$$= \sinh \pi\xi \cosh(2\pi-\gamma+\alpha-\theta)\,\xi$$
$$+ \sinh(\gamma-\pi)\,\xi \cosh(2\pi-\alpha-\theta)\,\xi.$$

The value of U thus found is finite and continuous, vanishes at an infinite distance, k_ξ being then zero, and satisfies the boundary conditions. It is therefore the solution required.

Case of $\gamma < \pi$.

8. The proof contained in the foregoing articles is expressed for the case when the angle in the region containing the electrified point is greater than two right angles. Taking, however, the figure in § 3, and placing the electrified point C in the acute angle, we shall have the same proof identically, provided angles are now measured positively when described from OA in the direction opposite to the motion of the hands of a watch.

Writing for a moment θ' for the longitudinal angle, we have the same expressions for $f(z, r, \theta')$ and $g(z, r, \theta')$, and the potential U due to the induced charges in the region containing the electrified point will be given by

$$U = f(z, r, \theta') + g(z, r, \theta'-2\pi+\gamma),$$

γ now being the acute angle AOB, and θ' the angle measured positively from OA to a point in the acute angle.

Putting θ for $2\pi-\theta'$ and observing that

$$f(z, r, 2\pi-\theta) = f(z, r, \theta),$$

we have then $$U = f(z, r, \theta) + g(z, r, \gamma-\theta),$$

where θ is now measured positively from OA in the direction of the hands of a watch, so that, in this case,

$$U = \sqrt{\frac{R}{r}} \int_0^\infty \frac{N}{\cosh \pi\xi \sinh \gamma\xi} k_\xi \, d\xi, \tag{9}$$

where $N = \cosh(\pi-\theta)\,\xi \sinh(\gamma-\alpha)\,\xi + \cosh(\pi-\gamma+\theta)\,\xi \sinh \alpha\xi$,

or, by transformation,

$$\sinh \pi\xi \cosh(\gamma-\alpha-\theta)\,\xi - \sinh(\pi-\gamma)\,\xi \cosh(\theta-\alpha)\,\xi.$$

Case of two Parallel Planes.

9. An interesting deduction from the general case in § 8 is that of two parallel planes with an electrified point between them.

In the general case, let the point O be moved to infinity, so that γ becomes infinitely small, and therefore also θ and α, and put

$$\frac{\alpha}{\gamma} = \frac{a}{a+b},$$

$$\frac{\theta}{\gamma} = \frac{x}{a+b},$$

where a, b are the distances of the charge from OA, OB respectively.

If, in the expression for U, we put

$$\gamma\xi = (a+b)\,\zeta,$$

and then make γ small, we shall obtain

$$U = \frac{a+b}{\gamma}\int_0^\infty \frac{N'}{\sinh\,(a+b)\,\zeta}\,k_{(\zeta)}\,d\zeta, \qquad (10)$$

where $$N' = e^{-x\zeta}\sinh b\zeta + e^{-(a+b-x)\,\zeta}\sinh a\zeta,$$

or, on the removal of the origin to the electrified point, by putting

$$x = X + a,$$

we have, by an easy transformation,

$$N' = \cosh X\zeta\left[\sinh\,(a+b)\,\zeta - 2\sinh a\zeta\,\sinh b\zeta\right] + \sinh X\zeta\,\sinh\,(a-b)\,\zeta. \qquad (11)$$

10. In regard to k_ξ, if σ be small, we may put

$$\cosh\sigma = 1 + \sigma^2/2,$$

and $$\frac{\pi}{\sqrt{2}}\,k_\xi = \sqrt{2}\int_0^\sigma \frac{\cos\xi v}{\sqrt{\sigma^2 - v^2}}\,dv.$$

Therefore $$k_\xi = J_0\,(\xi\sigma). \qquad \text{(Heine, II., § 60.)}$$

Now $$\sigma^2 = \frac{z^2 + (R-r)^2}{Rr}$$

$$= \frac{z^2 + y^2}{R^2} \qquad (R \text{ and } r \text{ large})$$

$$= \frac{\rho^2}{R^2}, \text{ say.}$$

Hence
$$\sigma\xi = \frac{(a+b)\,\zeta\rho}{\gamma R}$$
$$= \zeta\rho. \tag{12}$$

11. The presence of $1/\gamma$ outside the integral makes the expression for U infinite, but the charge R is now also infinite. If we reduce the charge to a finite amount R', we must multiply the expression for U by R'/R, and thus, since

$$R\gamma = a+b,$$

finally obtain for U the value

$$R'\int_0^\infty \frac{N'J_0(\rho\zeta)}{\sinh\,(a+b)\,\zeta}\,d\zeta.$$

This represents the potential at a point whose coordinates are X, ρ, due to the electricity induced in the two planes by the charge $-R'$.

The particular case when $a = b$ is

$$U = R'\int_0^\infty \frac{\cosh X\zeta e^{-a\zeta}\,J_0(\rho\zeta)}{\cosh a\zeta}\,d\zeta. \tag{13}$$

Potential at a Point outside the Inverted Surface.

12. Referring to the figure in § 3, we invert with regard to the point C, the radius of inversion being R.

The intersection of the planes OA, OB will invert into a circle through C and O in a plane perpendicular to the plane of the paper; the planes themselves into segments of spheres cutting in this circle at an external angle γ. The new conductor will be at potential unity. If we denote the potential at any point Q outside of it by $\overline{U}$, we shall have, by the ordinary rules for inversion,

$$\overline{U} = \frac{R}{(OQ^2CQ^2-R^2z^2)^{\frac{1}{4}}}\int_0^\infty \frac{\overline{N}}{\cosh\pi\xi\,\sinh\gamma\xi}\,k_\xi\,(\cosh\overline{\sigma})\,d\xi, \tag{14}$$

in which $\overline{N}$ denotes the same expression as in § 7 or § 8, the angle θ now being the angle between the segment corresponding to OA and the segment passing through Q. The angle θ is measured from the former segment to the latter in the opposite direction from that in which it is measured in § 7 or § 8.

The quantity $\cosh\bar{\sigma}$ denotes

$$\frac{CQ^2+OQ^2}{2\sqrt{CQ^2 . OQ^2-R^2z^2}},$$

where z is the perpendicular from Q to the plane of the paper.

Since $\overline{U}$, as may be easily shown from the above expression, and is otherwise obvious, is a symmetrical function about the axis of the inverted conductor, we may put $z=0$, and consider only points Q in the plane of the paper.

13. For points on the axis of the conductor

$$\overline{U}=\frac{R}{CQ}\int_0^\infty \frac{\overline{N}}{\cosh\pi\xi\sinh\gamma\xi}\,d\xi.$$

As a particular case, let $\gamma=2\pi$; then, § 7,

$$\overline{U}=\frac{R}{CQ}\int_0^\infty \frac{\cosh(\pi-\theta)\,\xi\cosh(\pi-a)\,\xi}{\cosh^2\pi\xi}\,d\xi.$$

This integral can be evaluated by expanding $\operatorname{sech}^2\pi\xi$ in powers of $e^{-\pi\xi}$. The result of integrating can then be arranged so as to exhibit its equality to

$$\frac{R}{2\pi CQ}\left\{\frac{\pi-\frac{1}{2}(\theta+a)}{\sin\frac{1}{2}(\theta+a)}+\frac{\frac{1}{2}(\theta-a)}{\sin\frac{1}{2}(\theta-a)}\right\}.$$

This gives the potential at any point of the axis of a spherical bowl charged to potential unity.

Capacity.

14. The capacity of the inverted conductor may be determined by means of a result given in my paper on "Ellipsoidal Harmonics" (*Phil. Trans.*, Vol. CLXXXII., 1891) to the following effect:—If P represent the potential at the point of inversion due to the electricity induced in any conductor A by a charge equal to $-R$, where R is the radius of inversion, then the capacity of the conductor inverted from A is PR.

Thus, in the general case above, since for the point of inversion $r=R$, $z=0$, and therefore $\sigma=0$, $k=1$, if we put $\theta=2\pi-a$ in § 7, and $\theta=a$ in § 8, we find that, whether the angle γ is greater or less than π, the capacity is given by

$$R\int_0^\infty \frac{\sinh\pi\xi\cosh(\gamma-2a)\,\xi+\sinh(\gamma-\pi)\,\xi}{\cosh\pi\xi\sinh\gamma\xi}\,d\xi. \qquad (17)$$

Difference of Charges on the Two Segments.

15. It is but a variation of the proof of the theorem quoted in § 14 to extend it to separate parts of the inverted conductor, provided we know the potentials due to the corresponding distributions before inversion. Thus, in the case on hand, the charge on the surface into which OA inverts is given, § 7, by

$$\begin{aligned} E_a &= Rf\,(O,\, R,\, 2\pi-\alpha) \\ &= Rf\,(O,\, R,\, \alpha), \end{aligned} \tag{18}$$

and similarly
$$E_b = Rg\,(O,\, R,\, \gamma-\alpha). \tag{19}$$

From (18) and (19), on substituting the values of f and g, and observing that

$$\cosh\,(\pi-\alpha)\,\xi\,\sinh\,(\gamma-\alpha)\,\xi-\cosh\,(\gamma-\pi-\alpha)\,\xi\,\sinh\,\alpha\xi$$

reduces to
$$\cosh\,\pi\xi\,\sinh\,(\gamma-2\alpha)\,\xi,$$

we have
$$\begin{aligned} E_a-E_b &= R\int_0^\infty \frac{\sinh\,(\gamma-2\alpha)\,\xi}{\sinh\,\gamma\xi}\,d\xi \\ &= R\,.\,\tfrac{1}{2}\,\frac{\pi}{\gamma}\cot\frac{\alpha}{\gamma}\,\pi. \end{aligned} \tag{20}$$

This theorem I formerly proved for γ less than π. It is now shown to be true for any value of γ from 0 to 2π, the extreme cases corresponding to these angles being two spheres in contact and a spherical bowl.

Note on Mr. W. D. Niven's paper on the Electric Capacity of a Conductor formed by Two Intersecting Spheres. By H. M. Macdonald. Communicated December 10th, 1896.[*]

In his paper "On the Electrical Capacity of a Conductor bounded by Two Spherical Surfaces cutting at any angle," *Proc. Lond. Math. Soc.*, XII., 1880, and also in a further "Note on the Electric Capacity of a Conductor in the form of Two Intersecting Spheres," communicated to the Society on November 12th, 1896, Mr. W. D. Niven obtains the potential due to the distribution induced on a wedge-shaped conductor bounded by two intersecting planes by an electrified point, its value being expressed at any point on the circle having its plane perpendicular to the two intersecting planes, its centre on their line of intersection, and passing through the electrified point. The expression there given is $f(\theta)+g(\theta-2\pi+\gamma)$, which is equal to

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[*] It was ordered by the Council that Mr. Macdonald's note should follow Mr. Niven's paper.

$$\frac{1}{\gamma}\int_0^\infty \frac{\cosh\frac{\pi-\theta}{\gamma}\xi\sinh\frac{\gamma-\alpha}{\gamma}\xi+\cosh\frac{3\pi-\theta-\gamma}{\gamma}\xi\sinh\frac{\alpha\xi}{\gamma}}{\cosh\frac{\pi\xi}{\gamma}\sinh\xi}\,d\xi.$$

The potential at the point θ due to the inducing charge is

$$-\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}(\theta+\alpha)=-\frac{1}{\gamma}\int_0^\infty\frac{\cosh\frac{\theta+\alpha-\pi}{\gamma}\xi}{\cosh\frac{\pi\xi}{\gamma}}\,d\xi;$$

therefore the potential V at the point θ on the circle due to the whole system is given by

$$V=\frac{1}{\gamma}\int_0^\infty\frac{d\xi}{\cosh\frac{\pi\xi}{\gamma}\sinh\xi}\left\{\cosh\frac{\pi-\theta}{\gamma}\xi\sinh\frac{\gamma-\alpha}{\gamma}\xi\right.$$
$$\left.+\cosh\frac{3\pi-\theta-\gamma}{\gamma}\xi\sinh\frac{\alpha\xi}{\gamma}-\cosh\frac{\theta+\alpha-\pi}{\gamma}\xi\sinh\xi\right\},$$

that is, $$V=\frac{2}{\gamma}\int_0^\infty\frac{\sinh\frac{\alpha\xi}{\gamma}}{\sinh\xi}\sinh\frac{2\pi-\theta-\gamma}{\gamma}\xi\tanh\frac{\pi\xi}{\gamma}\,d\xi.$$

Now $$\tanh\frac{\pi\xi}{\gamma}=2\int_0^\infty\frac{\sin\frac{2\pi\xi\zeta}{\gamma}d\zeta}{\sinh\pi\zeta}$$

for all values of ξ; therefore

$$V=\frac{4}{\gamma}\int_0^\infty\int_0^\infty\frac{\sinh\frac{\alpha\xi}{\gamma}\sinh\frac{2\pi-\theta-\gamma}{\gamma}\xi\sin\frac{2\pi\xi\zeta}{\gamma}d\xi d\zeta}{\sinh\xi\sinh\pi\zeta},$$

ıat is, $$V=\frac{2}{\gamma}\int_0^\infty\int_0^\infty\frac{\sin\frac{2\pi\xi\zeta}{\gamma}d\xi d\zeta}{\sinh\zeta\sinh\pi\xi}$$
$$\times\left\{\cosh\frac{2\pi-\theta-\gamma+\alpha}{\gamma}\pi\xi-\cosh\frac{2\pi-\theta-\gamma-\alpha}{\gamma}\pi\xi\right\}.$$

Again, writing

$$I=\int_0^\infty \frac{\sin\kappa\pi\xi\cosh\pi\lambda\xi}{\sinh\pi\xi}\,d\xi=\frac{1}{2\pi\iota}\int_0^1\frac{(x^{-\kappa\iota}-x^{\kappa\iota})(x^{\lambda}+x^{-\lambda})}{x^{-1}-x}\,\frac{dx}{x},$$

that is
$$I=\frac{1}{2\pi\iota}\int_0^\infty\left\{\frac{x^{\lambda+\kappa\iota}-x^{-\lambda-\kappa\iota}}{x-x^{-1}}-\frac{x^{\lambda-\kappa\iota}-x^{-\lambda+\kappa\iota}}{x-x^{-1}}\right\}\frac{dx}{x},$$

whence
$$I=\frac{1}{4\iota}\left[\tan(\lambda+\kappa\iota)\frac{\pi}{2}-\tan(\lambda-\kappa\iota)\frac{\pi}{2}\right],\ 1>\lambda>0;$$

we have
$$\int_0^\infty\frac{\sin\kappa\pi\xi\cosh\pi\lambda\xi}{\sinh\pi\xi}\,d\xi=\tfrac{1}{2}\,\frac{\sinh\kappa\pi}{\cosh\kappa\pi+\cos\lambda\pi},\ 1>\lambda>0.$$

It follows that

$$2\int_0^\infty\frac{\sin\dfrac{2\pi\xi\zeta}{\gamma}\cosh\dfrac{2\pi-\theta-\gamma+a}{\gamma}\pi\xi}{\sinh\pi\xi}\,d\xi=\frac{\sinh\dfrac{2\pi\zeta}{\gamma}}{\cosh\dfrac{2\pi\zeta}{\gamma}+\cos\dfrac{2\pi-\theta-\gamma+a}{\gamma}\pi},$$

provided
$$1>\frac{2\pi-\theta-\gamma+a}{\gamma}>0,$$

or
$$1>\frac{\theta+\gamma-2\pi-a}{\gamma}>0;$$

hence

$$V=\frac{1}{\gamma}\int_0^\infty\frac{d\zeta}{\sinh\zeta}\left\{\frac{\sinh\dfrac{2\pi\zeta}{\gamma}}{\cosh\dfrac{2\pi\zeta}{\gamma}-\cos\dfrac{2\pi-\theta+a}{\gamma}\pi}-\frac{\sinh\dfrac{2\pi\zeta}{\gamma}}{\cosh\dfrac{2\pi\zeta}{\gamma}-\cos\dfrac{2\pi-\theta-a}{\gamma}\pi}\right\}$$

for all values of θ in the dielectric. This expression agrees with the general value of V given in my paper Vol. XXII., p. 160; and I therefore desire to withdraw my previous conclusion that it was incorrect. Putting in my expression for the potential $z=0$, $\gamma=\gamma'$, and therefore $\eta=0$, q being $-\gamma'$, it becomes

$$V=\frac{1}{2a}\int_0^\infty\frac{d\zeta}{\sinh\dfrac{\zeta}{2}}\left\{\frac{\sinh\dfrac{\pi\zeta}{a}}{\cosh\dfrac{\pi\zeta}{a}-\cos\dfrac{\pi(\theta+\theta')}{a}}-\frac{\sinh\dfrac{\pi\zeta}{a}}{\cosh\dfrac{\pi\zeta}{a}-\cos\dfrac{\pi(\theta-\theta')}{a}}\right\},$$

the same as the above, remembering that a in the last expression is γ in the former, θ is $\theta-2\pi+\gamma$, and θ' is $\gamma-a$.

An Essay on the Geometrical Calculus. By E. Lasker.

Received July 15th, 1896. Communicated November 12th, 1896, by Mr. Tucker.

Introduction.

In 1844 a remarkable book was published, entitled *Die Ausdehnungslehre*, by H. Grassmann. It deals with a new species of magnitudes —such as have a number of extensions, and which are composed of a number of independent units—and teaches how to interpret their addition and multiplication.

The book did not excite much interest, probably because its author omitted to demonstrate that his calculus, although quite different from ordinary algebra, was nevertheless, if properly interpreted, nothing but a way of writing algebraical identities. Had he done so, he might have considerably reduced the volume of his book, enlarged its sphere, and would probably have found the keystone to modern algebra.

In 1847 an essay of his, on the geometrical calculus of Leibnitz, took the prize of the Jablonowsky Society at Leipzig. And, in 1862, prompted by some of the few mathematicians of renown who had studied his writings, he brought out a second edition of his *Ausdehnungslehre.* This second edition was far superior to the former, and gained a small, but select, circle of friends. Since that time the literature of the subject has grown to a considerable extent, and it is yet continually on the increase.

It is to be regretted that Grassmann made very little practical use of his calculus. It is surprising that, though he had been for many years in the possession of a new and powerful calculus, he failed to contribute in any way to the development of the mathematical ideas which interested his contemporaries. Yet the time was full of mathematical life and vigour, under the influence of such men as Hesse, Clebsch, Riemann, Sylvester, Cayley, not to mention the older generation of which Gauss and Cauchy were still alive.

The author's essay attempts to demonstrate that Grassmann's *Ausdehnungslehre* is a shape into which projective geometry or modern algebra may be thrown; that it is coextensive with these two branches of mathematics, and that its symbolism embodies probably the shortest and clearest and most suggestive manner of expressing the truths of these sciences. The manner of deduction is purely geometrical, based on a few assumptions concerning the nature of plane spaces of any manifoldness.

1. We shall assume in the following an abstract universe, a space S which shall contain any kind of space whose definition is faultless. S itself will have no property to distinguish it in any way. The Euclidean properties of our space of three manifoldness are consistent with each other. S will therefore contain the straight line joining any two of its points, the plane determined by any three, and the space of three dimensions determined by any four of its points.

If, now, any point P outside of a space of three dimensions S_3 is joined by straight lines with all the points of the S_3, then a space will be originated which we shall call a S_4, a plane space of four dimensions, or of four manifoldness. Let A, B be any two points of the S_3, and A', B' any other two points upon the lines PA, PB. Then the line $A'B'$ will be contained by the plane PAB; which, in accordance with its generation, must itself be contained by the S_4. The S_4 will therefore contain the line joining any two of its points, the plane joining any three and the S_3 joining any four of the points of the S_4. A S_4 is determined by any S_3 contained by it, and any point P besides, not belonging to the S_3. It follows that it is uniquely determined by any five of its points not situated in one S_3. Continuing the process by which the possibility of the geometrical existence of a S_4 was evolved, we demonstrate in the same way the (abstract) existence of a S_n, a plane space of n manifoldness. Its fundamental properties will be or are assumed to be—

(1) That it is uniquely determined by any $n+1$ of its points not belonging to any S_k where k is smaller than n;

(2) That it contains any S_h with which it can be shown to have $h+1$ points in common not situated in a S_l where l is smaller than h;

(3) That it is continuous everywhere;

(4) That no part of it is distinguished by itself from any other part of it which is identically constructed; and

(5) That any S_{n-1} cuts a S_n into two compartments A, B which are identically constructed, and such that any point moving in a con-

tinuous line (for instance, a straight one) from A into B must of necessity pass the S_{n-1}.

Let A, B, C, D, E be any five points fixing the position of a S_4. Join AB and construct all the lines passing through A and perpendicular to AB. Then the totality of these lines form a plane space S_3, which will be said to be perpendicular to AB. In S_3 $ABCD$ construct the plane e, containing A, perpendicular to AB; and similarly the plane c in the S_3 $ABDE$ and d in the S_3 $ABEC$. $ABCD$ and $ABDE$ have the plane ABD in common; consequently e and c the perpendicular δ on AB in A contained by ABD; similarly c and d the perpendicular ϵ in ABE; and d and e the perpendicular γ in ABC. δ and ϵ determine, in accordance with Euclid's axioms in space S_3, the plane c uniquely; and so ϵ and γ the plane d; γ and δ the plane e. c, d, e are therefore contained in a space Σ_3 of three manifoldness, fixed by A and one point each of γ, δ, ϵ. c, d, e divide Σ_3 into altogether four compartments connected in A, in such wise that any point moving in the Σ_3 would have to pass one of the planes c, d, e, in order to move from one compartment into another. Let us only consider two of these compartments L, M which are opposite to each other, but are otherwise identically constructed. Move one of the bordering planes c, d, e, say e, through one of the border lines, say δ, in L (and M) continuously. In its initial position e was perpendicular to AB. If the angle that e forms with AB should vary, if, for instance, the angle eAB should be larger than R, as far as L was concerned, and therefore smaller than R if its value is measured in M, until e comes in the course of its movement to coincide with a plane e' in L, M—when again the angle $e'AB$ may be R—then the spaces L', M' described by the moving plane will again be opposite corners, like L, M, and identically constructed.

But any plane through δ in L', M' will make with AB an angle larger than R in L', smaller than R in M'. This would be equivalent to a permanent property distinguishing L' from M' and must therefore be rejected. It follows, then, that all lines through A contained by the Σ_3 must be perpendicular to AB. Let us assume any line l through A perpendicular to AB, but not belonging to Σ_3. Any one of the points of l not coinciding with A may be called P. The S_4 in which we operate may be regarded as generated by the Σ_2 and point B in the manner originally described. B and P will therefore be collinear with some point Q of the Σ_3. AP, AQ, AB being all contained in the plane ABP, the two lines AP and AQ perpendicular to AB must coincide. The proposition to be demonstrated is therefore verified. In exactly the same manner we may prove by induction the more general theorem that the aggregate of lines l through A perpendicular to some line AB in space S_n constitute a plane space S_{n-1}, called perpendicular to AB.

Spaces perpendicular to the same line l are called parallel. They cannot have any point in common. For, were P any such point, A and B the points which the two spaces have in common with l, then PAB would be a triangle with two right angles A and B.

Any plane space may be moved into coincidence with any other plane space of the same manifoldness. This is a direct consequence of the fundamental properties (1), (2), (3), (4) of plane spaces.

Through any point P belonging to a S_k, itself contained in a S_{k+1}, a line l may be drawn in that S_{k+1}, perpendicular to the S_k. And only *one* such line is possible. To find l, draw any line l' in S_{k+1} anywhere; and at any one of its points P' erect the perpendicular S'_k. After that move the figure conceived to be rigid, so that P' coincides with P, S'_k with S_k. The position which l' assumes will indicate l.

From any point P a perpendicular can be let fall on any space Σ_k outside of it. To construct it erect the perpendicular at any point Q of Σ_k, in space $\Sigma_k P$; and draw in the plane of P and this perpendicular the parallel through P to it. Only one such perpendicular is possible, according to Euclid's parallel axiom.

$n+1$ points not situated in a S_h, where h is smaller than n, form a figure which will here be called a pyramid of n manifoldness. The points $A_1, \dots A_{n+1}$ will be called the corner-points, the lines $A_1A_2, A_1A_3, \dots A_cA_g, \dots$ the edges, the planes $A_1A_2A_3, \dots A_cA_gA_k \dots$ the border-planes, the spaces S_3 $A_1A_2A_3A_4$, $A_1A_2A_3A_5, \dots$ the border S_3 of the pyramid; and so on generally. A pyramid of n manifoldness will thus have $n+1$ corner-points, $\frac{n+1\,.\,n}{2}$ edges, $\frac{n+1\,.\,n\,.\,n-1}{1\,.\,2\,.\,3}$ border-planes ... $(n+1)_{k+1}$ border S_k.

Of any limited part of the space S_n we shall assume that it possesses magnitude. And we shall further assume that our ordinary conceptions concerning the addition and the measurement of geometrical magnitudes in space S_3 are applicable to them. One of our assumptions is therefore that magnitude in space S_n is expressible by a number multiplied by the n^{th} power e^n of the unit of length.

Let H and K be two spaces of manifoldness $n-1$ contained in a S_n; both perpendicular to some line l and therefore parallel to each other. In H draw any figure L, and by lines parallel to l project it on to K into the position M. Then the figure bordered by L, M, and the lines parallel to l is measured by the length l of the perpendicular bordered by the two spaces H, K multiplied by the measure of the magnitude $L\,(=M)$. This may be proved by a most elementary process of integration, in conformity with the assumption that our conception of the addition of parts of spaces should be applicable to parts of spaces in the S_n.

If P be any point, A and B any two parallel S_{n-1} in a S_n, and we draw from P a continuous aggregate of lines, which cut out of A (supposed to be nearer to P than B) a figure A', out of B a figure B', then the geometrical body of n manifoldness bordered by A', B', and the lines emanating from P is larger than $A'.h$ and smaller than $B'.h$; h being the distance of A' from B'.

It might be difficult to prove this when the foot of the perpendicular let fall from P on A and B falls outside the figures A' and B'. It is, however, sufficient for the present purpose to suppose that these points fall inside these two figures. Then, indeed, it is easily seen that the perpendicular projection of the figure A' on B' is totally enclosed by B', and that the projection of B' on A' encloses the figure A' on all sides. So, then, in conjunction with our last proposition, the truth of the theorem is at once seen to follow from this.

We are now enabled to prove the fundamental theorem: If Δ be the magnitude of any pyramid of n manifoldness and D be the magnitude of any one of its border pyramids of $n-1$ manifoldness, h the distance of that border S_{n-1} from the opposite corner, then

$$\Delta = \frac{D\,.\,H}{n}.$$

It will not detract from the generality of the theorem, if we prove it to be true only when the foot of the perpendicular from the vertex on the S_{n-1} is enclosed by the figure whose magnitude is D. For, if it be true under this restriction, it is easily seen that it must be generally true. We make therefore that assumption.

Cut h into any number m of equal parts, the points of intersection being A (the corner-point or vertex) and $H_1, H_2, \dots H_m$ (which last is the foot of the perpendicular let fall from A on the S_{n-1}). Through each H_i draw the space of $n-1$ manifoldness $S_{n-1}^{(i)}$ perpendicular to h, therefore parallel to the S_{n-1}. Then the whole pyramid is divided into m parts; each contained between two parallel spaces of $n-1$ manifoldness $S_{n-1}^{(i)}$ and $S_{n-1}^{(i+1)}$.

Now the pyramid of $n-1$ manifoldness in the $S_{n-1}^{(i)}$ is in all proportions similar to the S_{n-1} (as can be easily proved from the corresponding figure *in plano*). If the unit of length in the S_{n-1} is e, then the unit of length in the $S^{(i)}_{n-1}$ will be $e\,\frac{i}{m}$. If therefore D is the measure of the pyramid in the S_{n-1}, then $D\left(\frac{i}{m}\right)^{n-1}$ is the measure of the pyramid in the $S^{(i)}_{n-1}$ (applying supposition 4). It follows that, Δ being equal to the sum of the slices between $S^{(i)}_{n-1}$ and S^{i+1}_{n-1}, Δ is larger than $D\,\frac{h}{m}\,\Sigma\left(\frac{i}{m}\right)^{n-1}$, the i ranging from 0 up to $m-1$, and Δ is smaller than $D\,\frac{h}{m}\,\Sigma\left(\frac{i}{m}\right)^{n-1}$, the i ranging from 1 up to m.

If m now becomes indefinitely large, this becomes equivalent to the equation to be demonstrated, $\Delta = \frac{D.h}{n}$.

2. Let A, C, D, ... L be the corner-points of a pyramid; and let B be any point collinear with and intermediate between A and C. Then, from our conception of geometrical magnitude, the pyramid $ABD \ldots L$ + the pyramid $BCD \ldots L$ is equal to the pyramid $ACD \ldots L$. This equation we shall agree to write in an abbreviated form, thus,

$$AB + BC = CA,$$

implying the above.

The equation written down may now be understood to be quite generally true, whether B be intermediate between A and C or not, as long as B is collinear with A and C. This implies that the magnitude of a pyramid must also be capable of assuming negative values; that, to be definite,

$$AB = -BA,$$

and

$$AA = 0$$

In words: If two corner-points of a pyramid are transposed, its absolute value does not change; but in as far as the above theorem of addition is general it must change its sign. And, if two corner-points of a pyramid coincide, its value is $= 0$.

For the objects that we are attempting to attain the true values of the volumes of pyramids are not so much wanted as their proportions. We shall therefore, for the sake of brevity, designate by the words—"volume of the pyramid $A_1 A_2 \ldots A_{n+1}$" and denote by $[A_1 \ldots A_{n+1}]$ the

n! fold of its true volume. We then obtain the following equation:—

$[A_1 \dots A_{n+1}] = [A_1 \dots A_n]$ multiplied by the distance from A_{n-1} to the space $A_1 \dots A_n$.

$[AB \dots L]$ may be calculated by multiplying $[AB]$, the perpendicular from C on line AB, the perpendicular from D on plane ABC, &c. by each other. The perpendicular from a point upon a plane space, in which it is itself situated, is of course 0. We see therefore that

$$[AB \dots L] = 0$$

whenever any one of the corner-points of the pyramid is situated in the space of the others. If, conversely,

$$[ABC \dots KL] = 0,$$

then either

$$[ABC \dots K] = 0,$$

or else L must belong to the space $ABC \dots K$. From this it could be shown by induction that, whenever

$$[ABC \dots KL] = 0,$$

any one of the corner-points of the pyramid must be situated in the space of the others. If therefore $n+1$ is the number of the points $A, \dots L$,

$$[A \dots L] = 0$$

is the necessary and sufficient condition that the $A \dots L$ may be situated in plane space of less than n manifoldness.

If ξ, η are two border-spaces of the pyramid $A \dots L$ that contain all of its points together but have between themselves no corner-points in common, and if we denote by $[\xi]$ and $[\eta]$ the pyramids of the corner-points situated in ξ and η respectively, then, from the manner of forming the magnitude $[AB \dots L]$, it is clear that

$$[AB \dots L] = [\xi].[\eta].[\xi\eta],$$

where $[\xi\eta]$ is some factor that is only dependent on the relative situation of the two spaces ξ and η to each other and not on the special position that the corner-points occupy within ξ and η. In the same manner, if $\xi_1, \dots \xi_k$ are k border-spaces of $A \dots L$ comprising all of these points, but of which none has any corner-point in common with any other, then

$$[AB \dots L] = [\xi_1] \dots [\xi_k] \cdot [\xi_1 \xi_2 \dots \xi_k],$$

where $[\xi_i]$ denotes the volume of the pyramid of the corner-points situated in ξ_i and $[\xi_1 \xi_2 \dots \xi_k]$ a certain number determined only by the position of the *spaces* ξ_i relatively to each other.

If $[A \dots L]$ is different from zero, then also $[\xi\eta]$ is different from zero; and ξ and η will have no point in common with each other. If the point P belonged to both ξ and η, we might move the corner-points in ξ and η, so that one of them in either space ξ and η coincides with P, while the value of $[\xi]$ and $[\eta]$ remains unchanged. The value of the pyramid would not be altered by this process; but, two of its corner-points coinciding (in P), it must be zero. The necessary and sufficient condition that any two spaces ξ and η have any one point in common is consequently

$$[\xi\eta] = 0.$$

If $[\xi\eta]$ is different from zero, the space composed by the two (*i.e.*, the space of the $A \dots L$) will be denoted by $\xi\eta$. In the same manner, $\xi_1\xi_2 \dots \xi_k$ denotes the space composed by $\xi_1, \xi_2, \dots \xi_k$, *i.e.*, the space containing all the points of $\xi_1, \xi_2, \dots \xi_k$. We may easily verify that $[\xi_1 \dots \xi_k]$ is obtained by multiplying $[\xi_1 \xi_2]$ into the magnitude that ξ_3 forms with space $\xi_1 \xi_2$, this again into the magnitude that ξ_4 forms with space $\xi_1 \xi_2 \xi_3$, &c.; and that the necessary and sufficient condition that any ξ_i should have any point in common with the space composed by the other ξ is

$$[\xi_1 \xi_2 \dots \xi_k] = 0.$$

To define our terms: The space $\xi_1 \dots \xi_k$ will be called the composed space, the ξ_i the different components. If $[\xi] = 1$, ξ will be said to be in its normal form. If $[\xi] = c$, different from 1, c will be called the *weight* of ξ. And $[\xi_1 \dots \xi_k]$ will be called the factor of the composition. We thus may say: The weight of the composed space is equal to the product of the weights of the different components multiplied by the factor of the composition.

3. If we fix a point O on any straight line L and lay down a unit of length, all the points of the line may be determined by their distance from O. If, then, a is the measure of OA, c that of OB, then $[AB] = c - a$. Positive and negative sign are distinguished in accordance with the law

$$[AB] + [BC] = [AC].$$

From the algebraical identity

$$(b-a)(d-c)+(c-b)(d-a)+(a-c)(d-b)=0,$$

we conclude

$$[AB].[CD]+[BC].[AD]+[CA].[BD]=0,$$

where also $$[AB]+[BC]+[CA]=0.$$

Let us denote by k, l, m the values of

$$[AB], \quad [BC], \quad [CA];$$

then $$k[CD]+l[AD]+m[BD]=0.$$

D denotes here any point of L whatever. It is therefore natural to write the equation between $[CD]$, $[AD]$, $[BD]$ as an equation between the points C, A, B themselves, leaving D to be determined until the moment when the requirements of the work make it convenient.

The equation may be translated into words as follows:—Between any three points of a straight line a linear equation must exist. Or else any point C of a line may be expressed as a linear form of any two others A and B,

$$C=\alpha A+\beta B,$$

so that $$\alpha+\beta=1$$

(on account of $k+l+m=0$).

By such an equation C is uniquely determined. For it must imply, according to the meaning of such an equation, that

$$[CD]=\alpha[AD]+\beta[BD],$$

which becomes, when D coincides with C,

$$0=\alpha[AC]+\beta[BC].$$

The ratio in which C divides the segment AB, and therefore C itself, is known when α and β are given.

Let us now agree to use equations of this kind

$$aA+bB+\ldots+lL=0,$$

even when the points A, B, ... L are not situated upon one line. If the space S of these points is of n manifoldness, then we define the meaning of such an equation by laying down that, if valid at all, it must be equivalent to

$$a\,[AX]+b\,[BX]+\ldots+l\,[LX]=0,$$

where X may be any arbitrary S_{n-1} within the S_n of the points $A, \ldots L$.

We shall now successively prove (1), if an equation like the above holds good in space of n manifoldness S_n, it will not cease to be valid in any space containing that S_n.

(2) Between any $n+2$ points of a space of n manifoldness which are not situated in space of less than n manifoldness, there exists exactly one linear equation

$$aA+bB+\ldots+lL=0,$$

where
$$a+b\ \ +\ldots+l\ \ =0,$$

and (3) if $A \ldots L$ is a pyramid of non-vanishing magnitude in space of n manifoldness, and any point P of that space is given as a linear form of the corner-points of that pyramid

$$P=aA+\ldots+\lambda L,$$

then P is uniquely determined by this equation. .

To prove (1) let

$$a\,[AX]+b\,[BX]+\ldots+l\,[LX]=0,$$

where X is any S_{n-1} in the S_n containing all the points $A, B, \ldots L$. Let further Σ be a S_{n+1} containing the S_n; and let Y be any space of n manifoldness contained by Σ.

Y has with any straight line of Σ a point in common. This follows from the original definition of plane spaces (as well as by $[Yl]=0$ where l is any line of Σ, which is easy to prove). With a plane contained by Σ it will therefore have in common at least two points, consequently a straight line; and similarly, with the S_n of the points $A, \ldots L$, a S_{n-1} which may be called X. Let P be any point of Y not contained by X. Then

Y is composed by X and P;

therefore
$$[AY]=[AX].[(AX)\,P]$$
$$=[AX].[S_nP].$$

Similarly,
$$[BY]=[BX].[S_nP],$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$[LY]=[LX].[S_nP].$$

It follows that

$$a\,[AY]+b\,[BY]+\ldots+l\,[LY]=0.$$

From this the proposition (1) is verified by induction.

The truth of (2) is also demonstrated by induction. Assume it to be true when $n=k$. Then it will continue to hold good when $n=k+1$. A, B, C, ... L being a group of $k+3$ points in space of manifoldness $k+1$, the line AB will have one point P in common with the space S_{n-1} of the remaining points C, ... L. Now, we know that between A, B, P there exists some equation such as

$$\alpha A+\beta B-P=0, \quad \alpha+\beta=1,$$

and according to assumption there exists an equation such as

$$\gamma C+\delta D+\ldots+\lambda L-P=0, \quad \gamma+\delta+\ldots+\lambda=1.$$

The space S of the points A ... L comprises the spaces where these equations are valid; they are therefore both valid within it. It follows then that the two expressions

$$\alpha A+\beta B \quad \text{and} \quad \gamma C+\delta D+\ldots+\lambda L$$

must be the same; and it is incidentally noticed that

$$\alpha+\beta=\gamma+\delta+\ldots+\lambda.$$

One equation of the form

$$aA+bB+cC+\ldots+lL=0, \quad a+b+\ldots+l=0,$$

is therefore sure to exist. Assume the existence of another

$$a'A+b'B+c'C+\ldots+l'L=0.$$

Compose X by C ... L. Then

$$[CX]=[DX]=\ldots=[LX]=0,$$

and

$$a\,[AX]+C\,[BX]=0,$$

$$a'\,[AX]+C'\,[BX]=0,$$

showing that $a:b=a':b'$. Similarly

$$a:b:c:\ldots:l=a':b':c':\ldots:l'$$

proving (2) completely.

Let, finally,

$$P=\alpha A+\beta B+\ldots+\lambda L.$$

From what we know about points upon a straight line $\alpha A+\beta B$ is the $\alpha+\beta$-ple of some definite point A' upon the line AB,

$$\alpha A+\beta B+\gamma C=(\alpha+\beta)\,A'+\gamma C$$

is the $\alpha+\beta+\gamma$-ple of some definite point on the line $A'C$; *i.e.*, some definite point in the plane ABC. By induction, it is immediately clear that

$$\alpha A+\beta B+\ldots+\lambda L, \quad \text{where} \quad \alpha+\beta+\ldots+\lambda=1,$$

is some definite point in the space $AB \ldots L$.

The values $\alpha, \beta, \gamma, \ldots \lambda$ which fix the position of a point

$$P=\alpha A+\beta B+\gamma C+\ldots+\lambda L$$

will be called the "coordinates" of P, whenever $A, B, \ldots L$ is a fixed pyramid used all through in one piece of work.

From the theorems given many properties of plane spaces can at once be deduced. For instance, any S_k has with any S_h, both being contained in space S_n, where $k+h=n$, one point in common. For, assume any pyramid of $k+1$ points in the S_k $A_1 \ldots A_{k+1}$, and a similar pyramid in the S_h $B_1 \ldots B_{h+1}$. The total number of these points which are all situated in the S_n being $h+k+2=n+2$, one linear equation must exist between them (generally speaking)

$$a_1 A_1+\ldots+a_{k+1}\,A_{k+1}+b_1 B_1+\ldots+b_{h+1}\,B_{h+1}=0,$$

$$\Sigma a+\Sigma b=0.$$

If $\Sigma a=c$ then $a_1 A_1+\ldots+a_{k+1}\,A_{k+1}$ is the c-ple of the point common to both S_k and S_h.

Similarly it may be shown that a S_k and a S_h both of which are contained in space S_n must have a plane space of manifoldness $k+h-n$ in common ($k+h$ supposed to be larger than n).

If S_h, S_k, S_l are any three spaces contained in a S_n, and

$$h+k+l=n-1,$$

one straight line (generally speaking) can be drawn to cut all three spaces. Assuming

$h+1$ points in the S_h, $A_1 \ldots A_{h+1}$,

$k+1$ points in the S_k, $B_1 \ldots B_{k+1}$,

$l+1$ points in the S_l, $C_1 \ldots C_{l+1}$, we have $h+k+l+3 = n+2$ points in the S_n, between which therefore a linear equation will exist, viz.,

$$a_1A_1 + \dots + a_{h+1}A_{h+1} + b_1B_1 + \dots + b_{k+1}B_{k+1} + c_1C_1 + \dots + c_{k+1}C_{k+1} = 0.$$

Consequently the three points A, B, C belonging respectively to S_h, S_k, S_l, defined by

$$\alpha A = a_1A_1 + \dots + a_{h+1}A_{k+1},$$

$$\beta B = b_1B_1 + \dots + b_{k+1}B_{k+1},$$

$$\gamma C = c_1C_1 + \dots + c_{s+1}C_{l+1},$$

are collinear.

(Some theorems based on the propositions of this section are given in two articles by the author in *Nature*, August 8th and October 17th, 1895.)

The coordinates of a point have a certain geometrical significance.

Let $$a_1A_1 + a_2A_2 + \dots + a_{n+2}A_{n+2} = 0$$

for $n+2$ points $A_1, \dots A_{n+2}$ in a S_n.

From this

$$a_1[A_1X] + a_2[A_2X] + \dots + a_{n+2}[A_{n+2}X] = 0.$$

Assume for X the space $A_3A_4 \dots A_{n+2}$; then

$$a_1[A_1A_3 \dots A_{n+2}] + a_2[A_2A_3 \dots A_{n+2}] = 0.$$

If, therefore, $$a_1 = [A_2A_3 \dots A_{n+1}],$$

then $$a_2 = -[A_2A_3 \dots A_{n+1}],$$

and generally a_k is found by replacing, in $[A_2A_3 \dots A_{n+2}]$, a_k by a_1 and changing the sign of the whole.

In conformity with the rule of signs, we shall obtain

$$a_1 = [A_2A_3 \dots A_{n+2}],$$

$$a_2 = -[A_1A_3 \dots A_{n+2}],$$

$$a_3 = [A_1A_2A_4 \dots A_{n+2}],$$

$$a_4 = -[A_1A_2A_3A_5 \dots A_{n+2}],$$

&c.

4. Let $$A_{n+2} = a_1A_1 + \ldots + a_{n+1}A_{n+2}.$$

In accordance with the meaning of such equations, we obtain, by adding any combination $X = P_1P_2 \ldots P_n$ of n points in space $A_1 \ldots A_{n+1}$ to each member,

$$[P_1P_2 \ldots P_nA_{n+2}] = a_1[P_1P_2 \ldots P_nA_1] + \ldots + a_{n+1}[P_1P_2 \ldots P_nA_{n+1}].$$

Replacing A_{n+2} by its linear expression in the A_i, this assumes the form of an identity

$$[P_1P_2 \ldots P_n(a_1A_1 + \ldots + a_{n+1}A_{n+1}]$$
$$= a_1[P_1P_2 \ldots P_nA_1] + a_2[P_1P_2 \ldots P_nA_2] + \ldots .$$

Treating the P similarly, it is clearly shown that the values of pyramids whose points are linear forms of other points are found by treating them as if they were algebraical products.

Let A, B, P, Q be four points on one line, and

$$P = aA + bB,$$
$$Q = cA + dB;$$

then, according to the above,

$$[PQ] = ac[AA] + ad[AB] + bc[BA] + bd[BB].$$

If, now, the rule of signs is taken into consideration, this is seen to be

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} [AB].$$

Let, similarly, P, Q, R be three points in a plane, linearly expressed by means of three points A, B, C in the same plane whose triangle does not vanish

$$P = aA + bB + cC,$$
$$Q = a'A + b'B + c'C,$$
$$R = a''A + b''B + c''C.$$

Then it is easily seen that

$$\begin{aligned}[PQR] = {} & a\begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix}[ABC] \\ & + b\begin{vmatrix} c' & a' \\ c'' & a'' \end{vmatrix}[BCA] \\ & + c\begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}[CAB].\end{aligned}$$

But, according to the rule of signs, $[ABC]$, $[BCA]$, $[CAB]$ all denote the same value; therefore

$$[PQR] = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} [ABC].$$

It could, in exactly the same manner and in connection with some elementary properties of determinants, be shown that, *if* $n+1$ *points* $P_1, \ldots P_{n+1}$ *are expressed as linear forms of the* $n+1$ *corner-points of some pyramid* $A_1 \ldots A_{n+1}$ *in their space*

$$P_i = a_{i,1} A_1 + \ldots + a_{i,n+1} A_{n+1},$$

then $$[P_1 P_2 \ldots P_{n+1}] = \begin{vmatrix} a_{1,1} & \ldots\ldots & a_{1,n+1} \\ \vdots & & \\ a_{n+1,1} & \ldots\ldots & a_{n+1,n+1} \end{vmatrix} [A_1 \ldots A_{n+1}].$$

Let $A_1 \ldots A_{n+1}$ be a fixed pyramid, P a variable point. Put

$$P = x_1 A_1 + \ldots + x_{n+1} A_{n+1};$$

then the totality of points for which the x satisfy a linear equation

$$\mathrm{a}_1 x_1 + \ldots + \mathrm{a}_{n+1} x_{n+1} = 0$$

form a certain S_{n+1}. Indeed, let $P_1, \ldots P_{n+1}$ be $n+1$ points whose coordinates $x_{i,j}$ satisfy the given linear equation. Then from elementary properties of determinants their determinant $(x_{i,j})$ vanishes; therefore also $[P_1 \ldots P_{n+1}]$, showing that P_{n+1} belongs to the space of the other n points.

If Q, R are any two points, and

$$a_1 x_1 + a_2 x_2 + \ldots + a_{n+1} x_{n+1} = u = 0$$

be the equation of any S_{n-1}, then

$$uQ : uR,$$

or the proportion of the values obtained by inserting in place of the running coordinates contained in u those of Q and R is equal to

$$[S_{n-1} Q] : [S_{n-1} R],$$

that is, equal to the proportion of the perpendiculars from the two points on the S_{n-1}. Indeed, let $P_1, \ldots P_n$ be any n points fixing the S_{n-1}, X a variable point

$$X = x_1 A_1 + \ldots + x_{n+1} A_{n+1};$$

then $[P_1 \ldots P_n X]$ must be a multiple of u, both being homogeneous and linear in the x_i and expressing by their vanishing the same circumstance.

5. Let $A, B, \dots L$; $A', B', \dots L'$ be any two pyramids in the same space ξ. Let η be any other space. Then, from the formula of composition,

$$[AB \dots L\eta] = [AB \dots L] \, . \, [\xi\eta],$$

$$[A'B' \dots L'\eta] = [A'B' \dots L'] \, . \, [\xi\eta].$$

Should therefore

$$[AB \dots L] = k\,[A'B' \dots L'],$$

then also
$$[AB \dots L\eta] = k\,[A'B' \dots L'\eta].$$

This equation may again be written in an abbreviated form thus,

$$AB \dots L = k \, . \, A'B' \dots L'.$$

In accordance with this result we may lay down that by the words "space ξ" is meant any combination of points in that space whose pyramid is $= 1$; and by $k\xi$, k denoting any constant, any combination of points in that space whose pyramid is $= k$.

In order to express the fact that any two spaces S and T are identical, we write $S \equiv T$, saying S congruent to T. Thus the straight line L, joining two points A, B, is $L \equiv AB$. However, according to the above, L only $= AB$ when $[AB] = 1$. Generally $AB = [AB] \, . \, L$.

A linear equation between spaces of the same manifoldness $\xi_1, \xi_2, \dots \xi_k$,

$$\text{(E)} \quad c_1\xi_1 + c_2\xi_2 + \dots + c_k\xi_k = 0,$$

is defined as an abbreviation for the equation between numerical values

$$c_1\,[\xi_1\eta] + c_2\,[\xi_2\eta] + \dots + c_k\,[\xi_k\eta] = 0,$$

η denoting an arbitrary space, such that

$$\xi_1\eta \equiv \xi_2\eta \equiv \dots \equiv \xi_k\eta.$$

It is immediately seen that from (E) it follows that also

$$c_1\xi_1\vartheta + c_2\xi_2\vartheta + \dots + c_k\xi_k\vartheta = 0,$$

ϑ denoting an arbitary space.

If a linear equation exists between two spaces, they must be congruent.

Let, indeed, ξ and η be any two spaces, P any point of η. From

$$c_1\xi = c_2\eta,$$

we conclude
$$c_1\xi P = c_2\eta P = 0;$$

i.e., any point belonging to η also belongs to ξ; therefore $\xi \equiv \eta$.

If a linear equation exists between three spaces of k manifoldness ξ_1, ξ_2, ξ_3, they must have a S_{k-1} in common; and they will be contained in a S_{k+1}. And, conversely, if this is so, then a linear relation must exist between them.

Indeed, let $$c_1\xi_1 + c_2\xi_2 + c_3\xi_3 = 0.$$

If P is a point common to ξ_2 and ξ_3, then

$$\xi_2 P = 0 \quad \text{and} \quad \xi_3 P = 0;$$

but $$c_1\xi_1 P + c_2\xi_2 P + c_3\xi_3 P = 0;$$

consequently also $$\xi_1 P = 0;$$

i.e., a point common to two spaces ξ belongs also to the third.

Let Σ be the space common to $\xi_1\xi_2\xi_3$, A_1 any point of ξ_1 not contained in Σ, A_2 any point of ξ_2 not contained in Σ. Join A_1, A_2.

From $$c_1\xi_1 A_1A_2 + c_2\xi_2 A_1A_2 + c_3\xi_3 A_1A_2 = 0,$$

we conclude, since $$\xi_1 A_1 = 0, \quad \xi_2 A_2 = 0,$$

that also $$\xi_3 A_1A_2 = 0;$$

i.e., that the line A_1A_2 has a point in common with ξ_3. Let this point be denoted by A_3.

Let, now, A_1' be a point, if such a point exists, belonging to ξ_1, but not to ΣA_1. Join A_1' with A_2. Any line cutting ξ_1 and ξ_2 will, as we have seen, also cut ξ_3. Let $A_1'A_2$ cut ξ_3 in A_3'. A_1A_1' and A_3A_3' will be contained in the plane $A_1A_2A_1'$; therefore have a point P in common. P belonging to both ξ_1 and ξ_3 will be contained in Σ, *i.e.*,

$$\Sigma A_1 A_1' = 0.$$

We see therefore that all points of ξ_1 are contained in ΣA_1; so we obtain

$$\xi_1 \equiv \Sigma A_1, \quad \xi_2 \equiv \Sigma A_2, \quad \xi_3 \equiv \Sigma A_3.$$

Σ is therefore of manifoldness $k-1$, and the space containing ξ_1, ξ_2, ξ_3 is $\Sigma A_1 A_2$, of manifoldness $k+1$.

Conversely, if ξ_1, ξ_2, ξ_3 are three spaces of manifoldness k, having a S_{k-1} (Σ) in common and contained in a S_{k+1}, they will be cut by any line of the S_{k+1} not belonging to the S_{k-1} in one point each, denoted respectively by A_1, A_2, A_3,

$$\xi_1 \equiv \Sigma A_1, \quad \xi_2 \equiv \Sigma A_2, \quad \xi_3 \equiv \Sigma A_3.$$

A_1, A_2, A_3 being collinear, a certain relation will exist between them,

$$c_1 A_1 + c_2 A_2 + c_3 A_3 = 0;$$

therefore also $$c_1\Sigma A_1+c_2\Sigma A_2+c_3\Sigma A_3=0,$$

which may also be written

$$c_1[\Sigma A_1]\,\xi_1+c_2[\Sigma A_2]\,\xi_2+c_3[\Sigma A_3]\,\xi_3=0.$$

The theorem proved may be expressed differently thus:—

(1) If two spaces ξ_1, ξ_2 of k manifoldness have a space S_{k-1} in common, then any linear form of them $c_1\xi_1+c_2\xi_2$ is congruent to some space ξ_3, also containing the S_{k-1}.

(2) If two spaces ξ_1, ξ_2 of k manifoldness have not a space S_{k-1} in common, then any linear form of them $c_1\xi_1+c_2\xi_2$ cannot be congruent to any space, but will have a symbolical significance only.

If, for instance, l_1, l_2 are two lines in space which have no point in common, then $c_1l_1+c_2l_2$, or any expression equivalent to it, will not be represented by a line. But nature adds forces in the same manner as lines are added in the sense defined above, so that such expressions $c_1l_1+c_2l_2$ may very well be employed to express the effect of a system of forces acting upon a rigid body, or the instantaneous movement of such a body.

Let $\xi_1, \xi_2, \dots \xi_m$ be spaces of k manifoldness. Then any linear form of them

$$c_1\xi_1+c_2\xi_2+\dots+c_m\xi_m+\dots$$

will be called a form of manifoldness k.

Between the border S_k's of any pyramid no linear relation can possibly exist. But any space of k manifoldness in the space of the pyramid can be represented as a linear form of these border S_k's.

Let $A_1, \dots A_{n+1}$ be the corner-points of any pyramid in space S_n. To prove the first part of the proposition assume any linear relation between the border S_k of the pyramid

$$c_1A_1\dots A_{k+1}+\dots+c_j\xi_j+\dots=0,$$

the c denoting constants, the ξ_j border S_k's. Add the combination

$$\eta=A_{k+2}A_{k+2}\dots A_{n+1},$$

containing all corner-points but the $A_1 \dots A_{k+1}$; then

$$[\xi_j\eta]=0;$$

therefore $$c_1[A_1\dots A_{k+1}A_{k+2}\dots A_{n+1}]=0.$$

$[A_1\dots A_{n+1}]$ is different from zero according to hypothesis. Therefore $c_1=0$, and generally $c_j=0$.

In any space ξ of k manifoldness in the S_n, assume any $k+1$ points $P_1, P_2, \dots P_{k+1}$, whose pyramid is $=1$. Then, according to definitions

$$\xi = P_1 P_2 \dots P_{k+1}.$$

But, the P being situated in the S_n fixed by the pyramid $A_1 \dots A_{n+1}$, they must be expressible as linear forms of these points A_i,

$$\begin{array}{l} P_1 \quad = c_{1,1} \quad A_1 + c_{1,2} \quad A_2 + \dots + c_{1,n+1} \quad A_{n+1}, \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ P_{k+1} = c_{k+1,1} A_1 + c_{k+1,2} A_2 + \dots + c_{k+1,n+1} A_{n+1}. \end{array}$$

$P_1 \dots P_{k+1}$ is therefore obtained by multiplying the linear expressions with each other, under consideration of the rule of signs, and the law that two identical factors give zero as result. The expression thus obtained defines ξ uniquely, as no other expression of the same form can be equal to it. The coefficients of the border S_k's are expressible in the form of determinants of the matrix

$$\left| \begin{array}{ccc} c_{1,1} & \dots\dots & c_{1,n+1} \\ \vdots & & \vdots \\ c_{k+1,1} & \dots\dots & c_{k+1,n+k} \end{array} \right|,$$

according to what has been shown before.

The most general form of manifoldness k in space S_n is consequently a linear form of the border S_k's of any pyramid situated in the S_n. As such it has altogether $(n+1)_{k+1}$ independent coefficients (coordinates).

The forms of manifoldness 2 can be given a certain shape which seems simpler than any other.

If a and b are any two lines, and $[ab]=0$, then a and b will have a point in common, and any linear form $ha+kb$ will therefore be congruent to some line. If, however, $[ab]$ is different from zero, such a reduction will not be possible, as was shown before.

Let A, B, C, D be the four corners of a pyramid in a S_3. The general form of the second order in the S_3 is

$$X = c_1 AB + c_2 AC + c_3 AD + c_4 BC + c_5 BD + c_6 CD,$$

i.e.,

$$= A\,(c_1 B + c_2 C + c_3 D) + c_4 BC + c_5 BD + c_6 CD.$$

But $c_1B+c_2C+c_3D$ is $\equiv$ some point,

$c_4BC+c_5BD+c_6CD$ is $\equiv$ some line,

in the plane BCD; therefore

$$X = ha+kb,$$

where a passes through an arbitrary point A, and b is totally situated in an arbitrary plane BCD (not containing A).

This, then, is the reduced form of the second order in space.

Similarly, $A_1, A_2, \dots A_5$ being the five corner-points of a pyramid in a S_4,

$$X = hA_1P+Y,$$

where X is the most general form of the second order in the S_4, and Y the most general form of manifoldness 2 in the S_3 $A_2A_3A_4A_5$, and P belongs to that S_3.

Put, then, $$Y = ka+lb,$$

so that a passes through P; then hA_1P+ka is again $\equiv$ some line c; and we finally obtain

$$X = mc+nb,$$

where m, n are some constants. The line of reasoning is thus indicated. We conclude: *The most general form of the second order in space* S_n *is*

$$X = c_1a_1+\dots+c_\nu a_\nu,$$

where the c are constants and the a lines, and ν is $=\frac{1}{2}n$ or $=\frac{1}{2}(n+1)$ according as n is even or odd.

If the a_i are all situated in a space Σ, we may say that X belongs to Σ. What, then, is the condition that X belongs to a space of manifoldness h, and not to space of lower manifoldness?

If $$X = ha+kb,$$

$$XX = X^2 = hkab,$$

aa and bb being zero, and

$$X^3 = 0.$$

If $$X^2 = 0,$$

then $$h\,.\,k = 0 \quad \text{or} \quad [ab] = 0,$$

from which we conclude that X must then simply be a line.

Therefore the necessary and sufficient condition that X should be a line is

$$X^2 = 0,$$

and that X should belong to a S_3 is

$$X^3 = 0,$$

and quite generally, concluding in the same way as above: *The necessary and sufficient condition that X should belong to a space of $2n+1$ manifoldness is*

$$X^{n+2} = 0.$$

And the space to which X belongs is $\equiv X^{n+1}$.

In the language of determinants this leads to the following theorem. Let, first of all, four points A_1, A_2, A_3, A_4 be the corner-points of a pyramid in space S_3, and

$$P = c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4,$$

$$Q = d_1 A_1 + d_2 A_2 + d_3 A_3 + d_4 A_4.$$

Then $$X = PQ = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} A_1 A_2 + \begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} A_1 A_3 + \dots$$

$$= \Delta_{1,2} A_1 A_2 + \Delta_{1,3} A_1 A_3 + \Delta_{1,4} A_1 A_4 + \Delta_{2,3} A_2 A_3$$

$$+ \Delta_{2,4} A_2 A_4 + \Delta_{3,4} A_3 A_4.$$

This not being the most general form of manifoldness 1 in S_3, the Δ must satisfy the relation expressed by

$$X^2 = 0.$$

Developed, this relation is seen to be

$$\text{(E)} \quad \Delta_{1,2} \Delta_{3,4} + \Delta_{1,3} \Delta_{2,4} + \Delta_{1,4} \Delta_{2,3} = 0.$$

Or, in the language of algebra, between the six determinants of the matrix

$$\begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

the above relation (E) holds good, and, if (E) be satisfied, then the magnitudes $\Delta_{i,j}$ can be expressed as determinants of such a matrix.

Let, now, quite generally $A_1 \dots A_{n+1}$ be any pyramid in space S_n,

$$\begin{array}{lll} a_1 \text{ be any line} & \equiv \Delta^{(1)}_{1,2} A_1 A_2 + \Delta^{(1)}_{1,3} A_1 A_3 + \dots & + \Delta^{(1)}_{i,j} A_i A_j + \dots, \\ a_2 \quad ,, \quad ,, & \equiv \Delta^{(2)}_{1,2} A_1 A_2 + \dots\dots\dots\dots & + \Delta^{(2)}_{i,j} A_i A_j + \dots, \\ a_\nu \quad ,, \quad ,, & \equiv \Delta^{(\nu)}_{1,2} A_1 A_2 + \dots\dots\dots\dots & + \Delta^{(\nu)}_{i,j} A_i A_j + \dots, \end{array}$$

$$X = h_1 a_1 + h_2 a_2 + \dots + h_\nu a_\nu = \Delta_{1,2} A_1 A_2 + \dots + \Delta_{i,j} A_i A_j + \dots.$$

If ν is smaller than $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, then X is not the most general form of the second order in space $A_1 \dots A_{n+1}$. The $\Delta_{i,j}$ must therefore satisfy a number of conditions. They are all expressed by the equation

$$\text{(E)} \quad X^{\nu+1} = 0,$$

when X^ν is supposed to be different from 0. In algebraic form

$$\text{(E}'\text{)} \quad \Sigma \Delta_{i_1, j_1} . \Delta_{i_2, j_2} . \Delta_{i_3, j_3} \dots \Delta_{i_{\nu+1}, j_{\nu+1}} = 0,$$

where the summation is to be extended over all indices

$$i_1, j_1, \dots, i_{\nu+1}, j_{\nu+1},$$

different from each other, but which belong to a circle of $2\nu+2$ integers; (E′) then is a series of necessary and sufficient conditions that magnitudes $\Delta_{i,j}$ shall be expressible as one and the same linear form

$$h_1 \Delta^{(1)}_{i,j} + h_2 \Delta^{(2)}_{i,j} + \dots + h_\nu \Delta^{(\nu)}_{i,j}$$

of the determinants $\Delta^{(k)}_{i,j}$ of matrices

$$\left| \begin{matrix} c^{(k)}_1 & \dots\dots & c^{(k)}_{n+1} \\ d^{(k)}_1 & \dots\dots & d^{(k)}_{n+1} \end{matrix} \right| .$$

We might put this again into determinant form, but we leave the matter here, as it lies too far apart from the object here pursued.

Forces acting on a rigid body have a certain line of action a, and a certain intensity h, so that, by identifying them with $h . a$, they are perfectly defined.

Two forces in the same plane have the same effect as one force according to the parallelogram of forces. Let a, b be the two lines of the forces f_1, f_2, P the point of intersection of a and b. Further, determine A on a, and B on b, so that

$$h = [PA],$$

and

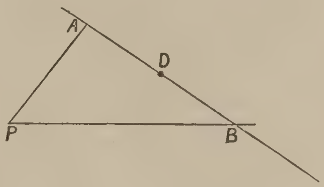
$$k = [PB],$$

where

$$f_1 = h . a = PA,$$

and

$$f_2 = k . b = PB.$$



The resultant of f_1 and f_2 is double PD, where D is the centre of the finite line BA; therefore

$$D = \tfrac{1}{2}(A+B),$$

according to the parallelogram of forces. So, then, the resultant of f_1 and f_2 is

$$= 2PD = PA+PB = f_1+f_2.$$

This is also true when a and b are parallel, and even when

$$h+k=0$$

(the two forces then forming a couple); only that then the line of action of the force is a certain exceptional line which will afterwards be spoken of as the line infinity, and which would again be characterized by f_1+f_2, in situation as well as in regard to a certain factor (the magnitude connected with the couple).

By applying the calculus to the formulæ of mechanics, the same result would be attained for forces acting on rigid bodies in space. The effect of a system of forces upon a rigid body would then be seen to be expressible by a form of manifoldness 1. The resultant of a system of forces would simply be their sum (in the sense I defined above), and the corresponding infinitesimal motion of the body would also be determined by the same form of manifoldness 1.

6. We have not in the preceding sections again mentioned the possibility that presented itself in the introduction, namely, that two spaces might be parallel. Projective geometry shows how to connect parallelism with the general theory. Parallel spaces *in plano*, or in space, are such as intersect in a certain line or plane, the line or plane at infinity. The same is true for spaces of any degree of manifoldness.

For three points on a line we had

$$[AB]+[BC]+[CA]=0,$$

similarly for four points in a plane

$$[ABC]-[BCD]+[CDA]-[DAB]=0,$$

and generally, $A_1, A_2, \ldots, A_{n+1}$, denoting any pyramid fixing a space

S_n, and P any point of that space

$$[A_2 \ldots A_{n+1}P]-[A_1A_3 \ldots A_{n+1}P]+[A_1A_2A_4 \ldots A_{n+1}P]-\ldots$$
$$\ldots \pm [A_1A_2 \ldots A_{n+1}] = 0.$$

This formula, indeed, is obtained by means of the last proposition of § 3, identifying A_{n+2} with P, and applying

$$a_1+a_2+\ldots+a_{n+2}=0.$$

The equation may also be written, the space

$$A_2 \ldots A_{n+1}-A_1A_3 \ldots A_{n+1}+A_1A_2A_4 \ldots A_{n+1}, \text{ \&c.},$$

being denoted by I,

$$[IP] = \mp [A_1A_2 \ldots A_{n+1}].$$

I is defined as a certain form of manifoldness $n-1$ in space of manifoldness n. Two S_{n-1}'s contained in a S_n have a S_{n-2} in common. Therefore a linear form of such S_{n-1}'s is again some S_{n-1}; and consequently I must be some S_{n-1}. The above equation, however, shows that all points P form with I one and the same magnitude. This is an apparent contradiction which has to be explained.

Let A, B, C, D be the four corners of a parallelogram. The diagonals bisecting each other, their point of intersection must be

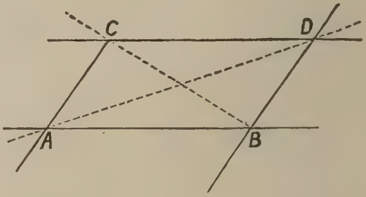
$$\tfrac{1}{2}(A+D) \text{ and also } \tfrac{1}{2}(B+C)$$

showing that

$$A+D=B+C,$$

or

$$A-B=C-D.$$



We found that $\alpha A+\beta B$ expressed the $\alpha+\beta$-ple of a point on the line AB. Here we see that, when $\alpha+\beta=0$, the point in question also belongs to any line parallel to AB. The significance of $\alpha+\beta=0$ is shown by letting $\alpha+\beta$ assume continuously varying values, with zero as limit. Let, for instance, α be stable $=1$, β approach -1 as limit. The point $P=\alpha A+\beta B$ will then travel away from the points A, B; the more nearly β approaches -1, the farther away P will move, and the more nearly the value of $[PA]:[PB]$ will approach unity. In the limit $\beta=-1$, P will be at infinity; *$A-B$ is therefore $\equiv$ the point at infinity of the line AB.*

The equation

$$[IP] = \mp [A_1 \dots A_{n-1}]$$

may now be interpreted as meaning

$$[IP] = [IQ],$$

where Q is any other point of the S_n; or else

$$[I(P-Q)] = 0,$$

showing that I contains all the points at infinity, belonging to any line PQ in space S_n. I is accordingly called the S_{n-1} infinity.

We shall define I by the form

$$I = A_2 A_3 \dots A_{n+1} - A_1 A_3 \dots A_{n+1} + \dots$$

with the condition, however, that the value of $\mp [A_1 \dots A_{n+1}]$ giving $[IP]$ must $= 1$. Hence $[IP] = 1$, where P is any finite point;

therefore $\quad A_2 A_3 \dots A_{n+1} - A_1 A_3 + \dots A_{n+1} + \dots = c . I,$

where c is the value of $\mp [A_1 \dots A_{n+1}]$.

Two S_{n-1} in the S_n, say S and T, which are parallel, intersect in I.

Indeed let AB be any line in S. From A let fall a perpendicular on T, cutting it in A'. In the plane BAA' draw the parallel $A'B'$ to AB through A', which, being perpendicular to AA', must belong to T. But AB and $A'B'$ have in common their point of intersection with I. AB being perfectly arbitrary, it is evident that the S_{n-2} in which S intersects I must also belong to T.

S, T, I are therefore connected by a linear equation

$$aS + bI = T.$$

To find the significance of a and b we compose this equation with P, where P is any point of S.

Thus we obtain $\qquad b = [TP],$

showing that b is the perpendicular distance of S from T. Composing with any point D of I,

$$a[SD] = [TD].$$

It will afterwards be shown that $[SD]$ is the sine of the angle that S forms with the lines that pass through D. So then

$$a = 1,$$

since parallel spaces form the same angle with any line, and

$$bI = T - S$$

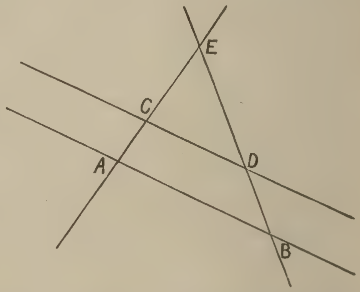
(where T and S are in their normal form).

Let $A, B, \dots L$ be any pyramid fixing a S_n. Since $A-B, A-C, \dots$ $\dots K-L$, all belong to I, it is evident that I contains all the points

$$xA+yB+\dots+zL,$$

for which $$x\;+y\;+\dots+z=0.$$

The latter is therefore the equation of I when coordinates are used.



Let A, B, C D be four points in a plane connected by the relation

$$A-B=\lambda\,(C-D)\,;$$

AB must be parallel to CD. From the above we obtain

$$A-\lambda C = B-\lambda D,$$

showing that the point E of intersection of AC and BD divides the segments AC and BD in the same ratio $\lambda : 1$, which is also $=\frac{[AB]}{[CD]}$. This makes it evident that the symbol $A-B$ is expressive of a certain direction as well as a certain magnitude; the magnitude being $[AB]$ and the direction being marked by the point at infinity of the line AB.

$A-B$ denotes therefore a certain "sect," parallel to AB and equal to it in length; $\lambda\,(A-B)$ denotes, similarly, a sect parallel to AB and equal to $\lambda\,[AB]$ in length (although, to be quite strict, such symbols should be regarded as denoting the point at infinity $A-B$ common to all lines parallel to AB, multiplied into a certain magnitude).

If P is a point, D any sect $=\lambda\,(A-B)$, then $P+D$ is a point, easily constructed by drawing through P in the plane PAB the line PQ parallel to AB and equal in length to $\lambda\,[AB]$. The magnitude of a sect may be positive or negative, according to the law of signs introduced in § 2. It may be denoted by the symbol $[\dots]$. Thus $[D]$ denotes the magnitude of the sect D. In applications where only its absolute value is considered, regardless of the sign, that value may be written, in conformity with a notation already in vogue, thus, $|\,D\,|$.

Let $D_1, D_2, \dots D_n$, be any n sects; $Q=P+D_1+D_2+\dots+D_n$ is found, as is easily seen, by describing a polygon whose one corner is P

and whose sides are successively parallel and equal in length to $D_1, D_2, \dots D_n$. The ultimate point of this polygon is Q. It closes when

$$D_1+D_2+\dots+D_n = 0.$$

For some applications of the preceding theory (although not affecting our present purpose) it may be well to remark that

$$| D_1+D_2+\dots+D_n |$$

is, at most, equal to

$$| D_1 | + | D_2 | +\dots+ | D_n |;$$

and that $$D_1+D_2+\dots+D_n+\dots, \lim n = \infty,$$

converges towards a definite sect of finite length whenever the series $| D_1 | + | D_2 | +\dots+ | D_n | +\dots$, $\lim n = \infty$, is convergent; and *vice versa*. This is immediately seen by considering the geometrical significance of $D_1+D_2+\dots+D_n$.

Points at I obey the same laws, so far as their composition is concerned, as points in the finite portion of the space S_n. So, then, D_1D_2 denotes the line at infinity joining the points at infinity D_1, D_2. We may represent D_1D_2 by means of triangles of a certain magnitude whose plane is parallel to a certain plane. The magnitude in question is $[P(P+D_1)(P+D_2)]$; the plane is PD_1D_2, P denoting any point whatever.

Generally the geometrical substrate of $D_1D_2 \dots D_n$, *i.e.*, the space S_{n-1} of these points at infinity, is a pyramid of n manifoldness, whose space is parallel and which is equal in magnitude to the pyramid

$$P(P+D_1) \dots (P+D_n),$$

P being perfectly arbitrary. The magnitude of the pyramid

$$P(P+D_1) \dots (P+D_n)$$

may, for shortness, be denoted by

$$[D_1D_2 \dots D_n],$$

which is allowable, this value being quite independent of how P is chosen (as may, for instance, be shown by the theorem of composition).

If I' is any space, D any point, both at I, then $[I'D]$ or the magnitude formed with space I' by point D is the sine of the angle which the direction of D forms with that of the space I'; the angle DI'

being defined as the angle formed by any line PD with its perpendicular projection on space PI'. For this is exactly the magnitude of a pyramid whose base is a pyramid of magnitude 1 in PI' and whose vertex is $P+D$ (where D is of length 1). $[D_1D_2 \dots D_n]$ is therefore calculated by multiplying the $[D_1][D_2] \dots [D_n]$ by the sine of the angle D_1D_2, this again by the sine of the angle that D_1D_2 forms with D_3 &c., according to the theorem of composition.

The conclusions of § 3 applied to sects show that the finite points of the space $PD_1 \dots D_n$ are expressible in the form

$$P+x_1D_1+\dots+x_nD_n,$$

where the x_i may assume any values whatever.

If S is any S_{n-1} in a S_n, then $[SP]$ denotes the length of the perpendicular from P on S. Interpreted in this manner, $[IP]$ would not $=1$, but be infinite. The explanation is that I belongs to a class of spaces (of which it is the only real representative) to which the conception of normal form as originally given does not apply. The reason for this will very soon appear.

If P is any finite point, D a variable point at I in its normal form, then $P+D$ will cover one half of the surface of a spherical manifoldness whose centre is P, the other half being represented by $P-D$. The geometry on the surface of a spherical manifoldness is therefore identical, in metrical as well as projective relations, with the geometry of points at I.

7. The calculus whose outlines have been laid down in the preceding paragraphs may be divested of its geometrical meaning; and it will then become a calculus of linear forms and of determinants.

Indeed, let $A, B, \dots L$ be n linear forms in n homogeneous variables, and let $AB \dots L$ denote the corresponding determinant; AB a matrix of two rows, the other rows $C', D', \dots L'$ left indeterminate; ABC similarly a matrix of three rows, &c. And let any equation such as, for instance,

$$\mathrm{a}.AB+\mathrm{b}.CD+\mathrm{c}.EF=0,$$

if valid at all, be understood as an abbreviation of an identity between matrices (AB, CD, EF) where the rows left indeterminate in these matrices are supposed to be identical. Then it is, indeed, easily enough seen that the laws of the geometrical calculus are expressions of elementary properties of determinants. For instance,

$$AA=0$$

would signify that the equality of two rows in a determinant causes it to vanish,

$$AB = -BA$$

that the transposition of two rows makes it change its sign.

$$(A+B)\,C = AC+BC$$

is easily seen to follow from the elementary fact that a determinant is a linear function of the terms of each row or column. And, finally, the significance of

$$ABC \dots L = 0$$

is obviously that A, B, C, … L are connected by some linear equation; that they are not "linearly independent."

8. The calculus is applicable to the geometry whose elements are:

(1) Plane spaces S_k through a fixed S_{k-1}.

(2) Plane spaces S_{n-1} in a fixed space of n manifoldness.

Indeed, any two S_k having S_{k-1} in common may be linearly connected so as to form another S_k through that S_{k-1}. Any element of this geometry may be generated by composing the fixed S_{k-1} with points outside of it; any linear manifoldness of elements of this geometry by composing the S_{k-1} with plane spaces outside the S_{k-1}. A space S_n will obviously be of manifoldness $n-k$ in regard to the elements of this geometry, the fixed S_{k-1} being contained by the S_n. If

$$E_1 = S_{k-1}A_1,$$
$$E_2 = S_{k-1}A_2,$$
$$\dots \quad \dots \quad \dots$$
$$E_h = S_{k-1}A_h$$

are elements of this geometry, we need only identify

$$[E_1 \dots E_h] \text{ with } [S_{k-1}A_1 \dots A_h],$$
$$E_1 \dots E_h \text{ with } S_{k-1}A_1 \dots A_h,$$

and the whole theory of point geometry is at once transferred to this geometry.

(2) is proved by reference to § 5, whence it appears that the S_{n-1} of a S_n form a linear manifoldness of degree n; that they are linearly expressible by the $n+1$ border S_{n-1} of any non-vanishing pyramid in the S_n; and that any linear form of these border S_{n-1} represents again, unconditionally, a S_{n-1}.

As regards geometry (1) $[E_1 \dots E_h]$ (in the new sense) was found to be $= [S_{k-1} A_1 \dots A_h]$ (in the old sense). If S is any one of its spaces,

$$S = S_{k-1}\xi$$

in its normal form (according to the original definition), and E any element

$$E = S_{k-1}A,$$

also in its normal form, then

$$[SE] = [S_{k-1}\xi A]$$

is, according to the theorem of composition, the perpendicular distance of A from S. If then the angle formed by two lines emanating from a point in the S_{k-1} into S and E respectively, and perpendicular to the S_{k-1}, is called the angle $\angle SE$, then it is easily seen that

$$[ES] = \sin \angle SE.$$

If the S_{k-1} lies entirely at I, then $[SE]$ is similarly seen to be the perpendicular distance of the two parallel spaces S and E. It is a remarkable fact that the same is true in geometry (2), as will be shown by the following line of reasoning.

The geometry (2) may be called the "reciprocal" geometry, and its composition be denoted by a vertical line (so that A/B would designate the space composed in this geometry by A and B).

Let then $ABCDE \dots L$ be any pyramid in the fixed S_n; and take for definiteness

$$X = ABCDE,$$

$$Y = CDEFG \dots L.$$

Then X and Y will have the plane CDE in common, and no point besides; since, if

$$P = aA + bB + cC + dD + eE$$

were a point of X also contained by Y,

$$YP = 0$$

would necessarily imply

$$[AB \dots L] = 0,$$

or else

$$a = 0, \quad b = 0.$$

Let, further, $A_1 \dots A_{n+1}$ be another pyramid, and

$$\begin{aligned} A &= a_{1,1}A_1 + \dots + a_{1,n+1}A_{n+1}, \\ &\vdots \\ L &= a_{n+1,1}A_1 + \dots + a_{n+1,n+1}A_{n+1}. \end{aligned}$$

Then the coordinates of CDE expressed by the border planes of the A_i are determinants of the third order in the $a_{i,j}$. The coordinates of X and Y are determinants of order 5 and $n-1$ in the $a_{i,j}$. Now X/Y is formed in accordance with the rules of the calculus. If

$$X = c_1P_1 + c_2P_2 + \dots,$$

$$Y = c_1'P_1' + c_2'P_2' + \dots,$$

the c being constants and the P border-spaces of the A_i, then

$$X/Y = c_1c_1'P_1/P_1' + c_1c_2'P_1/P_2' + \dots .$$

The result must be CDE, as we found before, multiplied by some constant. This constant must be in the $a_{i,j}$ of order $n+1$. And it will never vanish so long as the assumption made is complied with, *i.e.*, so long as $[A \dots L]$ is different from zero. Therefore it cannot be different from $[A \dots L]$ itself.

If, then, BX is a S_{n-1}, A a point outside of it,

$$AX/BX = [ABX].X,$$

as is seen by considerations similar to the above. From this the proposition to be demonstrated (which might be called the sine theorem) follows exactly as in case (1). The factor of composition (in the reciprocal geometry) of ξ/η, where η is a S_{n-1}, and ξ any space, is the sine of the angle formed by ξ and η. If ξ is parallel to η, it is their distance; and the same is true when ξ is a point.

The S_{n-1} I forms with any finite S_{n-1} in its normal form the magnitude 1. Indeed, let A, B be two parallel S_{n-1} in their normal form, such that their distance is $= 1$.

Then $$I = A - B;$$

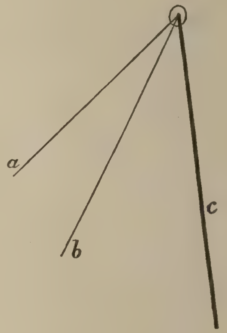
therefore $[IB]$ in the reciprocal sense

$$= [AB] \text{ in the reciprocal sense} = 1,$$

and, similarly, $$[IA] = -[BA] = 1.$$

With this the formal laws of the calculus are complete, since, by means of the theorem of composition in the original and reciprocal form, the coefficients occurring in any piece of work can always be determined.

9. Infinity is represented in the two geometries introduced in the last section in a manner very different from that in which it was expressed in point geometry.



Let O be a fixed point through which pass three rays a, b, c, situated in the same plane.

Let $\angle ab$ be denoted by γ,
$\angle bc$,, ,, α,
$\angle ca$,, ,, β,

a, b, c may be in their normal form. They are then connected by the relation

$$[bc]\,a+[ca]\,b+[ab]\,c=0,$$
$$\sin\alpha\,.\,a+\sin\beta\,.\,b+\sin\gamma\,.\,c=0.$$

If a, b are fixed, and c varies, then the angles α, β will vary. As long as α and β remain real, $\sin\alpha$ and $\sin\beta$ will also remain real and determinate quantities. There is no reason why we should restrict ourselves to real values only, the right of existence of imaginary quantities and geometrical entities in geometry having been long affirmed. If α and β assume, then, complex values, $\sin\alpha$ and $\sin\beta$ will still remain definite. This ceases only when α and β become infinite.

Let us now investigate the meaning of $\sin\alpha$ and $\sin\beta$ becoming infinite. α, β, γ being connected by the relation

$$\alpha+\beta+\gamma=0,$$

where γ is constant, $\sin\alpha$ and $\sin\beta$ will become infinite simultaneously. From

$$\sin^2\alpha+\cos^2\alpha=1,$$

$$\lim\frac{\cos\alpha}{\sin\alpha}=i \qquad (i^2=-1),$$

when

$$\lim\sin\alpha=\infty.$$

From $\sin\alpha\cos\gamma+\cos\alpha\sin\gamma+\sin\beta=0$, dividing by $\sin\alpha$,

$$\lim\frac{\sin\beta}{\sin\alpha}=-(\cos\gamma\pm i\sin\gamma),$$
$$\lim\sin\alpha=\infty.$$

In words: Let a, b be any two lines in a plane, so that $\lambda a+\mu b$, where λ and μ are any two constants, is $\equiv$ a line in their plane, and through their point of intersection. If γ is the angle formed by a and b, then $\lambda:\mu=-(\cos\gamma\pm i\sin\gamma)$ determines two lines of this pencil, which form with a and b, and therefore with any other line of the pencil, an infinite magnitude. These two lines are always distinct from each other, since $\sin\gamma$ must be different from zero. The two values $\lambda:\mu$ to which they belong are given by the equation

$$\lambda^2+2\lambda\mu\cos\gamma+\mu^2=0.$$

If $$\gamma=R,$$

then $$\cos\gamma=0,$$

and the equation becomes $$\lambda^2+\mu^2=0,$$

showing that the two exceptional lines—isotropic lines as they are called—divide any pair of lines a, b perpendicular to each other in a harmonic ratio. These two lines are therefore the double lines of an involution, determined by pairs of lines through O at right angles to each other. The involution of these lines is projected on to line I of plane ab into an involution of points independent of O; and there are therefore two points (the circular points) on I determined as the double points of the involution at I of points at right angles with each other. Through one of these two points all isotropic lines must pass.

According to one of our elementary propositions the totality of lines situated in space S_n which are perpendicular to a line l and pass through a certain point A on l form a plane space of manifoldness $n-1$. This may be put differently by considering only the I of the space thus: to any point D of I corresponds a certain Σ_{n-2} situated at I called perpendicular to D; and *vice versa*, Σ being given, D is uniquely determined. A correspondence of such a nature may be conceived, as is well known, as resulting from polarization upon some quadric surface. The quadric surface thus determined at I will be denoted by J. It has received different names, one of which is "the imaginary spherical manifoldness at infinity." But we shall avoid giving this quadric a special name, only reserving the letter J for it.

We are not dependent for its definition on projective geometry. The following is an independent investigation, to prove the (abstract) existence, and to show the significance of that formation.

Let a, b, c be three rays emanating from a point O in a space S_3. Then any other ray d in the S_3 through O is a linear form in the a, b, c,

$$d = \lambda a + \mu b + \nu c.$$

Each plane through O will contain two isotropic lines; the totality of the isotropic lines through O in the S_3 form therefore a cone, which is cut by any plane through its vertex in two lines, and is therefore of the second order. The angle a, b being γ, that of b, c being a, and that of c, a being β, the isotropic lines situated in the planes ab, bc, ca, respectively, are the three pairs defined by

$$\nu = 0, \quad \lambda^2 + \mu^2 + 2\lambda\mu \cos\gamma = 0,$$

$$\lambda = 0, \quad \mu^2 + \nu^2 + 2\mu\nu \cos a = 0,$$

$$\mu = 0, \quad \nu^2 + \lambda^2 + 2\nu\lambda \cos\beta = 0,$$

respectively. It follows that the equation of the isotropic cone must be

$$\lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \cos\gamma + 2\mu\nu \cos a + 2\nu\lambda \cos\beta = 0.$$

This cone will cut I in a conic, determined by the three point-pairs in which it is cut by the lines infinity of ab, bc, ca, respectively; and which is therefore quite independent from O. It is this conic which we designate by J. If D_1, D_2, D_3 are any three points at I, forming with each other angles γ, a, β, respectively, then the conic J will contain all points $\lambda D_1 + \mu D_2 + \nu D_3$, for which above equation is satisfied. More especially, if γ, a, β are all equal to R, then the equation of J will be

$$\lambda^2 + \mu^2 + \nu^2 = 0.$$

If D is any point at I in its normal form, then the "cond" that D should belong to J (by which we denote that function of the coordinates of the formations considered, or that magnitude, which must vanish whenever the condition in question is satisfied), if D does not belong to J, is $= 1$. Indeed, let D_1, D_2, D_3 be three points at right angles at I. Let

$$D = a_1 D_1 + a_2 D_2 + a_3 D_3.$$

Then the cond $$JD = a_1^2 + a_2^2 + a_3^2.$$

Now, if P be any finite point, $P+D=Q$ another, then

$$JD = J(Q-P)$$

gives, according to the theorem of Pythagoras in space, the square of the distance from Q to P, which is $=1$, since D is in its normal form.

In exactly the same manner it may be shown in regard to any space S_n.

(1) That the totality of the isotropic lines through any finite point P in a S_n cuts I in a certain surface J, of the second order, independent of P.

(2) That this surface contains all points

$$\lambda_1 D_1 + \lambda_2 D_2 + \ldots + \lambda_n D_n,$$

for which

$$\Sigma \lambda_i^2 = 0,$$

where the D_i are any sects in their normal form at right angles to each other (a configuration whose existence is easily shown by induction).

(3) That for any point D at I in its normal form we have

$$JD = 1,$$

with the exception, of course, of the points belonging to J (for which a normal form does not exist).

Let now D_1 and D_2 be any two points at I. If λ, μ be any two values, $J(\lambda D_1 + \mu D_2)$ is a homogeneous form of the second order in λ, μ; and, since

$$J(D_1) = J(D_2) = 1,$$

it must be

$$J(\lambda D_1 + \mu D_2) = \lambda^2 + \mu^2 + 2\lambda\mu \,.\, K,$$

where K is some constant depending solely on D_1 and D_2. It is this constant K that H. Grassmann calls "the inner product" of the two sects D_1 and D_2. To find its significance consider the values of $\lambda : \mu$ for which the quantic of the second order vanishes. Its two roots obviously indicate the position of the two points $\lambda D_1 + \mu D_2$ in which the line $D_1 D_2$ cuts J. But their equation is, as we know,

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos(D_1, D_2) = 0.$$

So, then,

$$K = \cos(\angle D_1 D_2).$$

If, generally, D_1, D_2 are any two sects, not necessarily in their normal form, then their "inner product" is the factor of $2\alpha\beta$ in the development of $J(\alpha D_1 + \beta D_2)$, according to powers of α, β. Geometrically, it is the length of the one multiplied by the perpendicular projection of the other upon it; or $[D_1][D_2] \cos \angle D_1 D_2$. Originally H. Grassmann introduced the sign × to denote the inner product. Later on he abandoned this way of writing. In mechanics, where the surface J and the inner product are probably destined to be of much use, such a short sign would have its advantages.

Let $D \times E$ denote the inner product of two sects D, E. Let F be any other sect. Then

$$(D+F) \times E = D \times E + F \times E.$$

This follows readily, for instance, from the algebraical definition, since obviously the factor of $2\alpha\beta$ in the development of

$$J\{\alpha(D+F)+\beta E\}$$

is the sum of the factors of $2\alpha\beta$ in the corresponding development of $J(\alpha D+\beta E)$ and $J(\alpha F+\beta E)$.

The equation of J may be written in a very simple form. Let $D_1, \dots D_n$ be any n linearly independent points at I. Then a point $\lambda_1 D_1 + \dots + \lambda_n D_n$ will belong to J if

$$(\lambda_1 D_1 + \dots + \lambda_n D_n) \times (\lambda_1 D_1 + \dots + \lambda_n D_n) = 0.$$

If the D_i are in their normal form, this is equivalent to

$$\Sigma \lambda_i^2 + \Sigma 2\lambda_i \lambda_j \cos \angle D_i, D_j = 0,$$

a form which might have been found by our original process.

To bring any sect $D = \lambda_1 D_1 + \dots + \lambda_n D_n$

to its normal form, it is necessary to divide it by the square root of JD, *i.e.*, by

$$\sqrt{\lambda_1^2 + \dots + \lambda_n^2 + 2\lambda_1\lambda_2 \cos \angle D_1 D_2 + \dots}.$$

The investigation carried on so far might be pursued on the same lines for the geometry of spaces S_k through a fixed S_{k-1}, or of S_{n-1} in a fixed S_n. We shall designate by the name of isotropic spaces the two spaces of any pencil that form an infinite magnitude with any other space of that pencil. If A, B are any two spaces of k manifoldness having a S_{k-1} in common, and φ the angle they form, then, just as before, $\lambda A + \mu B$ will be an isotropic space when

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos \phi = 0.$$

It is then shown, in exactly the same manner as before, that

$$\lambda_1 s_1 + \ldots + \lambda_{n+1} s_{n+1}$$

(the $s_1 \ldots s_{n+1}$ denoting a system of S_{n-1} in a fixed S_n) is an isotropic space, when

$$\lambda_1^2 + \ldots + \lambda_{n+1}^2 + \Sigma\, 2\lambda_i \lambda_j \cos \angle\, S_i S_j = 0,$$

$\angle\, S_i S_j$ denoting the angle formed by S_i and S_j. An isotropic space, it will be noticed, is one which cuts J in a quadric surface having a double point. The above is therefore, if projected into I, the reciprocal equation of J.

The "cond" that a space S_{n-1} in its normal form should touch J is 1. It will be sufficient to consider the state of things at I. Assume n S_{n-2} at I at right angles to each other $a, b, \ldots l$. Put

$$S_{n-2} = \lambda a + \mu b + \ldots + \nu l.$$

The "cond" in question is

$$= \lambda^2 + \mu^2 + \ldots + \nu^2.$$

The S_{n-2} and the $a \ldots l$ being supposed to be in their normal form

$$[ab \ldots l] \text{ is } = 1,$$

$$\lambda = [b \ldots l\, S_{n-2}], \quad \mu = -[ac \ldots l\, S_{n-2}], \text{ \&c.}$$

$[b \ldots l\, S_{n-2}]$ is simply the sine of the angle which the point $b \mid c \mid \ldots \mid l$ forms with S_{n-2}, &c. According to the "Pythagoras" for a space S_n, the value of $\lambda^2 + \mu^2 + \ldots$ is seen to be 1.

If we form the "cond" in question for $\alpha s + \beta t$, where α, β are any two constants, s and t any two S_{n-2} at I, the result will be a quadratic function of α, β of the form $\alpha^2 + \beta^2 + 2\alpha\beta K$. Similarly, as before, K may be denoted as the inner product of s and t, written $s \times t$. We have then, if s and t are in their normal forms,

$$s \times t = \cos \angle\, s, t,$$

and $$(as + b.s') \times t = as \times t + bs' \times t,$$

the a, b being constants. If the $a_1, a_2, \ldots a_n$, are the border-spaces of any pyramid at I, the reciprocal equation of J is simply

$$(\lambda_1 a_1 + \ldots + \lambda_n a_n) \times (\lambda_1 a_1 + \ldots + \lambda_n a_n),$$

which is $$= \Sigma \lambda_i^2 + \Sigma \lambda_i \lambda_j \cos \angle\, a_i a_j,$$

and it also follows that, to bring

$$\lambda_1 a_1 + \ldots + \lambda_n a_n$$

to its normal form, it is necessary to divide it by the square root of that expression.

Among the many properties of J none seem so interesting as the one which brings it into intimate connexion with the theory of the potential function W in any space S_n. It is that J is "apolar" to W.

10. To find the trigonometrical formulæ of plane and spherical manifoldnesses, and related problems, can now be easily solved. The following is a brief account of what might be said under this head.

Let $A, B, \ldots L$ be a pyramid in space of n manifoldness; $\bar{A}, \bar{B}, \ldots \bar{L}$ the border-spaces opposite to the corner-points $A, B, \ldots L$,

$$\bar{A} = \quad BCD \ldots L,$$

$$\bar{B} = -ABD \ldots L,$$

$$\bar{C} = \quad ABD \ldots L,$$

$$\&c.,$$

so that $$A\bar{A} = B\bar{B} = C\bar{C} = \ldots = \Delta = 1,$$

where the magnitude Δ of the pyramid is assumed for convenience equal to unity.

For any point P we shall then have

$$\pm P = [\bar{A}P]\,A + [\bar{B}P]\,B + \ldots + [\bar{L}P]\,L,$$

for $$\bar{A}A = \bar{B}B = \ldots = \pm 1;$$

therefore this will be true when $P \equiv A, B, \ldots L$; hence also when P is a linear form of multiples of $A, B, \ldots L$. Similarly, if s be any S_{n-1} in the S_n,

$$s = [As]\,\bar{A} + [Bs]\,\bar{B} + \ldots + [Ls]\,\bar{L}.$$

As a special case, $$IA = IB = \ldots = 1,$$

$$I = \pm (\bar{A} + \bar{B} + \ldots + \bar{L}).$$

If, then, we cut this by I,

$$\text{(E)} \quad \bar{A}/I + \bar{B}/I + \bar{C}/I + \ldots + \bar{L}/I = 0,$$

$\bar{A}, \bar{B}, \ldots \bar{L}$ are not in their normal form, but appear multiplied by the magnitude of the border-pyramid in their space. This is not altered by the intersection with I. Bringing $\bar{A}/I$ to the right-hand side and forming the inner product of each side with itself, we obtain

$$[\bar{A}]^2 = [\bar{B}]^2 + \ldots + [\bar{L}]^2 + 2\,[\bar{B}]\,[\bar{C}] \cos \angle \bar{B}\bar{C} + \ldots,$$

the generalized cosine theorem for point geometry. By treating (E) differently, it can of course be given different forms.

It may be shown without any difficulty that all the magnitudes connected with a pyramid are known when the $\dfrac{n \,.\, n+1}{2}$ quantities, representing the distances of any corner-point from any other, are known; and that these quantities are perfectly independent among themselves.

The cosine of the angles formed by the directions of any two edges which do not intersect can easily be discovered.

From $$A - B = A - D + B - C + D - C,$$

forming the "inner square" of each side,

$$\begin{aligned}[A-B]^2 = [A-D]^2 + [B-C]^2 + [D-C]^2, \\ +2\,[A-D]\,[B-C] \cos \angle AD, BC \\ +2\,[A-D]\,[D-C] \cos \angle ADC \\ +2\,[B-D]\,[D-C] \cos \angle BCD,\end{aligned}$$

giving $\cos \angle AD, BC$ in terms of known quantities.

The distance of any point

$$P = aA + \ldots + \lambda L$$

from any other $$Q = a'A + \ldots + \lambda L$$

is the square root of the inner product of $P - Q$; and therefore expressible by the $\dfrac{n+1 \,.\, n}{2}$ quantities.

The cosine of the angle formed by any two spaces

$$\lambda_1 a_1 + \ldots + \lambda_{n+1} a_{n+1},$$

$$\lambda_1' a_1 + \ldots + \lambda_{n+1}' a_{n+1},$$

the $a_1, \ldots a_{n+1}$ being supposed to be in their normal form, is

$$\cos \phi = \frac{\lambda_1 \lambda_1' + \ldots + \lambda_i \lambda_j' \cos \angle (a_i, a_j) + \ldots}{\sqrt{\lambda_1^2 + \ldots + 2\lambda_i \lambda_j \cos \angle (a_i, a_j)}\,\sqrt{\lambda_1'^2 + \ldots}},$$

where (a_i, a_j) denotes the angle formed by a_i, a_j. For this is the inner product of the two spaces after they have been brought to their normal form.

Connected with $n\ S_{n-1}$, having a point P in common, is a certain magnitude, which is the sine of the angle formed by any two of them, multiplied into the sine of the angle formed by any third one with the S_{n-2} common to the first two, &c. It is the factor of composition in the reciprocal geometry, the composed space being the point P. If the $n\ S_{n-1}$ are

$$\lambda_{i,1} a_1 + \lambda_{i,2} a_2 + \ldots \equiv S_i$$

(the a being supposed to be in their normal form), then P is found

$$P \equiv L_1 A_1 + L_2 A_2 + \ldots,$$

where the L are expressed according to the rules given by the $n+1$ determinants of a matrix of $n+1$ rows and n columns, whose coefficients are the $\lambda_{i,j}$. The factor of composition of P is therefore

$$x = \frac{L_1 + L_2 + \ldots + L_{n+1}}{\sqrt{\lambda_{1,1}^2 + \ldots + 2\lambda_{1,i}\lambda_{1,j}\cos(a_i, a_j)}\ \sqrt{\lambda_{2,1}^2 + \ldots}},$$

the n square roots being necessary to bring the $n\ S_{n-1}$ to their normal form. If, for instance, $n = 2$, x represents the sine of the angle formed by two lines.

The discriminant of

$$(x_1 D_1 + \ldots + x_n D_n) \times (x_1 D_1 + \ldots + x_n D_n)$$

considered as a quantic in the x_i, the D being sects of any kind, is the square of $[D_1 D_2 \ldots D_n]$. Indeed, if $E_2 \ldots E_n$ are sects in the space $D_1 \ldots D_n$ at right angles to each other, and of length 1, then

$$(x_1 E_1 + \ldots + x_n E_n) \times (x_1 E_1 + \ldots + x_n E_n) = x_1^2 + x_2^2 + \ldots + x_n^2,$$

a form whose discriminant may have the value 1. If we replace

D_1 by its linear equivalent in the E_i,
D_2 ,, ,, ,, E_i,
&c.,

then $$(x_1 D_1 + \ldots + x_n D_n) \times (x_1 D_1 + \ldots + x_n D_n)$$

will be expressed by the sum of n squares of forms y_i linear and homogeneous in the x_i, whose determinant is not different from the determinant of the linear forms by which the D_i were expressed by the E_i; the discriminant in question is therefore the square of this determinant. But $[E_1 \dots E_n] = 1$; therefore the discriminant is the square of $[D_1 \dots D_n]$.

Let it be required to find the distance of a point P from a line l in space S_3 by means of the coordinates of P and l.

If A, B, C, D are any pyramid in S_3,

$$P = Ap_1 + p_2B + p_3C + p_4D,$$

$$l = a_1AB + a_2AC + \dots + a_6CD,$$

then
$$lP = (a_1p_3 - a_2p_2 + a_4p_1)\, ABC + \dots.$$

If l is in its normal form, then the weight of lP will be $[lP]$. Therefore
$$[lP]^2 = c\,\{(a_1p_3 - a_2p_2 + a_4p_1)^2 + \dots\} = lP \times lP,$$
where c is a constant whose value is 1, when l is in its normal form, and solely dependent on the a_i.

The right-hand side will vanish only when this algebraical expression for the distance of P from l vanishes, that is, when lP is one of the isotropic planes of the pencil in the S_3 through l; then it will vanish always. The value of the perpendicular from P on l may therefore be found by constructing these two isotropic planes, and forming the "cond" that P may be contained by any one of them.

If P, Q, R are any three collinear points, l any line not intersecting PQ in the space S_3; if, further, lP forms with lQ the angle ϕ, and

$$[lP] = p, \quad [lQ] = q,$$

$l(\alpha P + \beta Q)$ will be an isotropic plane, when

$$\alpha^2p^2 + \beta^2q^2 + 2\alpha\beta pq \cos\phi = 0.$$

Should therefore $R = \lambda P + \mu Q$, then the square (r^2) of the distance of R from l is
$$r^2 = \lambda^2p^2 + \mu^2q^2 + 2\lambda\mu pq \cos\phi.$$

From
$$\lambda P + \mu Q = R$$
it follows that
$$\lambda lP + \mu lQ = lR.$$

If, therefore, the angles which lP and lQ form with lR are denoted by χ and ψ, then
$$\lambda p : \mu q = \sin\psi : \sin\chi.$$

In the same way, it is hardly necessary to mention, many other apparently more complicated metrical problems may easily be solved. In the remaining portion of this section we shall investigate the perpendicular distance of any two plane spaces that have no point in common.

If A and B are any two spaces of manifoldness h and k respectively, having no point in common with each other, then one point of A and one point of B are in such relation to each other that their distance is shorter in absolute length than that of any other two points belonging to A and B respectively. The line PQ joining them is perpendicular to A as well as B, and there is no other line of this nature. Indeed, the I of the space AB is of manifoldness $h+k$, I/A is of manifoldness $h-1$, and I/B is of manifoldness $k-1$. To I/A corresponds a space Σ_1 at I, of manifoldness k, whose every point is perpendicular to I/A; and similarly to I/B a Σ_2, of manifoldness h. Σ_2 and Σ_1 have a point D in common, perpendicular to I/A as well as I/B.

Through any point of the space AB it is possible to draw one line to cut A as well as B. This was shown in § 3. And only one such line can be constructed if $[AB]$ is different from zero, an assumption with which we started. The line l thus belonging to D is therefore the only line cutting A as well as B at right angles.

This line l will intersect A and B in points P, Q. The distance of P to Q is measured by the perpendicular distance of P from the space of manifoldness $h+k$ composed by B and I/A, parallel to A. This distance is the same from every point on A. It is, to write it symmetrically, the magnitude formed by $B\,I/A$ with $A\,I/B$, and thus, as a rule, easy to calculate when the coordinates of A and B are given.

11. To make this essay somewhat complete it will be necessary to discuss, in a few words, the theory of projection, or of linear transformation, as it presents itself in plane spaces of any manifoldness.

Many words are unnecessary on this subject, since it has long been exhaustively treated. Let any two spaces of the same manifoldness, for definiteness point-spaces, be put into a projective correspondence with each other. Then to any point A in the one space S corresponds one point A' in the other space S'. If any two pyramids are fixed in S, S', the coordinates of the points of S and S' are mutually expressible as linear functions of each other. It follows, that to any point A in S will generally correspond the multiple of some point A' of S'. And, if

$$\begin{array}{llcl} & A & \text{corresponds to} & aA', \\ & B & \text{,,} \quad \text{,,} & bB', \\ \text{then} & \alpha A+\beta B & \text{,,} \quad \text{,,} & \alpha aA'+\beta bB'. \end{array}$$

Let now

$$\begin{array}{llll} A & \text{correspond} & \text{to} & aA', \\ B & ,, & ,, & bB', \\ \ldots & ,, & ,, & \ldots, \\ L & ,, & ,, & lL', \end{array}$$

where $A \ldots L$ is any pyramid in S, $A' \ldots L'$ the corresponding one in S'.

Any point $M = \alpha A + \ldots + \lambda L$

will correspond to $M' \equiv \alpha a A' + \ldots + \lambda l L'$.

It is therefore seen that $n+2$ points determine the correspondence. For let

$$\begin{array}{llllll} A & \text{be} & \text{projectively} & \text{related} & \text{to} & A', \\ B & ,, & ,, & ,, & ,, & B', \\ \ldots & ,, & ,, & ,, & ,, & \ldots, \\ L & ,, & ,, & ,, & ,, & L', \\ M & ,, & ,, & ,, & ,, & M', \end{array}$$

$$M = \alpha A + \ldots + \lambda L,$$

$$M' = \alpha' A' + \ldots + \lambda' L';$$

then $$a = \alpha' : \alpha \ldots l = \lambda' : \lambda,$$

and everything is known.

If S and S' are brought to coincidence with each other, then one pyramid $PQR \ldots S$ will exist, whose corner points correspond to themselves.

Indeed, assume $$P = \alpha A + \ldots + \lambda L,$$

$$P' = \alpha a A' + \ldots + \lambda l L'.$$

If, then, P' and P coincide, for some value of ρ we must have

$$\alpha(\rho A - aA') + \beta(\rho B - bB') + \ldots + \lambda(\rho L - lL') = 0.$$

If such equation connects

$$\rho A - aA', \quad \rho B - bB', \quad \ldots,$$

then that pyramid must vanish,

$$\left[(\rho A - aA')(\rho B - bB') \ldots (\rho L - lL')\right] = 0,$$

an equation in ρ of order $n+1$, which has therefore $n+1$ roots.

To each value of ρ will belong a system of values $\alpha, \beta, \ldots \lambda$, connecting

$$\rho A - aA' \ldots \rho L - lL',$$

so that the corner-points P of the pyramid whose existence was asserted can be found. In general, the ρ will be distinct from each other. If they are not, limiting processes will explain what then takes place. But this is not of much importance for the immediate objects of this essay.

Thursday, December 10*th,* 1896.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Present—twenty-two members and four visitors.

The following gentlemen were elected members :—John Borthwick Dale, B.A., late Scholar of St. John's College, Cambridge; Charles Samuel Jackson, M.A., Instructor in Mathematics, Royal Military Academy, Woolwich; Arthur William Ward, M.A. St. John's College, Cambridge, Professor of Mathematics and Physics, Canning College, Lucknow, India. Mr. S. S. Hough was admitted into the Society.

The Auditor (Mr. Terry), having read his report, complimented the Treasurer on the way in which he had performed his duties. Mr. Kempe moved, and Mr. Bickmore seconded, a vote of thanks to the Auditor for his services. The vote was carried unanimously. A motion was then made by the President, and seconded by Lt.-Col. Cunningham, and carried unanimously, for the acceptance of the Treasurer's report. Dr. Larmor suitably acknowledged the compliment.

Major MacMahon gave a sketch of the result arrived at in Prof. Sylvester's "Note on a Discovery in the Theory of Denumeration." In connexion with this paper the President announced that Prof. Sylvester had given permission to the Society to publish the "Outline of Lectures on the Partitions of Numbers," which he read at King's College, London, in 1859, and which had never been published; and that the Council had arranged to print the "Outlines" as a companion to the late President's Valedictory Address.

Mr. Burbury communicated a paper "On the Stationary Motion of a System of Equal Elastic Spheres of Finite Diameter."

Mr. Hough read a paper "On the Influence of Viscosity on Waves and Currents."

Mr. Macfarlane Gray gave a description of his Multiplying apparatus. Messrs. C. V. Boys, Dewar, and Greenhill joined in a discussion of points connected with the subject of Mr. Gray's communication, and a cordial vote of thanks was passed to these gentlemen.

The multiplying apparatus consists of two principal parts, a sole frame and a grid. In the sole frame the product cards for the multiplicand are set up in order. These are the same as what are called "Napier's rods," being each the products in one column of the multiplication table, up to 9 times 9, with a card for the 0 column. The grid is a frame fitted with a number of sliders, each of the same breadth as the product cards. Each slider has at mid length a pane of glass. The edges of the sliders coincide with the vertical centres of the cards, when superposed, so that each pane lies over the unit place of one card, and the place of tens in the adjacent card. The sliders are set to bring the panes each over the product lines for one figure in the multiplier, taking the figures in the order the reverse of that in which the multiplicand has been set up. The grid frame is fitted to slide over the card frame upon stepped guides, the steps insuring the proper relative positions when reading the products. The sliders may be of leather with the glass panes cemented on. There is a figured plate for setting the sliders by.

To obtain the product of two multidigital numbers, the cards for the figures in one of them are set up on the sole frame, and the sliders in the grid are set for the figures in the other. The grid is then moved linearly over the sole frame, moving one figure at a time, and at each step the components of the products in one of the vertical columns of the ordinary multiplication are exhibited at the panes of the multiplier sliders. These are added together, giving one figure of the required product; the grid is then slid on to the next figure, and the next vertical column is then shown. In this way the final product is obtained without transcribing the intermediate products.

						0
	6		3		7	1
1	2		6	1	4	2
1	8		9	2	1	3
2	**4**	**1**	2	2	8	4
3		1	5	3	5	5
3	6	1	**8**	**4**	2	6
4	2	2	1	4	9	7
4	8	2	4	5	**6**	8
5	4	2	7	6	3	9

shows three cards of the multiplicand 637 being multiplied by 864. The multiplier is set to the required figures by adjusting the sliders till the figures in the column to the right (under 0) are seen through the windows.

Each figure in the product is the sum of the black figures seen through the windows. Through each window two figures are seen, but they are not on the same card. The 3 in the product is given in this position.

By the Machine.

$$\begin{array}{rrrrrr} & & & 6 & 3 & 7 \\ & & & 8 & 6 & 4 \\ \hline & & & \mathbf{4} & 2 & 8 \\ & & 2 & \mathbf{1} & 2 & \\ \hline & & 6 & \mathbf{8} & 2 & \\ & 3 & 1 & \mathbf{4} & & \\ \hline & 8 & 4 & \mathbf{6} & & \\ 4 & 2 & 5 & & & \\ \hline 5 & 5 & 0 & 3 & 6 & 8 \end{array}$$

By Ordinary Multiplication.

$$\begin{array}{rrrrrr} & & & 6 & 3 & 7 \\ & & & 8 & 6 & 4 \\ \hline & & 2 & 5 & 4 & 8 \\ & 3 & 8 & 2 & 2 & \\ 5 & 0 & 9 & 6 & & \\ \hline 5 & 5 & 0 & 3 & 6 & 8 \end{array}$$

Lt.-Col. Cunningham stated some results arrived at in his paper "On the Connexion of Quadratic Forms." Upon a portion of these results Mr. Bickmore made some supplementary remarks.

The following papers were taken as read :—

Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holoedrically Isomorphic with the Symmetric and the Alternating Substitution Groups on k Letters: Prof. E. H. Moore.

On a Series of Co-Trinodal Quartics: Messrs. H. M. Taylor and W. H. Blythe.

On Finite Variations: Mr. E. P. Culverwell.

The following presents were received for the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xx., St. 10; Leipzig, 1896.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxx., Liv. 3; Harlem, 1896.

"Wiskundige Opgaven met de Oplossingen door de Leden van het Wiskundig Genootschap," Deel vii., St. 2; Amsterdam, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. iii., No. 2; New York, 1896.

"Festschrift der Naturforschenden Gesellschaft in Zürich, 1746–1896," Teile 1, 2; Zürich, 1896.

"Bulletin des Sciences Mathématiques," Tome xx., Oct., 1896; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. ii., Fasc. 8–10; Napoli, 1896.

"Transactions of the Canadian Institute," Vol. v., Pt. 1, No. 9; Toronto, October, 1896.

"Rendiconti del Circolo Matematico di Palermo," Tomo x., Fasc. 5; 1896.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. v., Fasc. 9, 10; Roma, 1896.

"Journal of the College of Science, Tokyo," Vol. x., Pt. i.; 1896.

"Journal für die reine und angewandte Mathematik," Bd. cxvii., Heft 2; Berlin, 1896.

"Annales de la Faculté des Sciences de Toulouse," Tome x., Fasc. 3, 4; Paris, 1896.

"Educational Times," December, 1896.

"Indian Engineering," Vol. xx., Nos. 17–20, Oct. 24–Nov. 14, 1896.

Presented by Mr. J. Hammond:—

"Commercium Epistolicum D. Johannis Collins et aliorum de Analysi promota"; Londini, 1722.

"The Method of Increments, wherein the Principles are Demonstrated, and the Practice thereof shown in the Solution of Problems" (by W. Emerson); London, 1763.

On the Influence of Viscosity on Waves and Currents. *By* S. S. Hough, M.A., Isaac Newton Student in the University of Cambridge. Received December 9th, 1896. Read December 10th, 1896.

In the following paper my aim has been to present the solution of certain problems illustrative of the effects of viscosity on the motions of the sea. With this end in view, I have therefore had no hesitation in introducing such approximations as would be applicable in the case presented by nature. The loss of generality resulting from these approximations will be compensated for by corresponding simplicity of the analysis, while the results are more readily intelligible, in that they admit of being expressed in a form which may be at once converted into numbers.

The motions dealt with may be divided into the following classes: (1) large-scale currents, of which the most familiar illustration is to be found in the circulatory system of the North Atlantic Ocean; (2) tidal oscillations, either of the nature of stationary vibrations or consisting of progressive wave-motions with a wave-length large in comparison with the depth; and (3) deep-sea waves, in which the wave-length is very short compared with the depth. Each of these motions, if once started and then left free from external maintaining cause, would slowly subside under the influence of dissipative forces, and my object has been to evaluate for the various types of motion the modulus of decay, that is, the period in which the velocities in the current motions and the amplitudes of vibration in the periodic motions would be reduced in the ratio $1:e$, due to the combined action of internal viscosity and friction at the ocean bed. It might be anticipated *à priori*, and it is established in the present paper, that in the two former classes of motion the friction of the ocean bed is by far the more important influence in destroying the motion, whereas in the case of short waves at the surface of deep water the friction of the ocean bed is of no moment in comparison with internal viscosity.

To deal with bottom-friction, it has been necessary to introduce some hypothesis as to the nature of the action between the water and the solid bed with which it is in contact. The most probable hypothesis, and that which I have adopted, is that no slipping at all is possible; but, if this be not the true law, the effects stated may at least be regarded as the maximum results which could be produced with the assigned degree of internal viscosity, and the moduli of decay obtained may be treated as inferior limits to the moduli of decay which would appear from a more general supposition as to the action in question. As it is the large order of magnitude of these moduli, rather than their actual numerical values, on which the physical application of the results turns, the practical value of these results is therefore by no means diminished.

I hope to enter more fully into the physical bearings of the problems solved in a later paper; but it will not be out of place to make the following remarks as illustrative of my purpose. The existence of ocean currents has been variously attributed to the tendency of the winds in certain regions (*e.g.*, the trade-wind regions) to set in particular directions, and to differences of density arising from differences of temperature, salinity, &c. The opponents of each theory have urged that the energy derivable from these sources is totally inadequate to *generate* the large motions known to exist in the ocean. Were the ocean free from viscosity, however, and initially at rest, it follows that currents arising from the sources in question must inevitably be set up, and, the causes being continuous in their action, it only requires lapse of time for the motions to become sensible or even very large. The same will be true when there is a small amount of viscosity; but in the latter case a limit will ultimately be attained when the rate at which currents are generated by the causes in question is on the average equal to that at which they are destroyed by friction. After this state has been attained the motion will remain steady, but we see that no estimate of the amounts of currents that could be set up can be obtained from considerations of the amounts of energy involved in the sources apart from considerations of the rate at which energy is dissipated by friction. The extremely large values we have obtained for the moduli of decay of the current-motions imply that energy is dissipated very slowly, and thus, though no doubt an extremely long time would be necessary for the currents, starting from rest, to acquire their present magnitude, there appears no difficulty in supposing that the causes suggested are quite adequate to *maintain* these motions when once set up.

As regards the tidal oscillations it appears that in a system comparable with the actual Earth the moduli of decay of the principal free oscillations will be very large compared with the periods of the disturbing forces due to the Sun and Moon. Hence it follows that the tidal forces will produce their full dynamical effect, and that the conclusions derived from an equilibrium theory, except in so far as they coincide with those derived from a dynamical theory, are without foundation. In the case of the long-period tides, which have usually been supposed to follow the equilibrium law in consequence of viscosity, it is known that the equilibrium theory and the dynamical theory lead to different results,* and therefore it follows that an equilibrium theory must be at fault even for such tides as the solar semi-annual tide.

1. *On the Rate of Decay of Current-Motions.*

The equation of motion of a viscous liquid moving everywhere parallel to the axis of x with velocity u, and subject to uniform gravity parallel to the axis of z, is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}, \tag{1}$$

where ν is the kinematic coefficient of viscosity.

If we suppose $u \propto e^{-\alpha t}$, and put $k^2 = \alpha/\nu$, this equation becomes

$$\frac{\partial^2 u}{\partial z^2} + k^2 u = 0;$$

from which we determine u as a function of z in the form

$$u = A \cos kz + B \sin kz,$$

where A, B are functions of the time alone.

Expressing the time factor $e^{-\alpha t}$, we obtain as the general solution of (1) of the assumed type

$$u = (A \cos kz + B \sin kz)\, e^{-\alpha t}, \tag{2}$$

where A, B are now arbitrary constants to be determined from the boundary-conditions.

Let $z = 0$, $z = h$ be the equations to the ocean bed and to the free surface respectively.

At the former we suppose that no slipping is possible, so that $u = 0$ when $z = 0$; this leads to

$$A = 0. \tag{3}$$

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* Lamb, *Hydrodynamics*, § 210.

At the free surface the condition to be satisfied is that the tangential stress must vanish; or that $\frac{\partial u}{\partial z} = 0$ when $z = h$. Therefore

$$kB \cos kh = 0. \tag{4}$$

If k or B vanish, u will be everywhere zero, and no motion will be involved. Hence the admissible values of k, which determine the various types of free "laminar" motion of which the system is capable, are the roots of the equation

$$\cos kh = 0;$$

the roots of this equation are of the form $(2n+1)\,\pi/2h$, where n is integral, and therefore the appropriate values of a are found by giving n integral values in the formula

$$a = \frac{(2n+1)^2\,\pi^2\nu}{4h^2}.\text{*}$$

A particular solution of (1) satisfying the assigned boundary-conditions is therefore

$$u = Be^{-\frac{(2n+1)^2\pi^2}{4h^2}\nu t} \sin \frac{(2n+1)\,\pi z}{2h}, \tag{5}$$

where B is an arbitrary constant.

To determine the motion resulting from assigned initial circumstances we may express the initial velocity u_0 by means of a Fourier's series in the form

$$u_0 = \sum_{n=0}^{n=\infty} A_n \sin \frac{(2n+1)\,\pi z}{2h}. \tag{6}$$

The subsequent motion will then be given by

$$u = \sum_{n=0}^{n=\infty} A_n e^{-\frac{(2n+1)^2\pi^2\nu}{4h^2}t} \sin \frac{(2n+1)\,\pi z}{2h}. \tag{7}$$

For example, if the velocity is initially constant and equal to u_0, the series (6) becomes

$$u_0 = \frac{4u_0}{\pi} \Sigma \frac{1}{2n+1} \sin \frac{(2n+1)\,\pi z}{2h},$$

and the motion at time t is given by

$$u = \frac{4u_0}{\pi} \sum_{n=0}^{n=\infty} \frac{1}{2n+1}\, e^{-\frac{(2n+1)^2\pi^2}{4h^2}\nu t} \sin \frac{(2n+1)\,\pi z}{2h}.$$

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* *Cf.* Helmholtz, *Werke*, Vol. III., p. 289.

After a sufficiently long interval all the types of motion except that corresponding to $n=0$ may be supposed to have subsided, and the ultimate state of motion will be expressed by

$$u = \frac{4u_0}{\pi} e^{-\frac{\pi^2 \nu t}{4h^2}} \sin \frac{\pi z}{2h}.$$

The moduli of decay of the various types of free motion are

$$\frac{4h^2}{\pi^2\nu}, \quad \frac{4h^2}{3^2\pi^2\nu}, \quad \dots \quad \frac{4h^2}{(2n+1)^2\pi^2\nu}, \quad \dots .$$

For water the value of ν referred to C.G.S. units is about ·0178, and, if the depth be taken as 1 metre, we find that the modulus of decay for that type of motion which subsides least rapidly is about 63 hours.

For depths at all comparable with the depth of the ocean, the moduli of decay will be extremely large. Thus, if we take the depth as 4,000 metres, which is probably less than the true mean depth, we find a modulus of decay for the type $n=0$ slightly exceeding 100,000 years, while, even for the type $n=100$, the modulus will be nearly 3 years.

2. *Dynamical Equations for Wave-Motions in Two Dimensions.*

The equations of motion of a viscous liquid oscillating in two dimensions, under uniform gravity parallel to the axis of z, can be expressed in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \psi}{\partial x} + \nu \nabla^2 u, \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z} + \nu \nabla^2 w, \\ \frac{\partial u}{\partial x} &+ \frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \tag{8}$$

where u, w denote the velocity-components parallel to the axes of x, z respectively; ν is the kinematic coefficient of viscosity, and

$$\psi = \text{const.} - gz - p/\rho, \tag{9}$$

g denoting the acceleration due to gravity, p the pressure, and ρ the density.

To obtain solutions of these equations, suppose that u, w, ψ are each proportional to $e^{im(x-Vt)}$; they then become

$$\left.\begin{aligned} \left\{\frac{\partial^2}{\partial z^2}-m^2+imV/\nu\right\} u &= -\frac{im}{\nu}\psi, \\ \left\{\frac{\partial^2}{\partial z^2}-m^2+imV/\nu\right\} w &= -\frac{1}{\nu}\frac{\partial\psi}{\partial z}, \\ imu+\frac{\partial w}{\partial z} &= 0; \end{aligned}\right\} \tag{10}$$

whence, if we eliminate u, w, we obtain

$$\left(\frac{\partial^2}{\partial z^2}-m^2\right)\psi = 0. \tag{11}$$

The solution of (11) is

$$\psi = Ae^{mz}+Be^{-mz},$$

where A, B are functions of x and t.

Introducing this value of ψ into the right-hand members of (10), we find at once the particular integrals

$$\left.\begin{aligned} u &= -\frac{\psi}{V} &&= -\frac{1}{V}\left\{Ae^{mz}+Be^{-mz}\right\}, \\ w &= -\frac{1}{imV}\frac{\partial\psi}{\partial z} &&= \frac{i}{V}\left\{Ae^{mz}-Be^{-mz}\right\}. \end{aligned}\right\} \tag{12}$$

To these must be added complementary functions which satisfy the equations

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial z^2}-k^2\right) u &= 0, \\ \left(\frac{\partial^2}{\partial z^2}-k^2\right) w &= 0, \\ imu+\frac{\partial w}{\partial z} &= 0, \end{aligned}\right\} \tag{13}$$

where $$k^2 = m^2-imV/\nu. \tag{14}$$

From the second of equations (13), we have

$$w = Ce^{kz}+De^{-kz}, \tag{15}$$

and therefore, by means of the third,

$$u = \frac{ik}{m}\left(Ce^{kz} - De^{-kz}\right). \tag{16}$$

Combining these complementary functions with the particular integrals (12), and expressing the factor $e^{im(x-Vt)}$, we obtain as the general solution of the equations of motion of the assumed type

$$\left.\begin{aligned} \psi &= \left(Ae^{mz} + Be^{-mz}\right) e^{im(x-Vt)}, \\ u &= \left[-\frac{1}{V}\left(Ae^{mz} + Be^{-mz}\right) + \frac{ik}{m}\left(Ce^{kz} - De^{-kz}\right)\right] e^{im(x-Vt)}, \\ w &= \left[\frac{i}{V}\left(Ae^{mz} - Be^{-mz}\right) + \left(Ce^{kz} + De^{-kz}\right)\right] e^{im(x-Vt)}, \end{aligned}\right\} \tag{17}$$

where A, B, C, D are arbitrary constants, to be determined by the boundary-conditions.

If F, H denote the components of traction parallel to the axes of x, z across any plane $z =$ const., we have

$$F = \rho\nu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad H = -p + 2\rho\nu\frac{\partial w}{\partial z};$$

and, therefore,

$$F/\rho = \nu\left[-\frac{2m}{V}\left(Ae^{mz} - Be^{-mz}\right) + \frac{i\left(k^2 + m^2\right)}{m}\left(Ce^{kz} + De^{-kz}\right)\right] e^{im(x-Vt)}, \tag{18}$$

$$\begin{aligned} H/\rho &= \psi + gz + \text{const.} \\ &\quad + 2\nu\left[\frac{im}{V}\left(Ae^{mz} + Be^{-mz}\right) + k\left(Ce^{kz} - De^{-kz}\right)\right] e^{im(x-Vt)}, \end{aligned}$$

the latter of which, on introducing the value of ψ from (17) becomes

$$H/\rho = \text{const.} + \left[\left(1 + \frac{2\nu im}{V}\right)\left(Ae^{mz} + Be^{-mz}\right) + k\nu\left(Ce^{kz} - De^{-kz}\right)\right] e^{im(x-Vt)}. \tag{19}$$

3. *The Boundary-Conditions.*

Let ζ denote the height of the free surface above the plane $z = h$, and suppose that ζ is expressible in the form

$$ae^{im(x-Vt)}.$$

Then at the surface $z = h$ we must satisfy the kinematical condition

$$w = \frac{\partial \zeta}{\partial t},$$

which requires $\frac{i}{V}(Ae^{mh} - Be^{-mh}) + Ce^{kh} + De^{-kh} = -imVa.$ (20)

Again the stress-conditions at this surface may be expressed by equating the stress across the plane $z = h$ to a normal stress equal to the weight of the harmonic inequalities. This requires $F = 0$, $H = -g\rho\zeta$, when $z = h$. The non-periodic constant on the right of (19) must vanish identically when $z = h$, and we find

$$\left.\begin{aligned} -\frac{2m}{V}(Ae^{mh} - Be^{-mh}) + \frac{i(k^2+m^2)}{m}(Ce^{kh} + De^{-kh}) &= 0, \\ \left(1 + \frac{2i\nu m}{V}\right)(Ae^{mh} + Be^{-mh}) + 2\nu k(Ce^{kh} - De^{-kh}) &= -ga. \end{aligned}\right\} \quad (21)$$

If $z = 0$ be the equation to the bottom, the conditions to be satisfied at this surface will depend on the assumption we make as to the nature of the action between the water and the ocean bed. If we assume that no slipping is possible, we must have $u = 0$, $w = 0$ when $z = 0$, and therefore

$$\left.\begin{aligned} -\frac{1}{V}(A+B) + \frac{ik}{m}(C-D) &= 0, \\ \frac{i}{V}(A-B) + (C+D) &= 0. \end{aligned}\right\} \quad (22a)$$

If, on the other hand, we suppose that the bottom is perfectly smooth, we require $w = 0$, $F = 0$ when $z = 0$; these conditions lead to

$$\left.\begin{aligned} \frac{i}{V}(A-B) + (C+D) &= 0, \\ -\frac{2m}{V}(A-B) + \frac{i(k^2+m^2)}{m}(C+D) &= 0. \end{aligned}\right\} \quad (22b)$$

The elimination of the constants A, B, C, D, a from the equations (20), (21), and (22a) or (22b) will lead to an equation connecting V, the velocity of wave-propagation, with $2\pi/m$, the wave-length. The character of the resulting motion will depend on the nature of the roots of this equation. When ν is absolutely zero, the roots will be of the form

$$V = V_0,$$

where V_0 is a real quantity; in this case we have

$$\zeta = ae^{im(x-V_0t)},$$

or, retaining only the real part,

$$\zeta = a\cos m(x-V_0t).$$

The motion will therefore consist of a train of simple harmonic waves propagated with velocity V_0 in direction parallel to the axis of x.

On the other hand, if V be purely imaginary, say

$$V = -\frac{i}{m\tau},$$

where τ is real, we have

$$\zeta = ae^{im(x+it/m\tau)} = ae^{-t/\tau}e^{imx},$$

whence, discarding the imaginary part,

$$\zeta = ae^{-t/\tau}\cos mx.$$

The surface will at any instant be of the form of a curve of sines, subsiding without displacement of the nodal lines until it ultimately takes the equilibrium form $\zeta = 0$. This is the case which might be expected to occur for very large values of ν.

Lastly, if V is complex, it may be expressed in the form

$$V_1 - \frac{i}{m\tau},$$

and we shall have

$$\zeta = ae^{-t/\tau}\cos m(x-V_1t).$$

The motion will then consist of a train of waves of length $2\pi/m$ propagated with velocity V_1, the amplitude of vibration slowly declining and being reduced in the ratio $1 : e$ in a period τ. This is the case which may be expected to occur when the viscosity is very small; further, we may anticipate that V_1 will differ but slightly from V_0, its value when $\nu = 0$, and that τ will be very large, when the value of ν is very small.

We have assumed that the values of τ will be positive, which is a necessary consequence of the stability of the equilibrium in the zero configuration.

4. *Approximate Solution when the Viscosity is Small.*

We propose for the future to confine ourselves to the case where the viscosity is very small; we see from (14) that k will then be a large quantity of the order $\nu^{-\frac{1}{2}}$.

Let k denote that root of (14) which has its real part positive. Then e^{-hk} will be a very small quantity; but from (22a) we see that D cannot become large, while from (22b) we see that D cannot become large unless at the same time C becomes large of the same order but with opposite sign. In either case we may neglect De^{-hk} in comparison with Ce^{hk}, and therefore the first of equations (21) is approximately equivalent to

$$Ce^{kh} = -\frac{2im^2}{V(k^2+m^2)}(Ae^{mh}-Be^{-mh}). \tag{23}$$

On substituting this value for Ce^{kh} and neglecting De^{-kh} in (20), we obtain

$$Ae^{mh}-Be^{-mh} = -\frac{k^2+m^2}{k^2-m^2}mV^2a; \tag{24}$$

while from the second of equations (21) we obtain, with errors of the order $\nu^{\frac{3}{2}}$ only,

$$Ae^{mh}+Be^{-mh} = -\frac{gaV}{V+2im\nu}. \tag{25}$$

On eliminating a from (24), (25), we have

$$\frac{Ae^{mh}-Be^{-mh}}{Ae^{mh}+Be^{-mh}} = \frac{mV^2}{g}\,\frac{k^2+m^2}{k^2-m^2}\,\frac{V+2im\nu}{V}.$$

But, by (14), $\dfrac{k^2+m^2}{k^2-m^2} = \dfrac{2m^2-imV/\nu}{-imV/\nu} = \dfrac{V+2im\nu}{V}$;

whence $$\frac{Ae^{mh}-Be^{-mh}}{Ae^{mh}+Be^{-mh}} = \frac{mV^2}{g}(1+2im\nu/V)^2. \tag{26}$$

Take first the case where the bottom is perfectly smooth. We then have from (22b)

$$A = B, \quad C = -D;$$

and, therefore, from (26)

$$\frac{mV^2}{g}(1+2im\nu/V)^2 = \tanh mh,$$

an equation which may be readily solved by successive approximation. If, as a first approximation, we neglect ν, we find

$$V_0^2 = \frac{g}{m}\tanh mh; \tag{27}$$

thus verifying the well-known formula for the velocity of wave-propagation in a frictionless liquid.* Replacing V by V_0 in the small terms which contain ν, we deduce, as a second approximation,

$$V^2 = \frac{g}{m} \tanh mh \left(1 - \frac{4im\nu}{V_0}\right);$$

whence $$V = V_0 - 2im\nu.$$

To the order of approximation considered the velocity of wave-propagation is therefore unaltered by friction, while the modulus of decay is given by the formula

$$\tau = \frac{1}{2m^2\nu} = \frac{\lambda^2}{8\pi^2\nu}, \tag{28}$$

where λ denotes the wave-length. This agrees with the formula given by Prof. Lamb† for the case of waves in deep water. We see now that it holds for waves of any wave-length in water of any depth, provided that the bottom is perfectly smooth and that the internal viscosity is sufficiently small to allow of our approximations.

Dealing next with the case where no slipping is allowed at the bottom, we see from (23) that, since e^{kh} is large, C must be excessively small. Hence the equations (22a) which are applicable under these circumstances take the approximate forms

$$\left.\begin{aligned} A + B &= -\frac{ik}{m} VD, \\ A - B &= \quad iVD, \end{aligned}\right\}$$

from which we deduce

$$\frac{A}{B} = \frac{k-m}{k+m},$$

and hence from (26)

$$\frac{mV^2}{g}(1 + 2im\nu/V)^2 = \frac{k \sinh mh - m \cosh mh}{k \cosh mh - m \sinh mh}. \tag{29}$$

But from (14) we have, with errors of the orders $\nu^{\frac{3}{2}}$,

$$k = \pm\sqrt{\{-imV/\nu\}} = \pm\,(1-i)\sqrt{(mV/2\nu)},$$

and, since by hypothesis the real part of k is positive, we must take the upper sign; we therefore find

$$\frac{m}{k} = \frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V)}}. \tag{30}$$

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* Lamb, *Hydrodynamics*, p. 372.

† *L.c.*, p. 545; or *Proc. Lond. Math. Soc.*, Vol. XIII., p. 62.

The right-hand member of (29) is approximately equal to

$$\tanh mh\left(1-\frac{m}{k}\coth mh\right)\left(1+\frac{m}{k}\tanh mh+\frac{m^2}{k^2}\tanh^2 mh+\dots\right)$$

$$=\tanh mh\left(1-\frac{2m}{k}\,\frac{1}{\sinh 2mh}-\frac{m^2}{k^2}\,\frac{1}{\cosh^2 mh}+\dots\right).$$

Hence equation (29) may be written

$$V^2\,(1+2im\nu/V)^2=V^2\left(1-\frac{2m}{k\sinh 2mh}\right);$$

or

$$V\;(1+2im\nu/V)\;=V_0\;\left(1-\frac{m}{k\sinh 2mh}\right);$$

where we have retained only the most important terms on the right. As a first approximation we find, as before, on omitting all small terms involving ν or $\nu^{\frac{1}{2}}$,

$$V=V_0.$$

Using this value of V in small terms, we obtain as a further approximation

$$\begin{aligned}V&=V_0\left\{1-\frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}/(1+2im\nu/V_0)\\&=V_0\left\{1-\frac{(1+i)\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}(1-2im\nu/V_0)\\&=V_0\left\{1-\frac{\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\}-2im\nu-\frac{i\sqrt{(mV_0\nu)}}{\sqrt{2}\sinh 2mh}\\&=V_1-\frac{i}{m\tau},\end{aligned}$$

where

$$\left.\begin{aligned}V_1&=V_0\left\{1-\frac{\sqrt{(m\nu)}}{\sqrt{(2V_0)}\sinh 2mh}\right\},\\\frac{1}{\tau}&=2m^2\nu+\frac{\sqrt{(m^3V_0\nu)}}{\sqrt{2}\sinh 2mh}.\end{aligned}\right\}\qquad(31)$$

From these equations we see that the velocity of wave-propagation is slightly retarded by friction with the ocean bed. Of the two terms in $1/\tau$ the first alone appears when we neglect bottom-friction. Hence we may attribute these terms to internal viscosity and to bottom-friction respectively. In general for small values of ν the second term, involving $\nu^{\frac{1}{2}}$, will be far more important than the first, which involves only ν; in other words, the friction at the bottom will be far more efficacious than internal viscosity in destroying the motion. But, if the depth be large in comparison with the wave-length, $\sinh 2mh$ will assume a very large value, and therefore the second term of $1/\tau$ becomes relatively unimportant. For waves of short wave-length (in comparison with the depth) the nature of the action at the bottom will be of no account, and we shall obtain the same formula (28) whatever assumption we make as to this action.

For "long waves" we may replace $\tanh mh$ by mh, and $\sinh 2mh$ by $2mh$. The formulæ (27), (31) then give

$$\left.\begin{aligned} V_0^2 &= gh, \\ V_1 &= \sqrt{gh} - \frac{\nu^{\frac{1}{2}} g^{\frac{1}{4}}}{2\sqrt{2}\, m^{\frac{1}{2}} h^{\frac{3}{4}}}, \\ \tau &= \frac{2\sqrt{2}\, h^{\frac{3}{4}}}{m^{\frac{1}{2}} g^{\frac{1}{4}} \nu^{\frac{1}{2}}}. \end{aligned}\right\} \qquad (32)$$

The approximations we have used require m to be small compared with V_0/ν and e^{-kh} small compared with unity.

The first condition requires ν to be small compared with V_0/m; the second requires ν to be small compared with $m^2h^2\,(V_0/m)$.

In the case of deep sea waves, both conditions will be satisfied provided ν is small compared with V_0/m. Taking

$$\nu = \cdot 0178,$$

the value for water, the requisite condition will be satisfied provided the wave-length is large compared with ·04 of a centimetre.

On the other hand, if the waves are of the nature of "long waves," so that mh is small, the two conditions will be satisfied provided ν is small compared with

$$m^2h^2\,(V_0/m).$$

With the same value of ν this requires the wave-length to be small compared with

$$10^4\, h^{\frac{5}{2}} \text{ centimetres},$$

where the depth h is to be expressed in centimetres.

We see, then, that no serious limitations are introduced on the range of applicability of our results, provided we are dealing with a liquid of such a small degree of viscosity as water.

For waves of 100 metres in a depth of 1 metre, the formulæ (32) will give a modulus of decay of about 1 hour 20 minutes, while there will be no sensible retardation in the velocity of wave propagation.

If there were no friction at the bottom, the modulus of decay in this case would be about $2\frac{1}{4}$ years.

Except in the case of deep sea waves, where it is necessary to take into account the tangential forces in the neighbourhood of the surface, we may put

$$Ce^{kh} = 0,$$

and *à fortiori* $C = 0$. Referring back to equations (17), we therefore have, with a high degree of approximation,

$$\psi = (Ae^{mz} + Be^{-mz})\, e^{im(x-Vt)},$$

$$u = \left[-\frac{1}{V}(Ae^{mz} + Be^{-mz}) - \frac{ik}{m} De^{-kz}\right] e^{im(x-Vt)},$$

$$w = \left[\frac{i}{V}(Ae^{mz} - Be^{-mz}) + De^{-kz}\right] e^{im(x-Vt)}.$$

These equations indicate that the motion is the same as if there were no friction, except through the region in the immediate neighbourhood of the bottom over which the term De^{-kz} remains sensible. If there is no friction at the bottom, D will be small of the same order as C, and hence the motion will be sensibly the same throughout as if there were no friction.

5. *Preliminary Analysis applicable to the Case of a Spherical Sheet of Water.*

The differential equations for the three-dimensional oscillations of a mass of viscous liquid can be expressed in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial \psi}{\partial x} + \nu\nabla^2 u, \\ \frac{\partial v}{\partial t} &= \frac{\partial \psi}{\partial y} + \nu\nabla^2 v, \\ \frac{\partial w}{\partial t} &= \frac{\partial \psi}{\partial z} + \nu\nabla^2 w, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &+ \frac{\partial w}{\partial z} = 0, \end{aligned}\right\} \tag{33}$$

where $$\psi = V - p/\rho + \text{const.}, \tag{34}$$

the notation used being the usual notation for such problems.

The general solution of these equations applicable for satisfying boundary-conditions at concentric spherical surfaces has been frequently discussed.* There is an inconvenience, however, attached to the use of the functions usually employed in the applications which we propose to make here, arising from the fact that these functions will always appear with large arguments. I have found it possible, however, to considerably simplify the analysis by utilizing a modified form of these functions applicable in the case where the total range of the arguments involved is small in comparison with the actual values of these arguments. Such will be the case if the distance between the internal and external bounding spheres is small in comparison with the radii of these spheres. We will commence by a recapitulation of the analysis used in the papers cited above, introducing where necessary the modifications referred to.

Take first the equation

$$(\nabla^2+k^2)\,\Phi = 0, \tag{35}$$

and suppose Φ is of the form $R\phi_n$, where R is a function of r only, and ϕ_n a solid harmonic of degree n. On substituting this form for Φ in (35), we deduce

$$\frac{d^2R}{dr^2}+\frac{2\,(n+1)}{r}\,\frac{dR}{dr}+k^2R = 0. \tag{36}$$

Now, suppose $r = a+h\xi,$

where h is small compared with a, and in the region to which our solution is required to apply ξ lies between 0 and 1. Changing the variable from r to ξ and neglecting terms of the order h^2/a^2, the above equation becomes

$$\frac{d^2R}{d\xi^2}+2\,(n+1)\,\frac{h}{a}\,\frac{dR}{d\xi}+h^2k^2R = 0\,;$$

or, with errors of the same order of magnitude,

$$\frac{d^2R}{d\xi^2}+2\,(n+1)\,\frac{h}{a}\,\frac{dR}{d\xi}+\left\{(n+1)^2\frac{h^2}{a^2}+h^2k^2\right\}R = 0. \tag{37}$$

The rigorous solution of this equation is

$$R = A\xi_n+B\Xi_n,$$

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* Lamb, *Proc. Lond. Math. Soc.*, Vol. XIII., pp. 51, 189; Love, *ibid.*, Vol. XIX., p. 170, &c.

where
$$\xi_n = e^{hk\xi i - \frac{(n+1)h\xi}{a}}, \qquad \Xi_n = e^{-hk\xi i - \frac{(n+1)h\xi}{a}}; \tag{38}$$

and therefore a solution of (35) is
$$\Phi = (A\xi_n + B\Xi_n)\,\phi_n. \tag{39}$$

From (38), we have
$$\left.\begin{aligned} \frac{d\xi_n}{dr} &= \frac{1}{h}\frac{d\xi_n}{d\xi} = \{ki - (n+1)/a\}\,\xi_n, \\ \frac{d\Xi_n}{dr} &= \frac{1}{h}\frac{d\Xi_n}{d\xi} = -\{ki + (n+1)/a\}\,\Xi_n; \end{aligned}\right\} \tag{40}$$

while, with errors of order $\dfrac{h^2}{a^2}$,
$$\left.\begin{aligned} \left(\frac{r}{a}\right)^n \xi_n &= e^{hk\xi i - h\xi/a} = \xi_0, \\ \left(\frac{r}{a}\right)^n \Xi_n &= e^{-hk\xi i - h\xi/a} = \Xi_0. \end{aligned}\right\} \tag{41}$$

Consider next the equations
$$\left.\begin{aligned} (\nabla^2 + k^2)\,u = 0, \quad (\nabla^2 + k^2)\,v = 0, \quad (\nabla^2 + k^2)\,w = 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{aligned}\right\} \tag{42}$$

By the preceding we see that, if ϕ_{n+1}, χ_n denote spherical solid harmonics of degree indicated by their suffixes, since
$$\frac{\partial \phi_{n+1}}{\partial x}, \quad \frac{\partial \phi_{n+1}}{\partial y}, \quad \frac{\partial \phi_{n+1}}{\partial z}, \quad y\frac{\partial \chi}{\partial z} - z\frac{\partial \chi_n}{\partial y}, \text{ \&c.}$$
are solid harmonics of degree n, the following will satisfy the first three of equations (42),
$$\left.\begin{aligned} u &= \xi_n\left[\frac{\partial \phi_{n+1}}{\partial x} + y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right], \\ v &= \xi_n\left[\frac{\partial \phi_{n+1}}{\partial y} + z\frac{\partial \chi_n}{\partial x} - x\frac{\partial \chi_n}{\partial z}\right], \\ w &= \xi_n\left[\frac{\partial \phi_{n+1}}{\partial z} + x\frac{\partial \chi_n}{\partial y} - y\frac{\partial \chi_n}{\partial x}\right]. \end{aligned}\right\} \tag{43}$$

These do not satisfy the last of equations (42), but, in virtue of (40), they make

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{(n+1)(kai-n-1)}{ar}\,\xi_n\phi_{n+1}.$$

Another set of solutions of the first three of (42) is

$$\left.\begin{aligned} u &= C\xi_{-n-3}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\ v &= C\xi_{-n-3}\frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\ w &= C\xi_{-n-3}\frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\end{aligned}\right\} \tag{44}$$

from which we deduce

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= -C\,\frac{(n+2)(kai+n+2)}{ar^{2n+4}}\,\xi_{-n-3}\phi_{n+1}\\ &= -C\,\frac{(n+2)(kai+n+2)}{a^{2n+4}r}\,\xi_n\phi_{n+1},\end{aligned}$$

in virtue of (41).

Hence, if we put $C=a^{2n+3}\dfrac{(n+1)}{(n+2)}\,\dfrac{kai-n-1}{kai+n+2}$,

and add together the particular solutions (43), (44), we obtain a set of values of u, v, w which satisfy all four of equations (42).

Another set of solutions may be found by using the functions Ξ_n instead of ξ_n; we therefore obtain solutions of the following types, which may be treated independently :—

$$\left.\begin{aligned} u &= \xi_n\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\sigma y}\right)+\Xi_n\left(y\frac{\partial X_n}{\partial z}-z\frac{\partial X_n}{\partial y}\right),\\ v &= \xi_n\left(z\frac{\partial\chi_n}{\partial x}-x\frac{\partial\chi_n}{\partial z}\right)+\Xi_n\left(z\frac{\partial X_n}{\partial x}-x\frac{\partial X_n}{\partial z}\right),\\ w &= \xi_n\left(x\frac{\partial\chi_n}{\partial y}-y\frac{\partial\chi_n}{\partial z}\right)+\Xi_n\left(x\frac{\partial X_n}{\partial y}-y\frac{\partial X_n}{\partial z}\right);\end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned}
u &= \xi_n \frac{\partial \phi_{n+1}}{\partial x} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} \xi_{-n-3} a^{2n+3} \frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial x} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial x}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right), \\
v &= \xi_n \frac{\partial \phi_{n+1}}{\partial y} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} \xi_{-n-3} a^{2n+3} \frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial y} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial y}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right), \\
w &= \xi_n \frac{\partial \phi_{n+1}}{\partial z} + \frac{(n+1)}{(n+2)} \frac{kai-n-1}{kai+n+2} a^{2n+3} \xi_{-n-3} \frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right) \\
&\quad + \Xi_n \frac{\partial \Phi_{n+1}}{\partial z} + \frac{(n+1)}{(n+2)} \frac{kai+n+1}{kai-n-2} a^{2n+3} \Xi_{-n-3} \frac{\partial}{\partial z}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right),
\end{aligned}\right\} \quad (46)$$

where χ_n, X_n, ϕ_n, Φ_n denote solid harmonics of degree n.

6. *On the Rate of Decay of Slow Currents in a Spherical Sheet of Water.*

Returning now to equations (33), suppose u, v, w, ψ each proportional to e^{-at}, and put

$$k^2 = a/\nu.$$

Then (33) reduce to

$$\left.\begin{aligned}
(\nabla^2+k^2)\, u &= -\frac{1}{\nu} \frac{\partial \psi}{\partial x}, \\
(\nabla^2+k^2)\, v &= -\frac{1}{\nu} \frac{\partial \psi}{\partial y}, \\
(\nabla^2+k^2)\, w &= -\frac{1}{\nu} \frac{\partial \psi}{\partial z}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &+ \frac{\partial w}{\partial z} = 0.
\end{aligned}\right\} \quad (47)$$

A set of particular integrals will be furnished by (45), provided

$$\psi = 0.$$

If F, G, H denote the components of surface-traction across any sphere $r=$ constant, we have

$$Fr = -px+\rho\nu\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)+\rho\nu\left(x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial x}+z\frac{\partial w}{\partial x}\right)$$

$$= -px+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)u+\frac{\partial}{\partial x}(ux+vy+wz),$$

&c. &c.

But the particular integrals in question make

$$ux+vy+wz=0;$$

and therefore

$$\left.\begin{aligned} Fr &= -px+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)u, \\ Gr &= -py+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)v, \\ Hr &= -pz+\rho\nu\left(r\frac{\partial}{\partial r}-1\right)w; \end{aligned}\right\}$$

or, from (45), with the aid of (40),

$$Fr = -px+\rho\nu\left\{(n-1)+\frac{r}{a}(kai-n-1)\right\}\xi_n\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)$$

$$+\rho\nu\left\{(n-1)-\frac{r}{a}(kai+n+1)\right\}\Xi_n\left(y\frac{\partial X_n}{\partial z}-z\frac{\partial X_n}{\partial y}\right),$$

&c. &c.

If there be no slipping at the bottom, we must suppose $u=0$, $v=0$, $w=0$, when $\xi=0$. These conditions will be satisfied if

$$\chi_n+X_n=0, \tag{48}$$

since ξ_n, Ξ_n both reduce to unity when $\xi=0$.

If there be no stress at the free surface, we must have $F=0$, $G=0$, $H=0$, when $\xi=1$ or $r=a+h$. This condition will be satisfied if

$$p=0,$$

$$\left[(n-1)+\frac{a+h}{a}(kai-n-1)\right]e^{hki-\frac{(n+1)h}{a}}\chi_n$$

$$+\left[(n-1)-\frac{a+h}{a}(kai+n+1)\right]e^{-hki-\frac{(n+1)h}{a}}X_n=0. \tag{49}$$

From (48), (49) we deduce, on eliminating χ_n, X_n,

$$e^{2hki} = \frac{a\,(n-1)-(a+h)(kai+n+1)}{a\,(n-1)+(a+h)(kai-n-1)}. \tag{50}$$

When the viscosity is small this equation may be solved by successive approximation. Putting k infinite, we deduce

$$e^{2hki} = -1,$$

or

$$2hk = (2n+1)\,\pi,$$

$$\alpha = k^2\nu = \frac{(2n+1)^2\,\pi^2\nu}{4h^2}, \tag{51}$$

which is the same as the value found in § 1.

We conclude that the rates of decay of the free current-motions are not sensibly affected by the curvature of the Earth's surface.

7. *On the Subsidence of Tidal Oscillations.*

The equations (47) will still be applicable if the motion is oscillatory. From them we deduce

$$\nabla^2\psi = 0, \tag{52}$$

with the particular integrals

$$u = -\frac{1}{\alpha}\frac{\partial\psi}{\partial x}, \quad v = -\frac{1}{\alpha}\frac{\partial\psi}{\partial y}, \quad w = -\frac{1}{\alpha}\frac{\partial\psi}{\partial z}. \tag{53}$$

A solution of (52) is

$$\psi = \psi_{n+1} + \Psi_{n+1}/r^{2n+3}, \tag{54}$$

where ψ_{n+1}, Ψ_{n+1} are solid harmonics of degree $n+1$; and the corresponding particular integrals are

$$\left.\begin{aligned} u &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial x} - \frac{1}{\alpha}\frac{\partial}{\partial x}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right), \\ v &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial y} - \frac{1}{\alpha}\frac{\partial}{\partial y}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right), \\ w &= -\frac{1}{\alpha}\frac{\partial\psi_{n+1}}{\partial z} - \frac{1}{\alpha}\frac{\partial}{\partial z}\left(\frac{\Psi_{n+1}}{r^{2n-3}}\right). \end{aligned}\right\} \tag{55}$$

To these must be added complementary functions of the type (46).

Now we have seen that in the case of long waves in two dimensions the tangential stresses are of no account except in the immediate neighbourhood of the bottom. In the present case we shall assume that the motion, except through a very thin layer, will be sensibly irrotational, and therefore will be represented by the particular integrals (55). The complementary functions selected should therefore be such that they vanish except for very small values of ξ. If we suppose that the real part of ik is negative, this condition will be satisfied provided we omit from the right-hand members of (46) the terms involving Ξ_n, Ξ_{-n-3} and retain only those involving ξ_n, ξ_{-n-3}. Thus we obtain as appropriate solutions of the equations of motion

$$\left.\begin{aligned}
\psi &= \psi_{n+1}+\Psi_{n+1}/r^{2n+3},\\
u &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial x}-\frac{1}{a}\frac{\partial}{\partial x}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial x}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\
v &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial y}-\frac{1}{a}\frac{\partial}{\partial y}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial y}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial y}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right),\\
w &= -\frac{1}{a}\frac{\partial\psi_{n+1}}{\partial z}-\frac{1}{a}\frac{\partial}{\partial z}\left(\frac{\Psi_{n+1}}{r^{2n+3}}\right)+\xi_n\frac{\partial\phi_{n+1}}{\partial z}+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\xi_{-n-3}\frac{\partial}{\partial z}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right).
\end{aligned}\right\} \quad (56)$$

The conditions $u=0$, $v=0$, $w=0$, when $r=a$, will therefore be satisfied if

$$\left.\begin{aligned}
&-\psi_{n+1}/a+\phi_{n+1}=0,\\
&-\Psi_{n+1}/a+\frac{(n+1)}{(n+2)}\frac{kai-n-1}{kai+n+2}a^{2n+3}\phi_{n+1}=0.
\end{aligned}\right\} \quad (57)$$

From equations (56) we find

$$ux+vy+wz=-\frac{n+1}{a}\psi_{n+1}+\frac{n+2}{a}\Psi_{n+1}/r^{2n+3}$$

$$+(n+1)\,\xi_n\phi_{n+1}-(n+1)\frac{kai-n-1}{kai+n+2}\frac{a^{2n+3}}{r^{2n+3}}\xi_{-n-3}\phi_{n+1},$$

which, by the aid of (57), reduces to

$$ux+vy+wz=(n+1)\left[(\xi_n-1)-\frac{kai-n-1}{kai+n+2}\frac{a^{2n+3}}{r^{2n+3}}(\xi_{-n-3}-1)\right]\phi_{n+1}.$$

At the free surface we may omit the small terms involving ξ_n, ξ_{-n-3}, and take

$$ux+vy+wz=-(n+1)\left[1-\frac{kai-n-1}{kai+n+2}\left\{1-\frac{(2n+3)h}{a}\right\}\right]\phi_{n+1}$$

$$=-(n+1)(2n+3)\left\{\frac{h}{a}\frac{kai-n-1}{kai+n+2}+\frac{1}{kai+n+2}\right\}\phi_{n+1}.$$

Let ζ denote the height of the waves at the free surface. Then we have the kinematical condition

$$r\frac{\partial\zeta}{\partial t} = ux + vy + wz,$$

when $r = a+h$; whence

$$a\zeta = \frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left\{1+(kai-n-1)\frac{h}{a}\right\}\phi_{n+1}. \tag{58}$$

Finally, the pressure at the surface $r = a+h$ must be equated to $g\rho\zeta$, where g denotes the value of the acceleration due to gravity. But from (34) we have

$$\psi = \text{const.} + V - p/\rho,$$

and therefore

$$[\psi] = \text{const.} + [V] - g\zeta,$$

where the square brackets indicate that surface values are to be understood.

If there be no external disturbing force, the periodic part of V will be the potential due to the harmonic inequalities ζ. Denoting the surface-value of this potential by v' and equating periodic parts in the two members of the last equation, we obtain

$$[\psi] = v' - g\zeta. \tag{59}$$

But from (58) we see that ζ is a surface-harmonic of order $(n+1)$, and therefore we obtain at once

$$v' = \frac{4\pi\rho\,(a+h)}{2n+3}\,\zeta;$$

while, if σ denote the mean density of the system under consideration, including both solid nucleus and liquid surface layer, we have

$$g = \tfrac{4}{3}\pi\sigma\,(a+h).$$

Thus

$$v' = \frac{3\rho}{(2n+3)\,\sigma}\,g\zeta,$$

and, if we denote by g_{n+1} the expression

$$g\left\{1 - \frac{3\rho}{(2n+3)\,\sigma}\right\}, \tag{60}$$

the equation (59) becomes

$$[\psi] = -g_{n+1}\zeta.$$

Introducing the values of ψ, ζ from (56), (58), we find

$$\psi_{n+1} + \Psi_{n+1}/r^{2n+3} = -\frac{g_{n+1}}{a}\,\frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left\{1+(kai-n-1)\frac{h}{a}\right\}\phi_{n+1}$$

when $r = a+h$; thus, on using (57), we obtain

$$1+\frac{(n+1)}{(n+2)}\,\frac{kai-n-1}{kai+n+2}\left\{1-(2n+3)\,h/a\right\}$$

$$= -\frac{g_{n+1}}{\alpha^2}\,\frac{(n+1)(2n+3)}{(kai+n+2)(a+h)}\left[1+(kai-n-1)\,h/a\right];$$

or
$$kai+1-(n+1)(kai-n-1)\,h/a$$

$$= -\frac{(n+1)(n+2)\,g_{n+1}}{\alpha^2\,(a+h)}\left[1+(kai-n-1)\,h/a\right];$$

whence, finally,

$$\frac{\alpha^2}{g_{n+1}} = -\frac{(n+1)(n+2)}{a+h}\,\frac{khi+1-(n+1)\,h/a}{kai-(n+1)\,khi+1+(n+1)^2h/a}. \tag{61}$$

On putting k infinite it follows that, as a first approximation,

$$\alpha^2 = -\frac{(n+1)(n+2)\,g_{n+1}\,h}{(a+h)\left\{a-(n+1)\,h\right\}}.$$

or, with sufficient accuracy,

$$\alpha^2 = -\beta^2,$$

where
$$\beta^2 = \frac{(n+1)(n+2)\,g_{n+1}\,h}{a^2}.\text{*} \tag{62}$$

From this we obtain

$$k^2 = \alpha/\nu = i\beta/\nu\,;$$

whence
$$k = \pm\,(1+i)\,\sqrt{(\beta/2\nu)},$$

or
$$ik = \pm\,(i-1)\,\sqrt{(\beta/2\nu)}.$$

The upper sign must be taken, since by hypothesis the real part of ik is negative. From (61) we therefore find, as a second approximation,

$$\alpha^2 = -\beta^2\left[1+\frac{1}{khi}\right]$$

$$= -\beta^2\left[1-\frac{1+i}{\sqrt{2}}\sqrt{\frac{\nu}{\beta h^2}}\right],$$

whence
$$\alpha = i\beta\left[1-\tfrac{1}{2}\sqrt{\frac{\nu}{2\beta h^2}}-\frac{i}{2}\sqrt{\frac{\nu}{2\beta h^2}}\right].$$

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* *Cf.* Lamb, *Hydrodynamics*, p. 315.

Hence the speed of the oscillation is reduced by friction in the ratio

$$1-\tfrac{1}{2}\sqrt{\frac{\nu}{2\beta h^2}} : 1,$$

and the modulus of decay is

$$\surd(8h^2/\beta\nu). \tag{63}$$

Using C.G.S. units and taking

$$\sigma/\rho = 5\tfrac{1}{2}, \quad \nu = \cdot 0178, \quad a = 6\cdot 357\times 10^8, \quad h = 4\times 10^5,$$

we deduce as the moduli of decay for the types corresponding to $n = 0$, $n = 1$, 42·6 and 31·7 years respectively, while, when $n = 100$, we find a modulus of decay of 4·8 years.

Even with considerably smaller depths, the moduli of decay of the principal types of oscillation will still be long. Thus, if we take

$$h = 2\times 10^4,$$

which implies a depth of 200 metres, or rather more than 100 fathoms, the moduli of decay for the types $n = 0$, $n = 1$ are 4·5 and 3·3 years respectively.

These results indicate how slight can be the effects of viscosity on the motions of the sea except possibly in shallow confined waters. It seems that wherever the depth exceeds a very moderate amount, say 100 fathoms, the rise and fall of the waters due to the disturbing influence of the Sun and Moon will not be appreciably affected by friction. It may be urged that we have a direct contradiction of this statement in the fact that the phases of the tides even at islands in the open sea often differ widely from the phases of the corresponding equilibrium tides. According to Sir G. Airy,* the acceleration or retardation of the semi-diurnal tide on the Moon's transit "does not at one port in a hundred agree in any measure with the result of this [the equilibrium] theory," and the want of agreement is attributed by him entirely to the effects of friction. The explanation seems to have been generally accepted by his successors, and, if it be true, will entirely invalidate our present results. A different explanation of the phenomenon in question has, however, been given by Newton,† and, in spite of the criticisms to which this has been subjected by Airy,‡ it appears to me that the results as stated by Newton are substantially correct, and that an agreement in phase at all places between the dynamical tides and the equilibrium tides, even without viscosity, could only be expected under very special circumstances as to the distribution of land and water on the globe. In support of this view, I may quote some examples worked out by Airy himself in later sections of the paper referred to above, the bearing of which does not seem to have been fully appreciated by the author. Thus, if we take the case of a continuous equatorial canal subjected to the disturbance of a luminary moving uniformly round the equator, it is known that the forced tide will consist of a progressive wave following the motion of the disturbing body, and that at any place high or low water will always accompany the transit of the luminary across the meridian. The circumstances will, however, be totally different if the canal, instead of being continuous so as to return into itself, is of limited extent. In the latter case,* to find the complete motion arising from the disturbing force, we must superpose on the primary wave, due directly to the attraction of the luminary, secondary positive and negative waves due to the repeated reflection of the former at the extremities of the canal. These secondary waves, which have, as it seems to me, been erroneously described by Airy as "free waves," will be co-periodic with the primary wave, but will give rise to an entire re-adjustment of phase. Even in the case of continuous canals the examples worked out by Airy† indicate that an agreement in phase with the phase of the equilibrium tide can only be regarded as fortuitous in character.

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* *Encyc. Metrop.*, Art. "Tides and Waves," § 62.

† *Principia*, Book III., Prop. 24.

‡ *L.c.*, §§ 16, 19.

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* Airy, *l.c.*, § 296.

† *L.c.*, §§ 431 *et seq.*

Connexion of Quadratic Forms. By Lt.-Col. ALLAN CUNNINGHAM, R.E., Fellow of King's College, London. Received and Read December 10th, 1896.

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1. *Quadratic Forms.*—The function

$$f(m, n) = (mX^2+nY^2)$$

is styled a *Quadratic Form* or sometimes (for shortness) simply a *Form;* the numbers m, n are the distinctive *characters* of the form itself, the numbers X, Y merely determining the particular number N expressed in the form. When either of m, $n = 1$, the form is styled *simple;* when neither of m, $n = 1$, it is styled *non-simple.*

2. *Object of this Paper.*—In this paper a method is developed (Art. 24) whereby from two given *distinct* forms of the same number a new and *distinct* form may be derived under certain conditions. Each of the three so "allied forms" will be shown to be in certain cases similarly *derivable* from the other two. The process depends on two known processes here styled *conformal multiplication* and *conformal division;* to render it clear, it has been necessary to develop these in considerable detail, especially with regard to the conditions of possibility of the latter (the conformal division); this occupies Art. 6–23. The new process of "derivation" occupies Art. 24–35, followed by examples Art. 36–41.

3. *Notation.*—The following notation is used uniformly throughout this paper. All symbols used denote *integers.*

$N =$ a *positive* integer; $p =$ a *prime*;

f, F, ϕ, Φ denote quadratic forms (usually*) of *positive* numbers;

$$f(m,n) = mt^2+nu^2; \qquad f(1, mn) = t^2+mnu^2;$$
$$F(m,n) = mT^2+nU^2; \qquad F(1, mn) = T^2+mnU^2;$$
$$\phi(m,n) = mx^2+ny^2; \qquad \phi(1, mn) = x^2+mny^2;$$
$$\Phi(m,n) = mX^2+nY^2; \qquad \Phi(1, mn) = X^2+mnY^2.$$

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* It is often convenient to use the *negative* unit-form (Art. 9), $f(m, -n) = -1$, instead of the *positive* form $f(n, -m) = +1$.

Thus $f(m, -n)$, $f(n, -m)$, $f(1, -mn)$, &c., usually denote *positive* numbers; also $f(m, \sim n)$ denotes the *ambiguous* $(mt^2 \sim nu^2)$, *i.e.*, the *positive* number $(mt^2 - nu^2)$ or $(nu^2 - mt^2)$.

4. *Isomorphs, Antimorphs, Symmorphs.*—Forms $f_1(m, n)$, $f_2(m, n)$, &c., containing the *same* m, n, are styled *Isomorphs*, (as being of *same* form); the two forms $f(m, n)$, $f(n, m)$ are also *isomorphs* when m, n are *both positive*. Two forms $f(m, -n)$, $f(n, -m)$ in which the m, n carry *opposite* signs, and are *interchanged in position*, are styled *Antimorphs;* thus $f(1, -n)$, $f(n, -1)$ are *simple antimorphs*. Two forms $f(m, n)$, $f(1, mn)$, wherein the mn of the simple form = the product of the m, n of the non-simple form, are styled *Symmorphs*, (as being closely *related*, as will appear later). When m, n carry opposite signs, the antimorphs $f(m, -n)$, $f(n, -m)$ are each *symmorphs* of both the *antimorphs* $f(1, -mn)$, $f(mn, -1)$.

5. *Conformals, Non-conformals.*—The three forms *Iso-*, *Anti-*, and *Sym-morphs* are so closely related (as will appear later on), that it is convenient to include them under a single name, viz., *Conformals*. Forms not so related are styled *Non-conformal*. The *simple* form $f(1, mn)$ may be a symmorph of several *non-simple* forms $f(m_1, n_1)$, $f(m_2, n_2)$, &c., which are non-conformal among each other, provided $mn = m_1 n_1 = m_2 n_2 = \&c.$

Note.—When either of m, n contains square factors, these factors may—if convenient—be cancelled from m, n, and *absorbed* into the adjacent squares X^2, Y^2. The properties of *iso-*, *anti-*, and *sym-morphism*, and of *conformality* are not affected by the presence of square factors (even though these factors be different) in the m, n of some of the forms; these properties are defined by the various m, n after *excluding* square factors from them.

6. *Conformal Multiplication.*—The following* process, whereby the product of two conformals (f, F) is expressed as a conformal (Φ), is styled *Conformal Multiplication;* there are two principal cases (marked i., ii.)

i. $$f(m, n)\, F(m, n) = (mt^2 + nu^2)(mT^2 + nU^2)$$
$$= X^2 + mnY^2 = \Phi(1, mn), \qquad (1)$$

where $X = mtT \pm nuU$, $Y = tU \mp uT$, (opp. signs in X, Y). (1*a*)

ii. $$f(m, n)\, F(1, mn) = (mt^2 + nu^2)(T^2 + mnU^2)$$
$$= mX^2 + nY^2 = \Phi(m, n), \qquad (2)$$

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* These results are well known: they are here given *in detail*, for reference further on.

where $X = tT \pm nuU,\ Y = mtU \mp uT$, (opp. signs in X, Y). (2a)

The results when the coefficients m, n carry the same, or opposite, sign are worth recording.

i. $f(m, n)\,F(m, n) = \Phi(1, mn)$; $f(m, -n)\,F(m, -n) = \Phi(1, -mn)$, (3)

ii. $f(m,n)\,F(1,mn) = \Phi(m,n)$; $f(m, -n)\,F(1, -mn) = \Phi(m, -n)$; (4)

with their sub-cases

i*a*. $f(m, -n)\,F(n, -m) = \Phi(mn, -1)$, (5)

ii*a*. $f(m, -n)\,F(mn, -1) = \Phi(n, -m)$. (6)

The simple cases, when either m or $n = \pm 1$, are of special importance:

i*b*, ii*b*. $f(1,n)\,F(1,n) = \Phi(1,n)$; $f(1, -n)\,F(1, -n) = \Phi(1, -n)$; (7)

with their sub-cases

i*c*. $f(m, -1)\,F(m, -1) = \Phi(1, -m)$, (8)

ii*c*. $f(m, -1)\,F(1, -m) = \Phi(m, -1)$. (9)

Thus, in all cases, the product of two conformals (f, F) can be expressed as a *conformal*; also, when the two factors f, F are *unequal*, it is expressible in *two* isomorphic ways, in consequence of the double sign in X, Y; and the signs used in X, Y are *opposite* or *similar*, according as the coefficients m, n carry the *same* or opposite signs.

[If m, n contain a common factor κ, so that $m = \kappa\mu$, $n = \kappa\nu$, then its square appears in the product $mn = \kappa^2 . \mu\nu$ in Results 1, 3, &c., and may be absorbed into the adjacent square (Y^2), if found convenient.]

Product of Non-conformals.—The product of two unequal non-conformals (f, F) is *sometimes* (arithmetically) expressible *in one way* as a conformal of *one* of the factors; but it does not seem possible to formulate this *algebraically*.

Ex.—Let $f = 17$, $F = 33 = 5 . 1^2 + 7 . 2^2$; here f cannot be expressed conformally with the given form of F; but the product $fF = 561 = 1^2 + 35 . 4^2$, which is a *conformal* (symmorph) of F.

7. *Degraded Conformals.*—Since the three numbers m, n, mn may be written

$$m = m . 1^2 + n . 0^2; \quad n = m . 0^2 + n . 1^2; \quad mn = 0^2 + mn . 1^2;$$

they may be looked on as *degraded conformals* of $f(m, n)$ and $f(1, mn)$. The intimate connexion of symmorphs is now seen, as they are interchanged by multiplication by m, n; for

$$m(mt^2+nu^2) = (mt)^2+mnu^2; \quad n(mt^2+nu^2) = (nu)^2+mnt^2;$$

$$m(t^2+mnu^2) = mt^2+n(mu)^2; \quad n(t^2+mnu^2) = m(nu)^2+nt^2.$$

In four cases *one* of the two product-forms Φ also degenerates, by one of its terms X, Y becoming zero,

Case i., when $mtT = nuU$; or $tU = uT$.

Case ii., when $tT = nuU$; or $mtU = uT$.

8. *Continued Conformal Multiplication.*—The above results are easily extended to the continued product of conformals; the formulæ are inconveniently heavy, but the results expressed in words are simple.

"The product of several conformals is a conformal." (10)

"That of several (r) non-simple isomorphs is an isomorph or symmorph, according as r is *odd* or *even*." (10a)

"That of several simple isomorphs is an isomorph." (10b)

In each case the final product-form Φ is—when the factors are *unequal*—expressible in *four* isomorphic ways when there are *three* factors, and generally in 2^{r-1} isomorphic ways when there are r factors, on account of the double sign involved in X, Y; and the signs used in X, Y are opposite or similar, according as the coefficients m, n carry the same or opposite signs.

9. *Unit-Forms.*—When m, n are such that values of τ, υ can be found, so that

$$f_0(m, n) = m\tau_0^2+n\upsilon_0^2 = \pm 1, \text{ (m, n must be of opp. signs)}, \quad (11)$$

then the powers of f_0 are all $= \pm 1$, and—being products of conformals—are expressible (Art. 6, 8) as conformals of f_0; thus (ω, ϵ denoting *odd* and *even* integers),

$$\{f_0(m, n)\}^{\epsilon} = \tau_\epsilon^2+mn\,.\,\upsilon_\epsilon^2 = f_\epsilon(1, mn) = (\pm 1)^\epsilon = +1, \quad (11a)$$

$$\{f_0(m, n)\}^{\omega} = m\tau_\omega^2+n\upsilon_\omega^2 = f_\omega(m, n) = (\pm 1)^\omega = \pm 1. \quad (11b)$$

Such quantities $f_0(m, n)$, $f_0(1, mn)$, and their powers are called *Unit-forms* of $f(m, n)$, $f(1, mn)$.

[Note that $f_0(1, -n) = \tau_0^2-n\upsilon_0^2 = +1$ is always possible; but $f_0(1, -n) = \tau_0^2-n\upsilon_0^2 = -1$, $f(m, n) = +1$ or -1 are only possible under certain conditions.]

10. *Automorphs.*—The product of a number N in a given form $N = f(m, n)$ by any of the unit-forms $N_0 = f_0(m, n)$ or $f_0(1, mn)$ thereof, or by their powers, is unchanged in value, and—being a product of conformals—is expressible as a conformal. All such forms of the *same number* N, derivable from one another by conformal multiplication by a unit-form (N_0) thereof, are styled *Automorphs* (of that form); and are considered *equivalent forms.* Forms of the same number—not so derivable from one another—are styled *Anautomorphs*, even though they be conformal (as conformals of composite numbers are not necessarily automorphs).

10*a*. *Use of unit-forms.*—Unit-forms are of great practical use (in computation) in modifying the *form* of a number N into an automorph thereof (*i.e.*, without changing the value of N), and chiefly in two ways :—

(1) In *changing the form* $f(m, n)$ or $f(1, mn)$ into the *symmorph* $f(1, mn)$ or $f(m, n)$; or $f(m, -n)$ into its *antimorph* $f(n, -m)$.

(2) In *reducing* a form $(mt^2 - nu^2)$ in which t, u are *large* numbers into an *isomorph* $(mt'^2 - nu'^2)$ in which t, u are smaller numbers.

Examples of both cases are given, Art. 39–41.

11. *Monomorphs, Dimorphs, Polymorphs.*—A form $f(m, n)$ of a particular number (N) is styled a *Monomorph*, *Dimorph*, or *Polymorph;* when N is expressible (anautomorphically) in *only one* way, in *two* ways, or in *several* ways in that form, or conformally therewith; automorphs (of that form) being here reckoned, as above stated, as *equivalent* forms.

Examples.—(1) Primes are always *monomorphs* of every form in which they are expressible: thus conformal forms of a prime are necessarily automorphs. (2) Products of two *unequal* conformals $f(m, n)$ and $F(m, n)$ or $F(1, mn)$ are always *dimorphs* (of that form). (3) Products of three *unequal* conformals are always *tetramorphs* (of that form). (4) Products of several (r) *unequal* conformals are always *polymorphs* of order 2^{r-1} (of that form). (5) Composite numbers (N) may be *monomorphs* of certain forms, and are always monomorphs of any form wherein m, n are both $+$, and either m or $n > \frac{1}{4}N$. (6) Thus dimorphism marks a composite number; whilst monomorphism is a necessary (but not a sufficient) mark of a prime.

12. *Powers of Forms.*—The r^{th} power of a form $N = f(m, n)$—being a product of conformals—is expressible conformally (Art. 6, 8), with $f(m, n)$, noting that, by (10*a*, *b*),

"The r^{th} power of a non-simple form $f(m, n)$ is an isomorph $\Phi(m, n)$, or symmorph $\Phi(1, mn)$ of $f(m, n)$, according as r is *odd* or *even*." (12)

But there is an apparent exception to the rules of Art. 11, as to the *poly*-morphism of the result (which are there given only for *unequal* factors). Thus, by the rules of conformal multiplication of

$$N = f(m, n) = (mt^2 + nu^2)$$

by itself, the two power-forms arising at each step from the double sign in X, Y are—writing $mt^2 + nu^2 = f$ (for shortness)—

$$N^2 = (mt^2 - nu^2)^2 + mn\,(2tu)^2 = (mt^2 + nu^2)^2 + mn\,.\,0^2. \qquad (13a)$$

$$N^3 = m\{t(mt^2 - 3nu^2)\}^2 + n\{u(3mt^2 - nu^2)\}^2 = m\,(tf)^2 + n\,(uf)^2. \qquad (13b)$$

$$N^4 = (m^2t^4 - 6mnt^2u^2 + n^2u^4)^2 + mn\{4tu\,(mt^2 - nu^2)\}^2$$

$$= \{(mt^2 - nu^2)\,f\}^2 + mn\{2tu\,.\,f\}^2 = (mt^2 + nu^2)^4 + mn\,.\,0^4. \qquad (13c)$$

$N^5 =$ &c., &c., &c.

It will be seen that there is only one new, really distinct, form at each step; the remaining forms of each step degenerating to simpler forms, including the original form $f(m, n)$ as a factor of both X, Y, and ending with

$$N^r = (f^r + mn\,.\,0^r), \text{ when } n \text{ is } even;$$

or

$$= m\,(tf^{(r-1)/2})^2 + n\,(uf^{(r-1)/2})^2, \text{ when } n \text{ is } odd. \qquad (13d)$$

In a certain sense, therefore, the powers of a form $f(m, n)$ may be said to be *monomorphs* of that form.

13. *Product of Automorphs.*—The continued product of several (say r) automorphs $f_1, f_2, f_3, \ldots f_r$ of any form $f(m, n)$ of a number N is of course expressible conformally with $f(m, n)$, since automorphs are only a special kind of conformals. Now, if f_0 be any unit-form of $f(m, n)$, then (Art. 10) each automorph of f differs from f only by some power of f_0 being included in it, so that

$$f_1 = f\cdot f_0^{\alpha}, \quad f_2 = f\cdot f_0^{\beta}, \quad f_3 = f\cdot f_0^{\gamma}, \text{ \&c.}$$

Hence

$$f_1\cdot f_2\cdot f_3 \ldots f_r = f^r\cdot f_0^{\alpha+\beta+\gamma+\ldots+\rho} = N^r. \qquad (14)$$

Thus the continued product of r automorphs of f differs from f^r only by some power of the unit-form f_0 being included, and all the forms in which N^r is thus expressible are automorphs of f^r itself, and therefore comprise no new distinct forms other than those arising from the involution of $f(m, n)$ by itself, and may therefore be said to be (in a certain sense) *monomorphic*, (Art. 12).

14. *Ratio of Conformals.*—The ratio of two conformals $F \div f$, &c., is generally expressible in an infinite number of ways as a ratio of two numbers $\Phi \div \phi$, each conformal with F, f by merely multiplying both numerator and denominator by the *same* number χ conformal with both. Thus

$$N = \frac{F(m, n)}{f(m, n)} = \frac{F(m, n)}{f(m, n)} \frac{\chi(m, n)}{\chi(m, n)} = \frac{\Phi(1, mn)}{\phi(1, mn)} \tag{15a}$$

$$= \frac{F(m, n)}{f(m, n)} \frac{\chi(1, mn)}{\chi(1, mn)} = \frac{\Phi(m, n)}{\phi(m, n)}. \tag{15b}$$

The simplest change of this sort is by use of the degraded (Art. 7) conformals m, n, or mn; thus

$$N = \frac{mT^2+nU^2}{mt^2+nu^2} = \frac{(mT)^2+mnU^2}{(mt)^2+mnu^2} = \frac{(nU)^2+mnT^2}{(nu)^2+mnt^2} = \frac{m(nU)^2+n(mT)^2}{m(nt)^2+n(mu)^2}. \tag{15c}$$

Similarly, the ratios of conformals

$$F(m, n) \div f(1, mn), \quad F(1, mn) \div f(m, n), \quad \text{\&c., \&c.,}$$

admit of similar automorphic changes.

15. *Conformal Division.*—The following* process, whereby the quotient (N) of two conformals ($F \div f$) is—under certain conditions—expressed as a conformal (ϕ), is styled *Conformal Division.* The conditions are

F to be a multiple of f, or $N = F \div f =$ integer. (16)

F to be dimorph, and f to be monomorph (of their form). (17)

The squares (t, u) in f, and the squares (T, U) in F, to be in each case prime-pairs. (18)

Then, dividing the product-identities $f \, . \, F = \Phi$, [(1), (2) of Art. 6], by f^2, so that $N = F \div f = \Phi \div f^2$, and writing out at length as below, it will be shown (below) that the X and Y of *one* of the two pairs X, Y in each of the three results (19, 20, 21) below—arising from the double signs in X, Y in the three formulæ (19a, 20a, 21a) are always both (arithmetically) divisible by their denominator (f), so that the $X \div f$, $Y \div f$ used below are integers; and, finally, the quotient N becomes in each case expressed *conformally* with F, f; thus three principal cases arise.

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* These results are already known: they are given here in detail, for use further on; it has been necessary to investigate the *conditions of possibility* of this division pretty fully here, as the new process of "derivation" (the object of this paper) is strictly limited by these.

[In forming Case iii. from Result (2), an obvious interchange of f with F, and of t, u with T, U of that result has had to be made.]

$$\text{i. } N = \frac{F(m,n)}{f(m,n)} = \frac{mT^2+nU^2}{mt^2+nu^2} = \frac{X^2+mnY^2}{(mt^2+nu^2)^2}, \text{ [by (1)]}$$

$$= \left\{\frac{X}{f(m,n)}\right\}^2 + mn\left\{\frac{Y}{f(m,n)}\right\}^2$$

$$= x^2+mn\,.\,y^2 = \phi(1, mn); \tag{19}$$

where $X = mtT \pm nuU$, $Y = tU \mp uT$; (opp. signs in X, Y). (19*a*)

$$\text{ii. } N = \frac{F(1,mn)}{f(m,n)} = \frac{T^2+mnU^2}{mt^2+nu^2} = \frac{mX^2+nY^2}{(mt^2+nu^2)^2}, \text{ [by (2)]}$$

$$= m\left(\frac{X}{f(m,n)}\right)^2 + n\left(\frac{Y}{f(m,n)}\right)^2$$

$$= mx^2+ny^2 = \phi(m, n); \tag{20}$$

where $X = tT \pm nuU$, $Y = mtU \mp uT$; (opp. signs in X, Y). (20*a*)

$$\text{iii. } N = \frac{F(m,n)}{f(1,mn)} = \frac{mT^2+nU^2}{t^2+mnu^2} = \frac{mX^2+nY^2}{(t^2+mnu^2)^2}, \text{ [by (2)]}$$

$$= m\left(\frac{X}{f(1,mn)}\right)^2 + n\left(\frac{Y}{f(1,mn)}\right)^2$$

$$= mx^2+ny^2 = \phi(m, n); \tag{21}$$

where $X = tT \pm nuU$, $Y = mTu \mp Ut$; (opp. signs in X, Y). (21*a*)

Argument.—The argument* (as to the divisibility of X and Y by f) is that unless one pair (X, Y) of the two pairs (X, Y) given by the double sign be arithmetically divisible by the denominator (f), the double sign would make the number of ways in which the product $f.F$ could be expressed in the form $\Phi(m, n)$ or $\Phi(1, mn)$ double of that in which the numerator F itself could be so expressed (when the X, Y of Φ are mutual primes, and the t, u of f are also mutual primes);

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* Case i., with its proof, is adapted from Mr. Bickmore's paper, "On the Numerical Factors of $a^u - 1$," Art. 8, in *Messenger of Mathematics*, Vol. xxv., p. 10. The proof of Case ii. was communicated to the author by Mr. Bickmore. The proofs of the three cases, being quite similar, have been combined into one, as above, by the author.

whereas, when f is a monomorph factor of the dimorph conformal F, this number is unchanged.

The cases when the coefficients m, n carry the *same*, or *opposite*, *sign* are worth writing down side by side for contrast and reference:

$$\text{i.}\quad F(m,n)\div f(m,n)=\varphi(1,mn);\quad F(m,-n)\div f(m,-n)=\varphi(1,-mn), \tag{22a}$$

$$\text{ii.}\quad F(1,mn)\div f(m,n)=\phi(m,n);\quad F(1,-mn)\div f(m,-n)=\phi(m,-n), \tag{22b}$$

$$\text{iii.}\quad F(m,n)\div f(1,mn)=\phi(m,n);\quad F(m,-n)\div f(1,-mn)=\varphi(m,-n); \tag{22c}$$

with their sub-cases

$$\text{i}a.\quad F(m,-n)\div f(n,-m)=\phi(mn,-1), \tag{23a}$$

$$\text{ii}a.\quad F(mn,-1)\div f(m,-n)=\phi(n,-m), \tag{23b}$$

$$\text{iii}a.\quad F(m,-n)\div f(mn,-1)=\phi(n,-m). \tag{23c}$$

The simple cases when either m or $n=\pm 1$ are of special importance;

$$\text{i., ii., or iii}b.\quad F(1,n)\div f(1,n)=\phi(1,n);\quad F(1,-n)\div f(1,-n)=\varphi(1,-n), \tag{24}$$

with their sub-cases

$$\text{i}c.\quad F(m,-1)\div f(m,-1)=\phi(1,-m), \tag{24a}$$

$$\text{ii}c.\quad F(1,-m)\div f(m,-1)=\phi(m,-1), \tag{24b}$$

$$\text{iii}c.\quad F(m,-1)\div f(1,-m)=\phi(m,-1). \tag{24c}$$

Thus, in all cases [under the conditions (16) to (18)], the quotient (N) of two conformals ($F\div f$) can be expressed as a *conformal* (ϕ). The quotient-results are exactly analogous with the product-results, and are worth comparing with them* in detail; but in the division process (except when the denominator $f=\pm 1$, Art. 16), one of the signs ($\pm$) involved in the X, Y leads to no result (as in every such case only one of the pairs of X, Y is divisible by its denominator f), so that the quotient-forms are all *unique* for the particular values of m, n, t, u, T, U in F, f.

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* Note that the single product, Case ii., with its Sub-cases ii. a, b, c, gives rise to *two* quotient Cases ii., iii., with corresponding Sub-cases ii. a, b, c, and iii. a, b, c.

16. *Conformal Division by a Unit-form.*—When the divisor f is a unit-form f_0 (Art. 9) of the numerator F, then both pairs of the X, Y are of course divisible by the unit-denominator ($f_0 = \pm 1$), so that *two* conformal quotients are here yielded, but these are merely automorphs or equivalent forms; this requires no formal proof.

17. *Conformal Division by an Automorph.*—When the divisor f is an automorph of the numerator F, the conformal quotient is *unique* and is always a unit-form of F, f; this requires no formal proof.

18. *Conditions of Conformal Division.*—The conditions (16, 17, 18) of conformal division need careful attention. The condition (16) of (arithmetical) divisibility of F by f is obviously necessary to any sort of division. The remaining conditions are needed to the argument (Art. 15) that one pair of the X, Y should be (arithmetically) divisible by f; these will be considerably relaxed in Art. 19–21.

The condition (17) that F should be dimorph when f is monomorph is the most important; it is needed from the fact that, in all the quotient-theorems of Art. 15, the numerator (F) = the product of two conformals (f, ϕ), and must therefore be dimorph (of their form), by Art. 6.

Ex.: $$N = 17 = \frac{561}{33} = \frac{1^2 + 35 \,.\, 4^2}{5 \,.\, 1^2 + 7 \,.\, 2^2} = \frac{F(1, mn)}{f(m, n)}; \quad (m = 5,\ n = 7).$$

Here $F \div f$ is an integer; F, f are conformals (symmorphs), and f is a *monomorph* of form $f(m, n)$; also the t, u of f and the T, U of F are mutual prime-pairs. Thus all the conditions of conformal division are satisfied, except that F is a *monomorph* (instead of dimorph); so that the conformal division of $F \div f$ is impossible, and N cannot be expressed in the form $\phi(5, 7)$.

Note.—Unfortunately the investigation of these conditions (dimorphism of F, *monomorphism* of f) is laborious when F or f respectively is a high number. In that case the best plan in practice is to examine the monomorphism of f only; and (when f is a monomorph) to proceed to the conformal division at once: if one pair of the X, Y turn out to be divisible by f, this shows that the conformal division is possible; whilst, if neither pair of X, Y is divisible by f, this shows that some condition of conformal division is violated. When f is not a monomorph, proceed as in Art. 20.

Exception.—In two cases the dimorphism of F is unnecessary, viz., (1) when f is a unit form of F, and (2) when f, F are automorphs; in both these cases F may be a monomorph (when f is monomorph); this is sufficiently evident from Art. 16, 17.

19. *Extension of Conditions of Conformal Division.*—One of the main conditions of conformal division of Art. 15 was that f should be a *monomorph.* This condition is, however, not essential.

In every case the numerator (F) is—by hypoth.—the product of two conformals ($f.\phi$); hence, when f itself is a polymorph, it follows that, for every distinct (*i.e.*, anautomorph) *conformal* partition of f, there are *two* distinct conformal partitions of F; and thus, if f is capable of r distinct conformal partitions, F has $2r$ distinct conformal partitions, only *two* of which *correspond* to any particular partition of f. Hence, when f is not a monomorph, the proper condition as to the form of F is—excluding the two exceptional cases of Art. 16, 17—

> "The form of the numerator (F) should be one of the two (anautomorphic) conformals *corresponding* to the particular form (f) of the divisor." (25)

Note.—When f is a high number, the *a priori* investigation as to whether this condition is satisfied would be very laborious. The simplest plan in practice would be to attempt the conformal division direct: if successful, this would be sufficient proof of its possibility; whilst, if neither of the X, Y are divisible by f, this would show that some condition of conformal division was violated.

If, however, the quotient N be really expressible* conformally with F, f, then the preceding analysis shows that the numerator must, in general—(omitting the exceptional cases of Art. 16, 17)—be expressible in *two* forms satisfying the above condition, but the analysis does not show how these two *new* forms of F are to be found from the given F, f. The labour involved would probably be prohibitory when F is a high number. This difficulty may, however, be entirely avoided by the process of the next article.

20. *Practical Conformal Division*, (f polymorph).—When f is not a monomorph (of the given form), this indicates that it is a *composite* number (Art. 11).

Let f be resolved into its *prime* factors, say $f_1, f_2, f_3, \dots f_r$; next, let each of these, or as many of them as possible, be expressed† in quadratic forms, conformal with f; these will necessarily be monomorphs. Any, not so expressible, should be combined into products of two or three, &c., at a time, and these products should then be expressed† in forms conformal with f; these will also be monomorphs. Thus finally f will have been resolved into *monomorphic conformal* factors, say $f = f_1'.f_2'.f_3'.\dots f_r'$.

The process of conformal division may now be applied to forming the following conformal quotients in succession:—

$$F_1 = \frac{F}{f_1'},\quad F_2 = \frac{F_1}{f_2'},\quad F_3 = \frac{F_2}{f_3'},\quad \text{\&c.};\quad \text{lastly,}\quad \phi = \frac{F_{r-1}}{f_r'}. \quad (26)$$

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[*] In many cases this is known to be possible *a priori*: *e.g.*, if

$$(T^2+nU^2) \div (t^2+nu^2) = p \text{ (a prime)},$$

then p is known to be of form x^2+ny^2, when $n = 1, \pm 2, \pm 3, -5$, &c.

[†] This should not be laborious, because the factors are supposed to be not large numbers.

The denominators all satisfy the necessary condition; if the conformal division of the *given* F by f be possible, the process *will succeed* at every step; but, if at any step the conformal division fail—*i.e.*, if it turn out that neither pair of the X_{z-1}, Y_{z-1} are (arithmetically) divisible by their denominator f_z—then the conformal division proposed of F by f is *impossible.*

Note.—The process, thus* conducted, is direct and certain—(effecting the conformal division when possible, and showing its impossibility when impossible)—involving no ambiguity beyond that due to the double sign in X, Y, which is removed at each step before proceeding to the next, so that the ambiguity is not cumulative.

21. *Common Factors of* F, f.—The arithmetical operations become of course easier, when the numbers m, n, t, u, T, U to be operated on can be reduced in magnitude. When F, f contain common factors, these can sometimes be cancelled out without destroying the conformality of F, f, and also without violating the conditions of conformal division. This is especially the case when some two or more of m, n, t, u, T, U contain common factors; such factors are easily recognised. The chief cases are:

(1) Factors common to t, u, T, U may be cancelled.

(2) Factors common to m, n may be cancelled in Case i.

(3) m may be cancelled when common to u, U in Case i.; or when common to u, T in Case ii.; or when common to t, U in Case iii.

(4) n may be cancelled when common to t, T in Cases i., ii., iii.

(5) Factors common to T, U, but not contained in t, u may be removed from F temporarily, and the conformal division will take effect upon the numerator so reduced; the factors removed are to be restored to the quotient after the operation.

Note that the cancelling of m, n in (3), (4) changes F, f into their symmorphs.

Exception.—When f is a divisor of both T^2, U^2 (in which case f must be a square) the mere cancelling of f out of both T, U yields at once a conformal quotient, and this is in general the only mode of procedure: if, however, this quotient (say F') be still divisible by f, the conformal division may be performed on the quantity $F' \div f$; and the result when multiplied by f will be a new conformal quotient-form of $F \div f$.

22. *Case when mn is* $-$.—When m, n carry *opposite signs*, the form being then $f(m, -n)$, $f(1, -mn)$, &c., it may happen that the con-

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* An example (in very high numbers) of the process conducted *direct* by Result (25) and another conducted as here described are given at end of Art. 41.

formal division indicated in Results (22*a*, *b*, *c*; 23*a*, *b*, *c*; 24*a*, *b*, *c*) will *fail*, *i.e.*, it may turn out that neither pair of the X, Y of formulæ 19*a*, 20*a*, 21*a* is arithmetically divisible by the denominator (f): this would show—provided that the precautions of Art. 18, 19, 20 have been attended to—that the forms of ϕ in question were *not possible forms.* In such a case the conformal division may still be possible if *either* the numerator (F), *or* denominator (f), can be converted into their *antimorphs*, say F' or f': by this conversion the following formulæ are interchanged (22*a*, 23*a*), (22*b*, 23*b*), (22*c*, 23*c*); (24*a*, 24*b* or *c*), and the quotients are of course the *antimorphs* of the original quotient-forms. The above conversion can always be done if either F or f possess an *antimorphic* unit-form F_0' or f_0' (by the conformal multiplication $F' = F \,.\, F_0'$, $f' = f \,.\, f_0'$, Art. 6), and also in other* cases not readily recognisable *a priori.*

Ex.:

$$N = 137 = \frac{2055}{15} = \frac{F}{f} = \frac{17\,.\,11^2 - 2\,.\,1^2}{17\,.\,1^2 - 2\,.\,1^2}.$$

Here $f = 15$ is a *monomorph*† of the form $(17t^2 - 2u^2)$, so that Result (22*a*) should be *directly* applicable; and yet the conformal division (22*a*) *fails* (for it will be found that neither pair of the X, Y of formula 19*a* is divisible by 15).

But, converting f into its *antimorph* $f = 15 = 2\,.\,4^2 - 17\,.\,1^2$, then

$$N = \frac{F}{f} = \frac{17\,.\,11^2 - 2\,.\,1^2}{2\,.\,4^2 - 17\,.\,1^2}; \text{ which falls under (23}a\text{), with (19}a\text{),}$$

$$= 34 \left(\frac{11\,.\,4 + 1\,.\,1}{15}\right)^2 - \left(\frac{17\,.\,11\,.\,1 + 2\,.\,4\,.\,1}{15}\right)^2$$

$$= 34\,.\,3^2 - 13^2 \quad \text{(the result required).}$$

23. *Conformal Division, Uses of.*—This process is of great use in effecting *directly* the quadratic partition‡ $\phi(m, n)$ of a *large* number N when given—as often happens in higher arithmetic—in the form of a quotient $F \div f$, when the corresponding partition $F(m, n)$ of the numerator is also *given*: in most cases the resolution of the divisor f into factors, and the expression of these in quadratic forms conformal with F, would be comparatively easy.

[Examples, some very large, will be given later (Art. 41).]

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* *E.g.*, when $-mn = 34, 146, 194, 205, 221, 305, 377, 386, 410, 466$, &c.

† Because its prime factors 3, 5 cannot be expressed in the form $(17t^2 - 2u^2)$.

‡ The problem is, in general, one of considerable labour, when the number N is large.

24. *Derivation.*—A method* will now be developed whereby, under certain conditions, from two given non-conformals of the same number N, a third form of that number, non-conformal with both the given forms, may be *derived:* the process will be called *Derivation*, the two given forms will be called *Base-forms*, and the new form will be called the *Derived form;* the three forms together will be called *Allied forms.*

Let N be a number expressed in the two non-conformal base-forms

$$N = \theta v^2 + mw^2 = \theta x^2 + ny^2, \quad (m \neq n). \tag{27}$$

Then N can always be reduced to the form

$$N = \theta \,.\, \frac{F(m, -n)}{f(m, -n)}, \text{ (where } F, f \text{ are isomorphs).} \tag{28}$$

For
$$N = \frac{\theta v^2 \,.\, ny^2 + mw^2 \,.\, ny^2}{ny^2} = \frac{\theta x^2 \,.\, mw^2 + ny^2 \,.\, mw^2}{mw^2}$$

$$= \theta \cdot \frac{m\,(wx)^2 - n\,(yv)^2}{mw^2 - ny^2}, \text{ which is of required form. (28)}$$

25. *Conditions of Derivability.*—When certain relations exist between θ, m, n, the expression $(\theta \,.\, F \div f)$ just found for N may be (algebraically) reduced to a *new* quadratic form which will be found to be conformal with F, f, but non-conformal with both base-forms. These conditions take four principal forms, with two cases in each (corresponding to a ± sign of θ), viz.,

$$\text{i. } \theta = 1; \quad \text{i}a.\ \theta = -1;$$

$$\begin{cases} \text{ii. } \theta v_0^2 + \mu w_0^2 = 1, & \textit{and} \quad \theta x_0^2 + ny_0^2 = 1; \\ \text{ii}a.\ \theta v_0^2 + \mu w_0^2 = -1, & \textit{and} \quad \theta x_0^2 + ny_0^2 = -1; \end{cases}$$

$$\text{iii. } \theta\sigma^2 = m\tau^2 - n\upsilon^2; \quad \text{iii}a.\ \theta\sigma^2 = n\upsilon^2 - m\tau^2;$$

$$\text{iv. } \theta\sigma^2 = \tau^2 - mn\upsilon^2; \quad \text{iv}a.\ \theta\sigma^2 = mn\upsilon^2 - \tau^2.$$

As the process involves a conformal division, the conditions of such division (Art. 18) must also be satisfied; the satisfaction of these conditions seems difficult to recognise *a priori*, but in any actual numerical case this will be sufficiently revealed—as explained in Art. 18–20—by the success or failure of the attempt at the conformal division itself. Moreover, as the result of conformal division is *unique* (Art. 15), only *one* new form can be "derived" from a given pair of base-forms.

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* This method—believed to be new—is the object of this paper; the methods of multiplication and division of a form f by any of its unit-forms, above developed, provides the means of finding the *automorphs* thereof, but not of finding *new* or *distinct* forms.

26. *Quadratic Triads.*—In certain cases the *allied forms* (Art. 24) are such that each one of the three can be similarly *derived* from the other two considered as base-forms. Such "allied forms" are called a *Quadratic Triad.* It will be found that the allied forms of Cases i., ii., iii., i*a.*, ii*a.*, iii*a.* are always quadratic triads; but that those of Cases iv., iv*a.* are a quadratic triad only under a certain further condition.

27. *Derivation.*—Cases i., i*a.* ($\theta = 1$, or -1; $m \neq n$).

In i., $$N = v^2 + mw^2 = x^2 + ny^2. \qquad (29)$$

In i*a.*, $$N = mw^2 - v^2 = ny^2 - x^2. \qquad (29a)$$

Here (28) reduces to

$$\text{i. } N = \frac{m(wx)^2 - n(yv)^2}{mw^2 - ny^2}; \quad \text{i}a.\ N = \frac{n(yv)^2 - m(wx)^2}{mw^2 - ny^2}, \qquad (30)$$

and, by* conformal division (when possible), these yield

either $$\text{i. } N = t^2 - mnu^2; \quad \text{i}a.\ N = mnu^2 - t^2; \qquad (31)$$

or (Art. 22), $$\text{i}'.\ N = mnu^2 - t^2; \quad \text{i}a'.\ N = t^2 - mnu^2. \qquad (31')$$

Thus, from any two single non-conformal base-forms (29, 29*a*) of the same number N, the third form (31) is *directly* derivable when it is a *possible form;* or the alternative form (31′) is *indirectly* derivable when the first form (31) is impossible (see Art. 22).

Further, the derived form (31), but not the alternative (31′), is identifiable with $(t^2 + lu^2)$ in Case i., by taking $l = -mn$; or with $(lu^2 - t^2)$ in Case i*a.*, by taking $l = mn$. The symmetry of the three forms shows that, if N be actually expressible in the three simple non-conformals

$$\text{i. } N = t^2 + lu^2 = v^2 + mw^2 = x^2 + ny^2, \qquad (32)$$

or $$\text{i}a.\ N = lu^2 - t^2 = mw^2 - v^2 = ny^2 - x^2, \qquad (32a)$$

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* If m, n contain a common factor, say $m = \kappa\mu$, $n = \kappa\nu$; then the product mn may in all such formulæ be replaced (Art. 5, Note) by $\mu\nu$, the square (κ^2) being absorbed into u^2.

where l, m, n are connected* by *any one* of the relations

$$\text{i. } l = -mn, \quad m = -nl, \quad n = -lm, \tag{33}$$

$$\text{i}a.\ l = mn, \quad m = nl, \quad n = lm, \tag{33a}$$

then each one of the three forms is *directly derivable* from the other two (as base-forms), so that the three forms (32) or (32a) are a *quadratic triad.*

[But when the two base-forms (29) or (29a) of N lead to the *alternative* derived form i.′ or ia.′ of (31′)—(which can only happen when the forms i., ia. of (31) are impossible)—then the three forms are merely allied forms, but *not quadratic triads;* as neither of the forms (29), (29a) would be *derivable* from the other pair taken as base-forms.]

Ex.—One of the most useful examples of Case i. is when $n = -m$, giving—when $(x^2 - my^2)$ is a possible form of N—the *quadratic triad*

$$N = \mathrm{a}^2 + \mathrm{b}^2 = v^2 + mw^2 = x^2 - my^2. \tag{34}$$

This includes all primes of form $p = 4m\varpi + 1$,

when $m > 1$, but < 11 (and also many cases when $m > 11$).

When $(x^2 - my^2)$ is not a possible form of N, then the two forms

$$\mathrm{a}^2 + \mathrm{b}^2 = v^2 + mw^2 \tag{34a}$$

would lead *indirectly* (Art. 22) to $(my^2 - x^2)$ as derived form, provided m be of form $m = \alpha^2 + \beta^2$ (a necessary condition of the coexistence of the three forms), and they then form a quadratic triad; this form is discussed in the Example at end of Cases iii., iii$a$., Art. 30.

28. *Cases* ii., ii$a$.

$$N = \theta v^2 + mw^2 = \theta x^2 + ny^2 \quad (\theta \neq 1;\ m \neq n), \tag{35}$$

where each of the base-forms admits of a unit-form (both $+1$, or both -1), *i.e.*,

$$\textit{Case}\ \text{ii.}\quad \theta v_0^2 + mw_0^2 = 1, \quad \text{and} \quad \theta x_0^2 + ny_0^2 = 1, \tag{36}$$

$$\textit{Case}\ \text{ii}a.\quad \theta v_0^2 + mw_0^2 = -1, \quad \text{and} \quad \theta x_0^2 + ny_0^2 = -1 \tag{36a}$$

(both involving "m, n of same sign, opposite to that of θ").

By conformal multiplication of each base-form by its own unit-form, each is reduced to a *simple* form (its own symmorph), so that ii., ii*a.* become

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* See previous note. Square factors in mn, nl, lm may be cancelled from the values of l, m, n respectively, and absorbed into u^2, w^2, y^2 respectively.

ii. $N = V^2-(-\theta m)\,W^2 = X^2-(-\theta n)\,Y^2$ $(-\theta m,\ -\theta n$ are both $+)$; (37)

ii*a.* $N = (-\theta m)\,W^2 - V^2 = (-\theta n)\,Y^2 - X^2$ $(-\theta m,\ -\theta n$ are both $+)$, (37*a*)

which now fall under Cases i., i*a.*; so that a third form, non-conformal with both base-forms (as below), is directly derivable in each case (when conformal division is possible); θ^2 being here absorbed under u^2,

either ii. $N = t^2 - mnu^2$; ii*a.* $N = mnu^2 - t^2$, (38)

or* (Art. 22) ii′. $N = mnu^2 - t^2$; ii*a*′. $N = t^2 - mnu^2$. (38′)

Hence, reasoning as in Cases i., i*a.* [noting that the base-forms are considered *equivalent* to their automorphs (Art. 10)], it follows that, if N be expressible in the three non-conformals below (subject to the conditions accompanying), then each set is a† *quadratic triad.*

ii. $N = t^2 + lu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (39)

where *one* of $l = -mn,\ m = -nl,\ n = -lm$, is satisfied, (40)

and *each* form admits of a unit-form $(= +1)$. (41)

ii*a.* $N = lu^2 - t^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (39*a*)

where *one* of $l = mn,\ m = nl,\ n = lm$, is satisfied, (40*a*)

and *each* form admits of a unit-form $(= -1)$. (40*a*)

[Note that the first form $(lu^2 \pm t^2)$ of each triad necessarily admits of the unit-form $\{= +1$ in (39), and $= -1$ in (39*a*)$\}$, when *one* of the conditions $l = \mp mn$, &c. is satisfied, since $mn,\ -\theta m,\ -\theta n$ are all $+$.]

29. *Cases* iii., iii*a.*, iv., iv*a.* $(\theta \neq 1,\ m \neq n.)$

Here $N = \theta v^2 + mw^2 = \theta x^2 + ny^2$, (41) subject to *one* of the conditions

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* As in Case i., conformal division leads *directly* to the forms (38) when these are *possible*, and only leads *indirectly* to the *alternative* forms (38′) when the former are impossible.

† The alternative forms (38′) are excluded from (39), (39*a*), as they would not form a *quadratic triad* with (35).

$$\text{Case iii. } \theta\sigma^2 = m\tau^2 - n\upsilon^2; \qquad \text{iii}a.\ \theta\sigma^2 = n\upsilon^2 - m\tau^2, \qquad (42, 42a)$$

$$\text{Case iv. } \theta\sigma^2 = \tau^2 - mn\upsilon^2; \qquad \text{iv}a.\ \theta\sigma^2 = mn\upsilon^2 - \tau^2. \qquad (43, 43a)$$

Now, in each case, the reduced form (28) of N may be written

$$N = (\theta\sigma^2)\frac{m(wx)^2 - n(yv)^2}{m(\sigma w)^2 - n(\sigma y)^2} = \Theta . \frac{F(m, n)}{f(m, n)}, \qquad (44)$$

wherein f, F, and $\Theta = \theta\sigma^2$ are *isomorphs* by virtue of the conditions (42, 42a, 43, 43a).

By conformal multiplication of Θ, F, the numerator can be in each case reduced to a conformal of f, F, viz.,

$$\text{iii. } \Theta . F = (m\tau^2 - n\upsilon^2)\{m(wx)^2 - n(yv)^2\} = T^2 - mnU^2,$$

$$\text{iii}a.\ \Theta . F = (n\upsilon^2 - m\tau^2)\{m(wx)^2 - n(yv)^2\} = mnU^2 - T^2,$$

$$\text{iv. } \Theta . F = (\tau^2 - mn\upsilon^2)\{m(wx)^2 - n(yv)^2\} = mT^2 - nU^2,$$

$$\text{iv}a.\ \Theta . F = (mn\upsilon^2 - \tau^2)\{m(wx)^2 - n(yv)^2\} = nU^2 - mT^2.$$

Next, by conformal division of the reduced ΘF by f (when the conditions of such division are satisfied), the quotient-form is obtained conformal with f, F; thus

$$\text{iii. } N = \frac{\Theta . F}{f} = \frac{T^2 - mnU^2}{m(\sigma w)^2 - n(\sigma y)^2} = mt^2 - nu^2, \qquad (45)$$

$$\text{or} \qquad = nu^2 - mt^2 \text{ (Art. 22)}, \quad (45')$$

$$\text{iii}a.\ N = \frac{\Theta . F}{f} = \frac{mnU^2 - T^2}{m(\sigma w)^2 - n(\sigma y)^2} = nu^2 - mt^2, \qquad (45a)$$

$$\text{or} \qquad = mt^2 - nu^2 \text{ (Art. 22)}, \quad (45a')$$

$$\text{iv. } N = \frac{\Theta . F}{f} = \frac{mT^2 - nU^2}{m(\sigma w)^2 - n(\sigma y)^2} = t^2 - mnu^2, \qquad (46)$$

$$\text{or} \qquad = mnu^2 - t^2 \text{ (Art. 22)}, \quad (46a')$$

$$\text{iv}a.\ N = \frac{\Theta . F}{f} = \frac{nU^2 - mT^2}{m(\sigma w)^2 - n(\sigma y)^2} = mnu^2 - t^2, \qquad (46a)$$

$$\text{or} \qquad = t^2 - mnu^2 \text{ (Art. 22)}. \quad (46a')$$

Thus, in each case, a third* form is derivable from the two base-forms, non-conformal with both.

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* By Art. 22, the first form (45, 45a, 46, 46a) of the above pairs of *alternative* forms is the correct form, whenever it is a possible form; the second form (15′, 45a′, 46′, 46a′) being used only when the former is an impossible form.

30. *Cases* iii., iii*a.*; *Quadratic Triads.*—It will now be shown that the two forms (41) along with their derived form (45) or (45*a*) [not (45′) or (45′*a*)] are in each case a *quadratic triad.* Taking Case iii. first:

Case iii.—It has been shown that, when N is *actually expressible* in the three *allied forms*

$$\left.\begin{array}{ll} & N = mt^2 - nu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2 \quad (\theta \neq 1,\ m \neq n); \\ \text{or} & N = f_1(m, -n) = f_2(\theta, m) = f_3(\theta, n), \end{array}\right\} \quad (47)$$

then $f_1(m, -n)$ is *directly derivable* from $f_2(\theta, m)$, $f_3(\theta, n)$ provided

$$\theta\sigma^2 = m\tau^2 - nv^2, \quad \text{or} \quad m\tau^2 = \theta\sigma^2 - (-n)\tau^2, \qquad (48)$$

and this latter is seen to be the necessary and sufficient condition that $f_3(\theta, n)$ should be *directly derivable* from $f_1(m, -n)$, $f_2(\theta, m)$.

The derivability of the form $f_2(\theta, m)$ from $f_1(m, -n)$, $f_3(\theta, n)$ is not, however, so obvious. The two forms

$$N = mt^2 - nu^2 = \theta x^2 + my^2,$$

combined as in Art. 24, give

$$N = \frac{\theta(ux)^2 + m(yt)^2}{u^2 + y^2} = \frac{F}{f} \text{ (suppose).}$$

Here $$f = u^2 + y^2 = \frac{1}{n}(mt^2 - \theta x^2), \text{ by the data,}$$

$$= \frac{v^2}{m\tau^2 - \theta\sigma^2}(mt^2 - \theta x^2), \text{ by (48),}$$

$$= t_1^2 - \theta m \,.\, u_1^2, \quad \text{(by conformal division)}$$

$$= t_2^2 + \theta m \,.\, u_2^2, \text{ (by Case i.),}$$

because the two forms $f = f(1, 1) = f(1, -\theta m)$ involve $f = f(1, \theta m)$, by Case i. Hence

$$N = \frac{\theta(ux)^2 + m(yt)^2}{t_2^2 + \theta m \,.\, u_2^2}$$

$$= \theta v^2 + mw^2 = f_2(\theta, m), \quad \text{(by conformal division).}$$

Thus the three forms (47) of Case iii., with the relation (48), are a quadratic triad.

Case iii*a*.—Interchanging m, n of Case iii., a similar procedure shows that, when N is actually expressible in the three allied forms

$$N = nu^2 - mt^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2, \qquad (47a)$$

subject to the condition $\theta\sigma^2 = nv^2 - m\tau^2, \qquad (48a)$

then the three forms (47*a*) are a *quadratic triad.*

[*Note.*—The preliminary hypothesis that N is *actually expressible* in the three forms (47, 47*a*) involves the *possibility* of all the conformal divisions required above, and also excludes the chance of any of the conformal quotients taking the *antimorphic* form of Art. 22.]

Ex.—One of the most useful examples of above is when $m = \pm 1 = -n$. Cases iii., iii*a*. coalesce, so that, when N is *actually expressible* in the three forms

$$N = \mathrm{a}^2 + \mathrm{b}^2 = \theta v^2 + w^2 = \theta x^2 - y^2, \qquad (49)$$

the coexistence of which involves

$$\theta = \tau^2 + v^2, \qquad (49a)$$

then the three forms (49) are *a quadratic triad.* [Compare (34*a*), Art. 26.]

Ex. 2.—When $m + n = 0$, the systems (47), (47*a*) are each divisible throughout by m (because the first form is): this case is then reducible to the last (49, 49*a*).

31. *Cases* iv., iv*a*. *may be quadratic triads.*—Let N be a number *actually expressible* in the three allied forms

$$\textit{Case iv.}\quad N = t^2 - mnu^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2, \qquad (50)$$

wherein $\theta\sigma^2 = \tau^2 - mnv^2, \qquad (51)$

or $\textit{Case iv}a.\quad N = mnu^2 - t^2 = \theta v^2 + mw^2 = \theta x^2 + ny^2, \qquad (50a)$

wherein $\theta\sigma^2 = mnv^2 - \tau^2; \qquad (51a)$

then it has been shown (Art. 29) that in each case the first form $\{\pm(t^2 - mnu^2)\}$ is *directly derivable* from the other two, as base-forms.

Now, if the first form in each case admits *either* of the symmorphic unit-forms

$$m\tau_0^2 - nv_0^2 = 1, \quad \text{or} \quad nv_0^2 - m\tau_0^2 = 1, \qquad (52)$$

then Cases iv., iv*a*., reduce to Cases iii., iii*a*. respectively. For N and $\theta\sigma^2$ are thereby changed as follows:—

iv. $m\tau_0^2 - nv_0^2 = 1$ changes $N = t^2 - mnu^2$ to $(mt'^2 - nu'^2)$,

and $\theta\sigma^2 = \tau^2 - mnv^2$ to $(m\tau'^2 - nv'^2)$;

iv*a*. $m\tau_0^2 - nv_0^2 = 1$ changes $N = mnu^2 - t^2$ to $(nu'^2 - mt'^2)$,

and $\theta\sigma^2 = mnv^2 - \tau^2$ to $(nv'^2 - m\tau'^2)$;

iv. $n\upsilon_0^2 - m\tau_0^2 = 1$ changes $N = t^2 - mnu^2$ to $(nu'^2 - mt'^2)$,

and $\theta\sigma^2 = \tau^2 - mn\upsilon^2$ to $(n\upsilon'^2 - m\tau'^2)$;

iv*a*. $n\upsilon_0^2 - m\tau_0^2 = 1$ changes $N = mnu^2 - t^2$ to $(mt'^2 - nu'^2)$,

and $\theta\sigma^2 = mn\upsilon^2 - \tau^2$ to $(m\tau'^2 - n\upsilon'^2)$.

Thus $m\tau_0^2 - n\upsilon_0^2 = 1$ changes Case iv. into iii., and iv*a*. into iii*a*.,

$n\upsilon_0^2 - m\tau_0^2 = 1$ changes Case iv. into iii*a*., and iv*a*. into iii.

Hence, with the added condition $(m\tau_0^2 - n\upsilon_0^2) = \pm 1$, the three allied forms (50, 50*a*) of Cases iv., iv*a*. are *quadratic triads*, but *not otherwise*.

Ex.—An interesting example of Case iv. is when $\sigma^2 = 1$, $n = -m$, giving

$$N = t^2 + (mu)^2 = \theta v^2 + mw^2 = \theta x^2 - my^2; \quad \text{and } \theta = \tau^2 + (m\upsilon)^2. \tag{53}$$

But these allied forms do not form a quadratic triad, except when $m = \pm 1$, as this is the only way in which the condition $m\tau_0^2 - n\upsilon_0^2 = \pm 1$ can be satisfied (viz., by $m = -n = \pm 1$, $\tau_0 = 1$, $\upsilon_0 = 0$), giving the quadratic triad (49) of Case iii.

32. *Conditions of Derivability.*—The four Cases (i., ii., iii., iv.) investigated are not the only cases in which a new form is derivable from two given base-forms of the same number (N). For, writing the reduced expression (28) to which these lead,

$$N = \theta \,.\, \frac{m\,(wx)^2 - n\,(yv)^2}{mw^2 - ny^2} = \theta\,\frac{F}{f}, \tag{54}$$

then—since F, f are conformal—it follows that, if F contain f as a factor, and if the conditions of conformal division are also satisfied, the quotient $F \div f$ is itself expressible as a conformal, say F', of F, f; excluding now the case of θ being conformal with F', and therefore with F, f, as being really a particular case of Case iii. (making $\sigma = 1$), there remains the case of the product (N) of the two non-conformals (θ, F') being conformally expressible with F' (*i.e.*, with F, f); this case is *sometimes* (arithmetically) possible, (Art. 6); but it seems difficult to formulate it algebraically, and it is doubtful whether the derived form in question would, along with the two base-forms, yield a quadratic triad.

33. *Practical Procedure.*—The procedure in Cases iii., iii*a*., iv., iv*a*. merits further examination. After reducing N to the form $\Theta f \div f$, it consists of two steps:—I., conformal multiplication of Θ, F; II., conformal division of $(\Theta \,.\, F)$ by f. This procedure is general, and certain of success when θ has the requisite form, and when the conformal division required is possible.

For practical computation, however, it is often much more convenient to invert the order of the above steps, performing the conformal divisions first (whenever possible), and following with the conformal multiplication, as by this procedure much lower numbers have to be dealt with than by the previous order. This may be done in the following cases (whenever the conformal divisions are possible):—

$$(1)\ f = \Theta f', \quad (2)\ F = fF', \quad (3)\ f = f_1 f_2, \quad \Theta = \Theta' f_1, \quad F = f_2 F'. \tag{55}$$

The order and nature of the operations are then as follows:—

Step.	Conditions.		
	(1) $f = \Theta f'$	(2) $F = fF'$	(3) $f = f_1 f_2$, $\Theta = \Theta' f_1$, $F = f_2 F'$
I. Conform. Divn. ...	$f \div \Theta = f'$	$F \div f = F'$	$\Theta \div f_1 = \Theta'$, $F \div f_2 = F'$
II. Conform. Multn. ...	... ...	$N = \Theta F'$	$N = \Theta' F'$
II. Conform. Divn. ...	$N = F \div f'$	... ...	... ...

The simplest case is, of course, when $\Theta = f$, for then N obviously reduces to

$$N = F = m\,(wx)^2 - n\,(yv)^2.$$

Note.—The condition $f = \Theta f'$ of Case (1) above is *always satisfied*, for the data [(27) Art. 24] give

$$mw^2 - ny^2 = \theta\,(x^2 - v^2),$$

whence

$$f = \Theta\,(x^2 - v^2);$$

but, as the conformal division may in this case be impossible, it is often convenient to use the other cases.

The denominator f being thus always composite (often highly composite), the procedure of Art. 20, viz., dissecting f into its monomorphic factors, should be adopted; and, to shorten the numerical work, it is important to strike out the obvious common factors of ΘF, f as explained in Art. 21: one case is especially simple, viz., when either of the pairs (m, n), (w, y) have common factors; thus, if

$$\frac{m}{n} = \frac{\mu}{\nu}; \quad \frac{w}{y} = \frac{\omega}{\eta}, \tag{56}$$

then the expression (28) for N is at once reduced to

$$N = \theta \,.\, \frac{\mu\,(\omega x)^2 - \nu\,(\eta v)^2}{\mu\omega^2 - \nu\eta^2}, \tag{57}$$

which has the practical advantage of containing lower numbers than (28).

34. *Repeated Derivation.*—The process of derivation, above applied to *two* non-conformals, may be extended to *three* or more non-conformals, in Cases i., i*a*., ii., ii*a*., say,

$$N = \theta x_1^2 + l_1 y_1^2 = \theta x_2^2 + l_2 y_2^2 = \theta x_3^2 + l_3 y_3^2 = \&c., \tag{58}$$

wherein θ is the same throughout, and $l_1 \neq l_2 \neq l_3 \neq \&c.$, (58*a*)

when the conformal divisions requisite are all possible. It will suffice to show this for *three* forms of Case i.

Case i. ($\theta = 1$).—Let N be expressed in the three non-conformals

$$N = x_1^2 + l_1 y_1^2 = x_2^2 + l_2 y_2^2 = x_3^2 + l_3 y_3^2. \tag{59}$$

By combining the second and third, the third and first, and the first and second forms in succession, the three new derived forms are obtained, when the conformal divisions are possible (excluding the alternative case of Art. 22),

$$N = x_{23}^2 - l_2 l_3 y_{23}^2 = x_{31}^2 - l_3 l_1 y_{31}^2 = x_{12}^2 - l_1 l_2 y_{12}^2, \tag{59a}$$

and, by combining any one of these with the non-corresponding original form, a new derived form results,

$$N = x_{123}^2 - l_1 l_2 l_3 y_{123}^2, \quad \text{or} \quad l_1 l_2 l_3 y_{123}^2 - x_{123}^2. \tag{59b}$$

Case i*a*. ($\theta = -1$).—This follows by merely slight alterations.

Cases ii., ii*a*.—Also follow by reduction to Cases i., i*a*., as in Art. 28.

Cases iii., iii*a*., iv., iv*a*.—Do not appear to be susceptible of this extension, without some additional conditions.

35. *Generalization.*—The process of derivation may be extended* to the more general quadratic forms

$$N = kv^2 + avw + bw^2 = kx^2 + cxy + dy^2, \tag{60}$$

wherein *one* of the coefficients (k) is common to both forms. Here

$$4kN = (2kv + aw)^2 + (4kb - a^2)\, w^2 = (2kx + cy)^2 + (4kd - c^2)\, y^2$$
$$= v'^2 + mw^2 = x'^2 + ny^2, \tag{60a}$$

by obvious substitutions. Hence, provided $m \neq n$ (so that these forms may be non-conformal), these yield (when the conformal division required is possible) the new form of $4kN$,

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* This extension is due to Mr. Bickmore.

$$4kN = t^2 - mn\,.\,u^2, \quad \text{or} \quad mn\,.\,u^2 - t^2, \tag{60b}$$

and, when $4k$ is conformal with $f(1, -mn)$, N itself is expressible [by conformal division of (60*b*) by $4k$] in the three forms (60*a*, *b*), which then form a quadratic triad.

36. *Numerical Examples.*—The power of the process of derivation is best seen with *very large* numbers; as with small numbers the partitions can often be more easily effected by direct trial. But it is well to begin with small numbers.

37. *Ex.*—The small prime $p = 241$ affords, from its numerous partitions (shown below), many easy examples; the antimorphs of the second and third lines are connected by the (negative) unit form shown in the fifth line

$$\begin{array}{rcccccccccc}
p = 15^2+4^2 & = & 13^2+2.6^2 & = & 7^2+3.8^2 & = & 5^2+6.6^2 & = & 14^2+5.3^2 & = & 9^2+10.4^2 \\
 & = & 21^2-2.10^2 & = & 17^2-3.4^2 & = & 25^2-6.8^2 & = & 31^2-5.12^2 & = & 41^2-10.12^2 \\
 & = & 2.11^2-1^2 & = & \dots\ \dots & = & \dots\ \dots & = & 5.7^2-2^2 & = & 10.5^2-3^2 \\
 & = & \dots\ \dots & = & \dots\ \dots & = & 3.9^2-2.1^2 & = & \dots\ \dots & = & \dots\ \dots \\
\left.\begin{array}{l}\text{Unit-}\\ \text{form}\end{array}\right\}, -1 & = & 1^2-2.1^2 & = & \dots\ \dots & = & \dots\ \dots & = & 2^2-5.1^2 & = & 3^2-10.1^2 \\
 & & & & & & & = & 1^2+15.4^2 & = & 11^2+30.2^2 \\
 & & & & & & & = & 16^2-15.1^2 & = & 19^2-30.2^2 \\
 & & & & & & & = & \dots\ \dots & = & \dots\ \dots \\
 & & & & & & & = & \dots\ \dots & = & 6.9^2-5.7^2.
\end{array}$$

Thus there are seven quadratic triads of the *simplest* form (34) of Case i.,

$$p = \mathrm{a}^2 + \mathrm{b}^2 = v^2 + mw^2 = x^2 - my^2,$$

given by $m = -n = 2, 3, 6, 5, 10, 15, 30$.

Of Case i. $(\theta = 1)$, and also of Case i*a*. $(\theta = -1)$, there are twenty-eight quadratic triads (Art. 27) given by the four interchanges of sign $(\pm m, \pm n)$ in the scheme

$$\begin{array}{rl}
\pm m & = 2,\ \ 2,\ \ 2,\ \ 3,\ \ 3,\ \ 6,\ \ 6, \\
\pm n & = 3,\ \ 5,\ 15,\ \ 5,\ 10,\ \ 5,\ 10, \\
l = \pm mn \text{ or } \pm\mu\nu & = 6,\ 10,\ 30,\ 15,\ 30,\ 30,\ 15.
\end{array}$$

Of Case ii. there are two quadratic triads given by

$$(\theta, m, n) = (-2, 1, 3) \quad \text{and} \quad (-5, 1, 6).$$

Of Cases iii., iv. there are four quadratic triads, given by

$$(\theta, m, n) = (3, 1, -2),\ (-2, 1, 3),\ (6, 1, -5),\ (-5, 1, 6),$$

these all falling under *both* Cases iii., iv., because $m = 1$ in each instance.

There are no instances of Cases ii*a*., iii*a*., iv*a*. in the above scheme (the conditions not being satisfied).

38. *Case* i.—Any two non-conformal simple forms of the small prime $p = 241$ will serve to prove the actual *working* of the process, *e.g.*,

$$p = 5^2+6 \,.\, 6^2 = 14^2+5 \,.\, 3^2\,;$$

here $\theta = 1$, $m = 6$, $n = 5$. Then, by (30),

$$p = \frac{6\,(6 \,.\, 14)^2-5\,(5 \,.\, 3)^2}{6 \,.\, 6^2-5 \,.\, 3^2} = \frac{F}{f}.$$

Here the factor 9 may be cancelled out of both F, f; leaving the reduced divisor $f \div 9 = (6 \,.\, 2^2-5 \,.\, 1^2) = 19$, a prime, and therefore a *monomorph* (see Art. 11, 21).

Hence $$p = \frac{6 \,.\, 28^2-5 \,.\, 5^2}{6 \,.\, 2^2-5 \,.\, 1^2}\,;$$

whence, by conformal division,

$$p = \left(\frac{6 \,.\, 2 \,.\, 28+5 \,.\, 5}{19}\right)^2-30\left(\frac{28+10}{19}\right)^2$$
$$= 19^2-30 \,.\, 2^2,$$

the required *derived* form.

[To illustrate the *working* of the cases when $\theta \neq \pm 1$, it is better to take larger numbers, giving $m \neq \pm 1$, $n \neq \pm 1$.]

39. *Case* ii.—Given $N = 42{,}641$ in the three allied forms

$$N = 2 \,.\, 542^2-7 \,.\, 279^2 = 11 \,.\, 277^2-2 \,.\, 633^2 = 11 \,.\, 384^2-7 \,.\, 475^2,$$

with the corresponding unit-forms (determining the derivability)

$$1 = 2 \,.\, 2^2-7 \,.\, 1^2 = 11 \,.\, 3^2-2 \,.\, 7^2 = 11 \,.\, 4^2-7 \,.\, 5^2.$$

By conformal multiplication of each form of N by its own unit-form, there follow

$$N = 215^2-14 \,.\, 16^2 = 279^2-22 \,.\, 40^2 = 271^2-77 \,.\, 20^2,$$

which now fall under Case i. Thus either set by itself, or any one form of either set, together with the non-conformal pair of the other set, is a *quadratic triad*.

40. *Cases* iii., iii *a*.—Take $N = 3317$.

$$\left.\begin{aligned} N &= 5 \,.\, 91^2-2 \,.\, 138^2 \\ &= 2 \,.\, 41^2-5 \,.\, 3^2 \end{aligned}\right\} = 13 \,.\, 12^2+5 \,.\, 17^2 = 13 \,.\, 15^2+2 \,.\, 14^2 \left\{\begin{array}{l} \text{Case iii.} \\ \text{Case iii}\,a. \end{array}\right.$$

Here $\theta = 13$, $m = 5$, $n = 2$; and taking $\sigma = 1$, the condition of derivability of both Cases iii., iii *a*. is fulfilled, viz.,

$$\theta\sigma^2 = 13 \,.\, 1^2 = 5 \,.\, 3^2-2 \,.\, 4^2 = 2 \,.\, 3^2-5 \,.\, 1^2,$$

the two cases being connected by the unit-forms

$$19^2-10 \,.\, 6^2 = 1 = 10 \,.\, 1^2-3^2,$$

by which the two leading forms are seen to be antimorphs, and the conditions themselves (48), (48*a*) are also antimorphic. Hence the two sets of "allied forms" are *quadratic triads*.

To show the *working* of the process of derivation, let N be given in the second and third forms, to derive the first form

$$N = 13 \,.\, 12^2+5 \,.\, 17^2 = 13 \,.\, 15^2+2 \,.\, 14^2,$$

giving $\theta = 13, \quad m = 5, \quad n = 2,$

with the condition $\theta\sigma^2 = 13 \,.\, 1^2 = 5 \,.\, 3^2 - 2 \,.\, 4^2.$

Here, by (28), $$N = 13 \,.\, \frac{5\,(17 \,.\, 15)^2 - 2\,(12 \,.\, 14)^2}{5 \,.\, 17^2 - 2 \,.\, 14^2} = \theta \,.\, \frac{F}{f}.$$

Here $$f = 1053 = 13 \,.\, 81,$$

so that θ divides out of f; whilst 9, being a factor of f and of *both* members of F, may be cancelled out of both F, f: the reduced denominator

$$f \div 13 \,.\, 9 = 9 = (7^2 - 10 \,.\, 2^2),$$

which is a *monomorph* (symmorphic with the reduced numerator, $F \div 9$). This reduces N to $$N = \frac{5 \,.\, 85^2 - 2 \,.\, 56^2}{7^2 - 10 \,.\, 2^2};$$

whence, by conformal division,

$$N = 5\left(\frac{7 \,.\, 85 + 2 \,.\, 2 \,.\, 56}{9}\right)^2 - 2\left(\frac{5 \,.\, 2 \,.\, 85 + 7 \,.\, 56}{9}\right)^2$$

$$= 5 \,.\, 91^2 - 2 \,.\, 138^2,$$

as the *derived* form.

41. *Partition of Large Primes.*—A frequently recurring necessity in the theory of residues is the partition of a *large* prime (p) into the simple form $p = t^2 + lu^2$, where l is some (given) *small* number ($l = +1, \pm 2, \pm 3$, &c.). Unless one of t, u be a *small* number (in which case the partition—by direct trial—is easy), this is often a laborious work. The properties of quadratic triads afford considerable help in this: for, if *any* two (non-conformal) partitions

$$p = v^2 + mw^2 = x^2 + ny^2 \tag{61}$$

can be discovered, forming a *quadratic triad* with the required form ($t^2 + lu^2$), the latter is at once* *derivable.*

Base-Forms.—Suitable base-forms (61) may often be discovered by the following tentative process. Subtract from the given p a number of† squares (say v^2, x^2) each $< p$; also subtract the given p from a number of† squares (say v^2, x^2) each $> p$. Resolve‡ the differences ($p \sim v^2$), ($p \sim x^2$) into their factors: some of these differences will be found to contain square factors, *i.e.*, will be of form mw^2, ny^2; these will give a *number* of partitions (61) with various values of m, n. If *any* pair (m, n) of these are such that *one* of the conditions

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* It is assumed that the required form ($t^2 + lu^2$) is a *possible* form of p (this would be known *a priori*): this being so appears to involve the possibility of the conformal division required (when conducted properly; see Art. 20).

† A *large* Table of squares is required for this part of the process: the extent of the Table available is a *practical* limit to the utility of the process. Barlow's Tables extend only to $10{,}000^2$: Ludolf's and Kulik's Tables both extend to $100{,}000^2$, but are both out of print, and are rare books now.

‡ The trial squares (v^2, x^2) should be so chosen that these differences should be resolvable into factors *at sight* from a factor-table (otherwise the labour of factorization is considerable). Barlow's *New Mathematical Tables* (1814) are very suitable, as they give the factors of *all* numbers < 10,000: they are unfortunately out of print.

$$l = -mn, \quad m = -nl, \quad n = -lm \tag{62}$$

is satisfied, then that pair will be suitable base-forms.

The advantage of this is that in that part of it which is *tentative* any two forms (61) which will satisfy *any one* of the conditions (62) will suffice; so that there is a good chance of a suitable pair being found with less labour than would be requisite for *directly* effecting the partition (t^2+lu^2). It sometimes happens that one or more forms of a given prime are determinable *algebraically*, and that in the course of the above tentative work a new form may be found suitable for use with it as a base-form.

Ex.—Let $p = 10{,}567{,}201$ (the large factor of $2^{75}-1$). Find the partitions

$$p = \mathrm{a}^2+\mathrm{b}^2 = \mathrm{c}^2+2\mathrm{d}^2.$$

As the last two figures (01) of p are the ending of the squares of four classes of numbers only, viz., of such as end in 01, 49, 51, 99, the trial values of v, x are conveniently assumed of these forms, because the differences $(p \sim v^2)$, $(p \sim x^2)$ will then all end in 00, and will be therefore more easily factorized. Starting from the critical value $\sqrt{p} = 4516$ nearly, and trying all possible values of v, x of above forms, both $< \sqrt{p}$ and $> \sqrt{p}$, forming the differences $(p \sim v^2)$, $(p \sim x^2)$, and factorizing these differences, there result (after a good many trials) the two suitable forms (for the a, b partition)

$$p = 2351^2+14\,.\,600^2 =^{*} 76{,}399^2-14\,.\,20400^2,$$

which fall under (34), Case i. Hence, by (28),

$$p = \frac{(76{,}399\,.\,600)^2+(2351\,.\,20400)^2}{600^2+20400^2} = \frac{76{,}399^2+(2351\,.\,34)^2}{1^2+34^2} = \frac{F}{f}$$

$$= \left(\frac{76{,}399\,.\,1+2351\,.\,34\,.\,34}{1157}\right)^2+\left(\frac{76399\,.\,34-2351\,.\,34}{1157}\right)^2, \text{ (by 19)}$$

$$= 2415^2+(34\,.\,64)^2 = \mathrm{a}^2+\mathrm{b}^2 \text{ (as required)}.$$

[*Note.*—The divisor $f = 1157 = 13\,.\,89 = 1^2+34^2 = 31^2+14^2$ is *not a monomorph*. The success of the conformal division in this instance is due to the fact that the forms of F, f here used chanced to be *corresponding* forms (Art. 19): the work has been here done with the non-monomorphic divisor only to show that it may sometimes succeed: the *safer* plan would be to proceed as in Art. 20.]

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* A much smaller solution of this form (x^2-14y^2) can be found from this one by conformal multiplication by the unit-form $15^2-14\,.\,4^2 = 1$; thus

$$p = (76{,}399^2-14\,.\,20{,}400^2)(15^2-14\,.\,4^2),$$

which, on reduction, gives $p = 3585^2-14\,.\,404^2$. This might have been used as the (x^2-14y^2) form above. [This was pointed out by Mr. Bickmore.]

Next, to effect the partition $p = c^2 + 2d^2$. In the course of the tentative work above, the form

$$p = 3251^2 - 2 . 30^2$$

was obtained (at the start): this, and the now known form $(a^2 + b^2)$, are suitable base-forms (since $a^2 + b^2$, $c^2 + 2d^2$, $e^2 - 2f^2$ are a quadratic triad), so that

$$p = 2415^2 + 2176^2 = 3251^2 - 2 . 30^2.$$

This falls under Case i. Hence, by (28),

$$p = \frac{(2176 . 3251)^2 + 2\,(2415 . 30)^2}{2176^2 + 2 . 30^2} = \frac{(2415 . 15)^2 + 2\,(544 . 3251)^2}{15^2 + 2 . 544^2} = \frac{F}{f}$$

(the factor 8 having been cancelled out of F, f).

Here $f = 592{,}097 = 11 . 19 . 2833 = (3^2 + 2 . 1^2)(1^2 + 2 . 3^2)(41^2 + 2 . 24^2)$,

so that the *safe* procedure is to divide conformally by each factor separately (writing $f_1 = 11$, $f_2 = 19$, $f_3 = 2833$), as in Art. 20. Omitting details, the steps are

$$F_1 = \frac{F}{f_1} = \frac{(2415 . 15)^2 + 2\,(544 . 3251)^2}{11 = (3^2 + 2 . 1^2)} = 331{,}433^2 + 2 . 479{,}037^2,$$

$$F_2 = \frac{F_1}{f_2} = \frac{331{,}433^2 + 2 . 479{,}037^2}{19 = (1^2 + 2 . 3^2)} = 133{,}831^2 + 2 . 77{,}544^2,$$

$$F_3 = \frac{F_2}{f_3} = \frac{133{,}831^2 + 2 . 77{,}544^2}{833 = (41^2 + 2 . 24^2)} = 623^2 + 2 . 2256^2 = p,$$

the required partition.

On a Series of Cotrinodal Quartics. *By* H. M. TAYLOR, M.A.

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1. Many well-known theorems in the geometry of the triangle relate to straight lines drawn at right angles to the sides of the triangle.

In attempting to generalize these theorems we are led to consider in what cases and in what circumstances it is possible by orthogonal projection to project a triangle and a given point in its plane into a triangle and its orthocentre.

2. A well-known theorem given by Lazare Carnot states that, if the angular points A, B, C of a triangle be joined to any two points O, O' in its plane by straight lines which cut the opposite sides of the triangle in the points D, D', E, E', F, F', then these six points lie on a conic. From the well-known properties of the nine-point circle it

appears that, if the point O were the centre of gravity of the triangle ABC, and if the conic $DD'EE'FF'$ were an ellipse, then the orthogonal projection which would project the ellipse into a circle would project the triangle ABC and the point O' into a triangle and its orthocentre.

It can be proved that, when the point O is the centre of gravity of the triangle ABC, the conic $DD'EE'FF'$ will be a parabola if the point O' lie on a side of the triangle or a side produced; the conic will be an ellipse if the point O' lie within the triangle or in a part of the plane reached by crossing two of the sides; and the conic will be a hyperbola if the point O' lie within a part of the plane reached by crossing one of the sides.

3. We shall now consider the more general problem to find, when one of the points O, O' is given, the locus on which the other point must lie in order that the conic $DD'EE'FF$ may have a given eccentricity.

If the terms of the second degree in the equation of a conic in rectangular coordinates be

$$Ax^2+2Hxy+By^2,$$

the eccentricity e of the conic is given by the equation

$$\frac{1-e^2}{(2-e^2)^2}=\frac{AB-H^2}{(A+B)^2},$$

or the equivalent equation

$$\frac{e^4}{(2-e^2)^2}=\frac{(A-B)^2+4H^2}{(A+B)^2}.$$

4. If the equation of a conic in trilinear coordinates

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\beta\gamma+2w'\alpha\beta=0$$

be transformed to rectangular coordinates by the usual transformation

$$\alpha=p_1-x\cos\alpha-y\sin\alpha,$$

$$\beta=p_2-x\cos\beta-y\sin\beta,$$

$$\gamma=p_3-x\cos\gamma-y\sin\gamma,$$

then

$$\begin{aligned} x^2&\{u\cos^2\alpha+v\cos^2\beta+w\cos^2\gamma+2u'\cos\beta\cos\gamma+2v'\cos\gamma\cos\alpha+2w'\cos\alpha\cos\beta\}\\ +xy&\{2u\cos\alpha\sin\alpha+2v\cos\beta\sin\beta+2w\cos\gamma\sin\gamma+2u'(\cos\beta\sin\gamma+\sin\beta\cos\gamma)\\ &\quad+2v'(\cos\gamma\sin\alpha+\cos\alpha\sin\gamma)+2w'(\cos\alpha\sin\beta+\cos\beta\sin\alpha)\}\\ +y^2&\{u\sin^2\alpha+v\sin^2\beta+w\sin^2\gamma+2u'\sin\beta\sin\gamma+2v'\sin\gamma\sin\alpha+2w'\sin\alpha\sin\beta\}\\ &\quad+\text{terms of first degree}=\text{const.}\end{aligned}$$

If this be written

$$Ax^2+2Hxy+By^2+\text{terms of first degree}=\text{const.},$$

then

$$\begin{aligned}
A+B &= u+v+w+2u'\cos(\beta-\gamma)+2v'\cos(\gamma-\alpha)+2w'\cos(\alpha-\beta)\\
&= u+v+w-2u'\cos A-2v'\cos B-2w'\cos C;\\
A-B &= u\cos 2\alpha+v\cos 2\beta+w\cos 2\gamma+2u'\cos(\beta+\gamma)\\
&\qquad\qquad +2v'\cos(\gamma+\alpha)+2w'\cos(\alpha+\beta)\\
&= u\cos 2\alpha+v\cos(2\alpha-2C)+w\cos(2\alpha+2B)+2u'\cos(2\alpha+B-C)\\
&\qquad\qquad -2v'\cos(2\alpha+B)-2w'\cos(2\alpha-C)\\
&= \cos 2\alpha\,\{u+v\cos 2C+w\cos 2B+2u'\cos(B-C)\\
&\qquad\qquad -2v'\cos B-2w'\cos C\}\\
&\quad +\sin 2\alpha\,\{v\sin 2C-w\sin 2B-2u'\sin(B-C)\\
&\qquad\qquad +2v'\sin B-2w'\sin C\};\\
2H &= u\sin 2\alpha+v\sin 2\beta+w\sin 2\gamma+2u'\sin(\beta+\gamma)\\
&\qquad\qquad +2v'\sin(\gamma+\alpha)+2w'\sin(\alpha+\beta)\\
&= u\sin 2\alpha+v\sin(2\alpha-2C)+w\sin(2\alpha+2B)+2u'\sin(2\alpha+B-C)\\
&\qquad\qquad -2v'\sin(2\alpha+B)-2w'\sin(2\alpha-C)\\
&= \sin 2\alpha\,\{u+v\cos 2C+w\cos 2B+2u'\cos(B-C)\\
&\qquad\qquad -2v'\cos B-2w'\cos C\}\\
&\quad +\cos 2\alpha\,\{-v\sin 2C+w\sin 2B+2u'\sin(B-C)\\
&\qquad\qquad -2v'\sin B+2w'\sin C\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(A-B)^2+4H^2 &= u^2+\&c.+4u'^2+\&c.+2uv\cos 2(\alpha-\beta)+\&c.\\
&\qquad +2u'v'\cos(\alpha-\beta)+\&c.+4uu'\cos(2\alpha-\beta-\gamma)+\&c.\\
&\qquad +2uv'\cos(\alpha-\gamma)+\&c.\\
&= u^2+\&c.+4u'^2+\&c.+2uv\cos 2C+\&c.-8u'v'\cos C+\&c.\\
&\qquad +4uu'\cos(B-C)+\&c.-4uv'\cos B+\&c.
\end{aligned}$$

If we take the triangle ABC as the triangle of reference, then, if the coordinates of O be α, β, γ, and those of O' be α', β', γ', and x, y, z be current coordinates, the equation of the conic $DD'EE'FF'$ is

$$\frac{x^2}{\alpha\alpha'}+\frac{y^2}{\beta\beta'}+\frac{z^2}{\gamma\gamma'}-yz\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)-zx\left(\frac{1}{\gamma'\alpha}+\frac{1}{\gamma\alpha'}\right)-xy\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)=0.$$

Comparing this equation with the equation

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0,$$

we see that each of the expressions $A+B$, $A-B$, $2H$ is of -1

dimension in $\alpha\beta\gamma$, and that therefore the equation

$$(A-B)^2+4H^2 = \frac{e^4}{(2-e^2)^2}(A+B)^2$$

admits only terms of the form

$$\frac{1}{\alpha^2},\ \frac{1}{\beta^2},\ \frac{1}{\gamma^2},\ \frac{1}{\beta\gamma},\ \frac{1}{\gamma\alpha},\ \frac{1}{\alpha\beta};$$

it follows therefore that, if the point O' $(\alpha', \beta', \gamma')$ be fixed, and the eccentricity e be given, the locus of the point O is a trinodal quartic having A, B, C for nodes.

It appears also from the equation

$$AB-H^2 = \frac{1-e^2}{(2-e^2)^2}(A+B)^2$$

that the locus $AB-H^2=0$ separates the parts of the plane in which O must lie in the two cases: (1) when the conic is an ellipse, and (2) when it is a hyperbola.

It may be remarked that, if the conic be a circle, the locus of the point O is reduced to a point.

5. The remaining part of this paper is concerned more especially with the case when the conic is a parabola. It is somewhat more convenient in discussing this problem to use triangular coordinates.

The condition that the triangular equation

$$\frac{x^2}{\alpha\alpha'}+\frac{y^2}{\beta\beta'}+\frac{z^2}{\gamma\gamma'}-yz\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)-zx\left(\frac{1}{\gamma'\alpha}+\frac{1}{\gamma\alpha'}\right)-xy\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)=0$$

should represent a parabola is

$$\frac{4}{\beta\beta'\gamma\gamma'}+\frac{4}{\gamma\gamma'\alpha\alpha'}+\frac{4}{\alpha\alpha'\beta\beta'}+\frac{4}{\alpha\alpha'}\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)+\frac{4}{\beta\beta'}\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)$$
$$+\frac{4}{\gamma\gamma'}\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)-\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)^2-\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)^2-\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)^2$$
$$+2\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)+2\left(\frac{1}{\alpha\beta'}+\frac{1}{\alpha'\beta}\right)\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)$$
$$+2\left(\frac{1}{\beta\gamma'}+\frac{1}{\beta'\gamma}\right)\left(\frac{1}{\gamma\alpha'}+\frac{1}{\gamma'\alpha}\right)=0,$$

so that

$$\left\{\frac{4}{\beta'\gamma'}+\frac{4}{\beta'\alpha'}+\frac{4}{\gamma'\alpha'}-\frac{2}{\beta'\gamma'}+\frac{2}{\alpha'^2}+\frac{2}{\alpha'\beta'}+\frac{2}{\gamma'\alpha'}\right\}\frac{1}{\beta\gamma}+\&\text{c.}$$
$$=\left(\frac{1}{\beta'}-\frac{1}{\gamma'}\right)^2\frac{1}{\alpha^2}+\left(\frac{1}{\gamma'}-\frac{1}{\alpha'}\right)^2\frac{1}{\beta^2}+\left(\frac{1}{\alpha'}-\frac{1}{\beta'}\right)^2\frac{1}{\gamma^2}.$$

Therefore $8\left\{\frac{1}{\gamma'\alpha}+\frac{1}{\alpha'\beta}\right\}\frac{1}{\beta\gamma}+\&c.$

$$=\left\{\left(\frac{1}{\beta'}-\frac{1}{\gamma'}\right)\frac{1}{\alpha}+\left(\frac{1}{\gamma'}-\frac{1}{\alpha}\right)\frac{1}{\beta}+\left(\frac{1}{\alpha'}-\frac{1}{\beta'}\right)\frac{1}{\gamma}\right\}^2.$$

Therefore $8\alpha'\beta'\gamma'\alpha\beta\gamma\left\{(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma\right\}$

$$=\left\{\alpha'(\beta'-\gamma')\,\beta\gamma+\beta'(\gamma'-\alpha')\,\gamma\alpha+\gamma'(\alpha'-\beta')\,\alpha\beta\right\}^2,$$

an equation which gives the locus of the point O for any given position $\alpha'\beta'\gamma'$ of the point O', when the conic $DD'EE'FF'$ is a parabola. The equation may for convenience be written $8\alpha'\beta'\gamma'\alpha\beta\gamma L = S^2$.

6. It may be noticed that the line $L = 0$ is the polar of the point O' with respect to the conic $\beta\gamma+\gamma\alpha+\alpha\beta = 0$, which is the circumscribing ellipse of minimum area.

The conic $S = 0$ is a hyperbola circumscribing the triangle ABC and passing through its centre of gravity G and also through the point O'.

The conic $S = 0$ is completely determined by the position of the point O', but any position of the point O' on the conic $S = 0$ would equally well determine the same conic $S = 0$. On the other hand, every different position of the point O' on the conic $S = 0$ gives rise to a different quartic $8\alpha'\beta'\gamma'\alpha\beta\gamma L = S^2$. It seems convenient, in discussing the different forms of the quartic, to assume the conic $S = 0$ determined by some point on it and then to trace the changes in the quartic curve for different positions of the point O', while it traces out the conic $S = 0$.

This has been done for me by my friend Mr. W. H. Blythe, who has kindly drawn a series of diagrams illustrative of the manner in which the shape and the position of the quartic curve change with the position of the point O'.

A paper by Mr. Blythe exhibiting these diagrams accompanies this paper.

7. It is easy to prove that the straight line $\beta+\gamma = 0$ meets the quartic twice at the point A, and is a tangent to the curve at the point of intersection with

$$\alpha\,(\alpha'\beta'+\alpha'\gamma'+2\beta'\gamma')-\beta\alpha'\,(\beta'-\gamma') = 0,$$

that is, at the point

$$\frac{\alpha}{\alpha'(\beta'-\gamma')}=\frac{+\beta}{\alpha'\beta'+\alpha'\gamma'+2\beta'\gamma'}=\frac{-\gamma}{\alpha'\beta'+\alpha'\gamma'+2\beta'\gamma'}=\frac{1}{\alpha'(\beta'-\gamma')}.$$

Similarly, the straight line $\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=0$ meets the curve twice at the point A, and it touches the quartic curve at the point where it meets

$$\alpha\,(2\alpha'+\beta'+\gamma')+\beta\,\frac{\alpha'}{\beta'}\,(\beta'-\gamma')=0,$$

that is, at the point

$$\frac{\frac{\alpha}{\alpha'}}{\beta'-\gamma'}=\frac{-\frac{\beta}{\beta'}}{2\alpha'+\beta'+\gamma'}=\frac{\frac{\gamma}{\gamma'}}{2\alpha'+\beta'+\gamma'}$$

$$=\frac{1}{\alpha'\,(\beta'-\gamma')-(\beta'-\gamma')\,(2\alpha'+\beta'+\gamma\,)}$$

$$=\frac{-1}{(\beta'-\gamma')(\alpha'+\beta'+\gamma')}=-\frac{1}{\beta'-\gamma}$$

8. We will now discuss in which of the portions of space, into which the plane is divided by the three sides of the triangle ABC and the line L, the curve $8\alpha'\beta'\gamma'\alpha\beta\gamma L=S^2$ lies.

It is clear from the form of the equation that the portions of space in which the curve lies are to be found from each other by crossing *two* or *four* of the straight lines $\alpha=0$, $\beta=0$, $\gamma=0$, $L=0$.

If, therefore, for any given position of the point $\alpha'\beta'\gamma'$, we determine the position of the line L and also the position of some point on the curve in one of the parts of the plane, we shall have determined in which parts of the plane the curve lies.

First, let us take the case where $\beta'+\gamma'$, $\gamma'+\alpha'$, and $\alpha'+\beta'$ are all positive, that is, where the point $\alpha'\beta'\gamma'$ lies within the triangle $A'B'C'$ formed by drawing through the angular points of the triangle ABC parallels to the opposite sides; then the expression L or

$$\alpha\,(\beta'+\gamma')+\beta\,(\gamma'+\alpha')+\gamma\,(\alpha'+\beta')$$

is positive if α, β, γ be all positive. Therefore, in this case, if α', β', γ' be all positive, there must be some part of the curve within the triangle ABC, but, if one of the three quantities α', β', γ' be negative, there is no part of the curve within the triangle ABC.

In this case, the triangle is not cut by the line L.

Next, let us consider the case where $\alpha'+\beta'$ and $\alpha'+\gamma'$ are both positive and $\beta'+\gamma'$ is negative; in this case, the line L will cut the sides AC and AB both internally in some points Q and R, and the quantity L will be negative for points on the same side of QR as A, and positive for points on the opposite side. It follows therefore that, if the product $\beta'\gamma'$ be positive, some part of the curve lies within the quadrilateral $RBCQ$, and, if the product $\beta'\gamma'$ be negative, no part lies within the quadrilateral $RBCQ$. Again, in the case where $\beta'+\gamma'$ is positive and both $\alpha'+\beta'$ and $\alpha'+\gamma'$ are negative, the line L cuts both the sides AC and AB internally in some points Q and R; and the quantity L is positive for points on the same side of QR as the point A, and negative for points on the opposite side. It follows therefore that some part of the curve lies within the quadrilateral $RBCQ$.

The tangents at the double point A are

$$8\beta\gamma\alpha\,\beta'\gamma'(\beta'+\gamma')=\gamma^2\beta'^2(\gamma'-\alpha')^2+\beta^2\gamma'^2(\alpha'-\beta')^2+2\beta\gamma\beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta').$$

These tangents will be real if

$$\{\beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta')-4\alpha'\beta'\gamma'(\beta'+\gamma')\}^2-\beta'^2\gamma'^2(\gamma'-\alpha')^2(\alpha'-\beta')^2$$

is positive, that is, if $\alpha'(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')$ is positive; that is, if the point $\alpha'\beta'\gamma'$ lies within the quadrilateral $BCB'C'$ or in any of the spaces reached from within the quadrilateral by crossing two of its sides.

9. For convenience, we will take the equation of the quartic to be

$$a^2\beta^2\gamma^2+b^2\gamma^2\alpha^2+c^2\alpha^2\beta^2=2\alpha\beta\gamma\,(l\alpha+m\beta+n\gamma).$$

When this equation is written in the form

$$(a\beta\gamma+b\gamma\alpha+c\alpha\beta)^2=2\alpha\beta\gamma\,\{(l+bc)\,\alpha+(m+ca)\,\beta+(n+ab)\,\gamma\},$$

we see that the straight line

$$(l+bc)\,\alpha+(m+ca)\,\beta+(n+ab)\,\gamma=0$$

is a double tangent to the quartic touching at the points where it meets the conic

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0.$$

As the above form of the equation of the quartic is equally true when the sign of one of the letters a, b, c is changed, it follows that the straight lines

$$(l+bc)\,\alpha+(m-ac)\,\beta+(n-ab)\,\gamma=0,$$

$$(l-bc)\,\alpha+(m+ac)\,\beta+(n-ab)\,\gamma=0,$$

$$(l-bc)\,\alpha+(m-ac)\,\beta+(n+ab)\,\gamma=0$$

are double tangents, their points of contact being the points where they meet the conics

$$-a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma-b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma+b\gamma\alpha-c\alpha\beta=0,$$

respectively.

The eight points of contact of these four double tangents lie upon the conic

$$(l^2-b^2c^2)\,\alpha^2+(m^2-c^2a^2)\,\beta^2+(n^2-a^2b^2)\,\gamma^2+2mn\beta\gamma+2nl\gamma\alpha+2lm\alpha\beta=0\,;$$

and the equation of the quartic may be written

$$\{(l+bc)\,\alpha+(m+ac)\,\beta+(n+ab)\,\gamma\}\,\{(l+bc)\,\alpha+(m-ac)\,\beta+(n-ab)\,\gamma\}$$
$$\times\{(l-bc)\alpha+(m+bc)\beta+(n-ab)\gamma\}\,\{(l-bc)\alpha+(m-ac)\beta+(n+ab)\gamma\}$$
$$-\{(l^2-b^2c^2)\,\alpha^2+(m^2-c^2a^2)\beta^2+(n^2-a^2b^2)\,\gamma^2+2mn\beta\gamma+2nl\gamma\alpha+2lm\alpha\beta\}^2$$
$$\equiv -a^2b^2c^2\{a^2\beta^2\gamma^2+b^2\gamma^2\alpha^2+c^2\alpha^2\beta^2-2a\beta\gamma\,(l\alpha+m\beta+n\gamma)\}=0.$$

Of the four conics

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$-a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma-b\gamma\alpha+c\alpha\beta=0,$$
$$a\beta\gamma+b\gamma\alpha-c\alpha\beta=0,$$

three are hyperbolas and one is an ellipse. They touch in pairs at the angular points of the triangle *ABC*.

10. When we compare this form with the earlier form

$$8\alpha'\beta'\gamma'\alpha\beta\gamma L=S^2,$$

we have

$$a=\alpha'(\beta'-\gamma'),\quad b=\beta'(\gamma'-\alpha'),\quad c=\gamma'(\alpha'-\beta'),$$
$$l=4\alpha'\beta'\gamma'(\beta'+\gamma')+\beta'\gamma'(\alpha'-\beta')(\alpha'-\gamma'),$$
$$l+bc=4\alpha'\beta'\gamma'(\beta'+\gamma'),$$
$$l-bc=4\alpha'\beta'\gamma'(\beta'+\gamma')+2\beta'\gamma'(\alpha'-\beta')(\alpha'-\gamma')$$
$$=2\beta'\gamma'\{2\alpha'\beta'+2\alpha'\gamma'+\alpha'^2-\alpha'\beta'-\alpha'\gamma'+\beta'\gamma'\}$$
$$=2\beta'\gamma'\{\alpha'^2+\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'\}$$
$$=2\beta'\gamma'(\alpha'+\beta')(\alpha'+\gamma').$$

Therefore $l^2-b^2c^2=8\alpha'\beta'^2\gamma'^2(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')$.

The equation of the conic through the eight points of contact of the four double tangents therefore is

$$\begin{aligned}
&4\alpha'\beta'\gamma'\left\{(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma\right\} \\
&\times 2\left\{\beta'\gamma'(\alpha'+\beta')(\alpha'+\gamma')\alpha+\gamma'\alpha'(\beta'+\alpha')(\beta'+\gamma')\beta+\alpha'\beta'(\gamma'+\alpha')(\gamma'+\beta')\gamma\right\} \\
&+2\alpha'\beta'\gamma'(\beta'-\gamma')(\gamma'-\alpha')(\alpha'-\beta')\left\{\alpha'(\beta'-\gamma')\,\beta\gamma+\beta'\,(\gamma'-\alpha')\,\gamma\alpha\right. \\
&\hspace{20em}\left.+\gamma\,(\alpha'-\beta')\,\alpha\beta\right\}=0;
\end{aligned}$$

or

$$\begin{aligned}
&4\,(\beta'+\gamma')(\gamma'+\alpha')(\alpha'+\beta')\left\{(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma\right\} \\
&\qquad\times\left\{\frac{\beta'\gamma'\alpha}{\beta'+\gamma'}+\frac{\gamma'\alpha'\beta}{\gamma'+\alpha'}+\frac{\alpha'\beta'\gamma}{\alpha'+\beta'}\right\}+(\beta'-\gamma')(\gamma'-\alpha')(\alpha'-\beta') \\
&\qquad\times\left\{\alpha'(\beta'-\gamma')\,\beta\gamma+\beta'(\gamma'-\alpha')\,\gamma\alpha+\gamma'(\alpha'-\beta')\,\alpha\beta\right\}=0.
\end{aligned}$$

The equations of the four double tangents of the quartic are

$$(\beta'+\gamma')\,\alpha+(\gamma'+\alpha')\,\beta+(\alpha'+\beta')\,\gamma=0,$$

$$2\beta'\gamma'\alpha+\gamma'\,(\alpha'+\beta')\,\beta+\beta'\,(\alpha'+\gamma')\,\gamma=0,$$

$$\gamma'\,(\alpha'+\beta')\,\alpha+2\alpha'\gamma'\beta+\alpha'\,(\beta'+\gamma')\,\gamma=0,$$

$$\beta'\,(\alpha'+\gamma')\,\alpha+\alpha'\,(\beta'+\gamma')\,\beta+2\alpha'\beta'\gamma=0.$$

The condition that the quartic should degenerate into two conics is identical with the condition that the conic

$$\begin{aligned}
&8\alpha'\beta'\gamma'\left\{\beta\gamma\,(\beta'+\gamma')+\gamma\alpha\,(\gamma'+\alpha')+\alpha\beta\,(\alpha'+\beta')\right\} \\
&\quad=\left\{\alpha\alpha'(\beta'-\gamma')+\beta\beta'(\gamma'-\alpha')+\gamma\gamma'(\alpha'-\beta')\right\}^2
\end{aligned}$$

should degenerate into two straight lines.

This is

$$\begin{vmatrix}
\alpha'^2\,(\beta'-\gamma')^2, & \begin{array}{r}\alpha'\beta'\,(\beta'-\gamma')(\gamma'-\alpha')\\ -4\alpha'\beta'\gamma'\,(\alpha'+\beta'),\end{array} & \begin{array}{r}\alpha'\gamma'\,(\beta'-\gamma')(\alpha-\beta')\\ -4\alpha'\beta'\gamma'\,(\gamma'+\alpha')\end{array} \\
\begin{array}{r}\alpha'\beta'(\beta'-\gamma')(\gamma'-\alpha')\\ -4\alpha'\beta'\gamma'\,(\alpha'+\beta'),\end{array} & \beta'^2\,(\gamma'-\alpha')^2, & \begin{array}{r}\beta'\gamma'\,(\gamma'-\alpha')(\alpha'-\beta')\\ -4\alpha'\beta'\gamma'\,(\beta'+\gamma')\end{array} \\
\begin{array}{r}\alpha'\gamma'(\beta'-\gamma')(\alpha'-\beta')\\ -4\alpha'\beta'\gamma'\,(\gamma'+\alpha'),\end{array} & \begin{array}{r}\beta'\gamma'(\gamma'-\alpha')(\alpha'-\beta')\\ -4\alpha'\beta'\gamma'\,(\beta'+\gamma'),\end{array} & \gamma'^2\,(\alpha'-\beta')^2
\end{vmatrix}=0,$$

which reduces to

$$\alpha'^2\beta'^2\gamma'^2\,(\beta'+\gamma')^2\,(\gamma'+\alpha')^2\,(\alpha'+\beta')^2 = 0.$$

The quartic therefore splits up into two conics only when the point $\alpha'\beta'\gamma'$ lies on a side, or a side produced, of one of the two triangles ABC, $A'B'C'$.

If $\alpha' = 0$, the equation of the quartic becomes

$$\beta'^2\gamma'^2\alpha^2\,(\beta-\gamma)^2 = 0,$$

that is, the quartic reduces to the two straight lines BC, AG, repeated twice.

If $\beta'+\gamma' = 0$, the equation of the quartic becomes

$$\left\{2\alpha'\beta'\beta\gamma+\beta'\,(\alpha'-\beta')\,\gamma\alpha+\beta'\,(\alpha'+\beta')\,\alpha\beta\right\}^2 = 0.$$

The quartic therefore reduces to the conic

$$2\alpha'\beta\gamma+(\alpha'-\beta')\,\gamma\alpha+(\alpha'+\beta')\,\alpha\beta = 0,$$

repeated twice. This conic circumscribes the parallelogram $ABA'C$, and has its centre at the middle point of BC. It is an ellipse, a degenerate parabola, or a hyperbola, according as the point $\alpha'\beta'\gamma'$ lies between the points B' and C', coincides with one of them, or lies outside them.

The quartic degenerates into a straight line and a cubic when the point $\alpha'\beta'\gamma'$ lies in one of the straight lines AG, BG, CG; thus, if $\beta' = \gamma'$, the equation of the quartic admits of the factor α; the quartic therefore in this case reduces to the line BC and the cubic

$$8\alpha'\beta\gamma\,\left\{2\beta'\alpha+(\alpha'+\beta')\,\beta+(\alpha'+\beta')\,\beta\right\} = \alpha\,\left\{(\alpha'+\beta')\,\beta-(\alpha'-\beta')\,\gamma\right\}^2.$$

The form of this cubic will change as the point O' moves along the line AG.

11. The following diagrams illustrate these results.* Let

$$S \equiv \beta\gamma\alpha'\,(\beta-\gamma)+\gamma\alpha\beta'\,(\gamma-\alpha)+\alpha\beta\gamma'\,(\alpha-\beta),$$
$$L \equiv \alpha\,(\beta'+\gamma')+\beta\,(\gamma'+\alpha')+\gamma\,(\alpha'+\beta').$$

A point $P\,[\alpha', \beta', \gamma']$ moves along a conic of the form S; it is required to trace the changes in the form of the quartic

$$S^2 = 8\alpha'\beta'\gamma'\alpha\beta\gamma L.$$

Take the triangle of reference as equilateral.

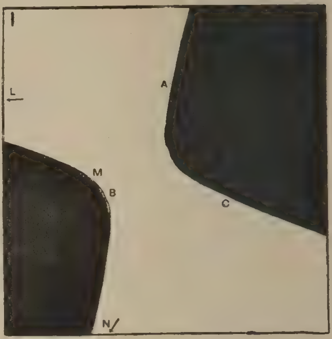
---

* The rest of the paper is due to Mr. W. H. Blythe; see § 6.

Figure 1 represents the locus of P, a hyperbola passing through A, B, C, the angular points of the triangle of reference, and through its centre of gravity.

This locus cuts the lines $\beta+\gamma=0$, $\gamma+\alpha=0$, $\alpha+\beta=0$ in the points L, M, N.

[The points L and N are too distant to be shown in the figure; their position is indicated by letters placed at the side of the diagram.]



In taking different positions of P on the locus given in Figure 1, we obtain all possible forms of the quartic; but the hyperbola in Figure 1 will in certain cases reduce to two straight lines, namely, a side of the triangle and its corresponding median, *i.e.*, $\alpha=0$ and $\beta=\gamma$. When P moves along $\alpha=0$, the quartic reduces to two coincident pairs of straight lines $\alpha=0$ and $\beta=\gamma$.

When P moves along the straight line $\beta=\gamma$, the quartic reduces to the straight line $\alpha=0$, together with a cubic having a node at A. The forms of these cubics will be given later on; see Figures 16 and 17.

It is shown in Salmon's *Higher Plane Curves* that each quartic represented by the above equation has a corresponding conic, such that, if α, β, γ be a point Q on the quartic, and x, y, z the corresponding point q on the conic, then

$$\alpha:\beta:\gamma::yz:zx:xy.$$

If one of these points be given, the other can be found from the fact that QA and qA make equal angles with

$$\beta+\gamma=0,$$

QB and qB with $\qquad \gamma+\alpha=0,$

and QC and qC with $\qquad \alpha+\beta=0.$

The position of the corresponding conic can also be determined geometrically, for it touches the six straight lines

$$\beta+\gamma=0, \quad \gamma+\alpha=0, \quad \alpha+\beta=0,$$

$$\alpha\alpha'+\beta\beta'=0, \quad \beta\beta'+\gamma\gamma'=0, \quad \gamma\gamma'+\alpha\alpha'=0.$$

It is only necessary to know the straight line

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0.$$

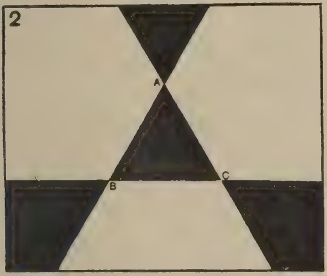
This line cuts the sides of the triangle *ABC* in the points at which

$$\alpha\alpha' + \beta\beta' = 0, \quad \beta\beta' + \gamma\gamma' = 0, \quad \gamma\gamma' + \alpha\alpha' = 0$$

meet them.

It will be found that the line at infinity corresponds to the circumscribing circle of *ABC*.

Figure 2.—When *P* is at the centre of gravity of the triangle, the corresponding conic is the circumscribing circle. The quartic is represented by the three sides of the triangle *ABC* and the line at infinity.



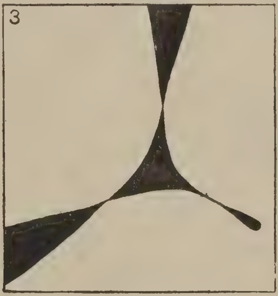


Figure 3.—Next, let *P* move to a point between the centre and the angle *C*.

The corresponding conic is an ellipse cutting all the sides of the triangle internally, and the circumscribing circle in four points, two on the arc *CA* and two on the arc *CB*. The quartic has therefore four asymptotes corresponding to the four points at which the conic meets the circle.

Two infinite branches pass through the angle *B* and two through *A*. Each pair have a corresponding hyperbolic branch having the same asymptotes.

These branches, which touch the lines

$$\alpha + \beta = 0, \quad \beta + \gamma = 0, \quad \text{and} \quad \gamma + \alpha = 0,$$

are too distant to appear in the figure, but are shown in Figures 4 and 5.

The part of the conic lying outside the triangle *ABC* between *AB* and the circumscribing circle corresponds to a loop of the quartic outside the angle *C*.

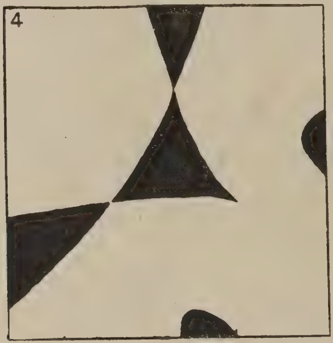
Since the conic touches

$$\alpha\alpha' + \beta\beta' = 0, \quad \beta\beta' + \gamma\gamma' = 0, \quad \gamma\gamma' + \alpha\alpha' = 0,$$

the quartic touches

$$\alpha\beta' + \beta\alpha' = 0, \quad \beta\gamma' + \gamma\beta' = 0, \quad \gamma'\alpha + \alpha'\gamma = 0.$$

Figure 4.—When *P* moves to the angle *C*, the corresponding conic, still an ellipse, touches the side *AB*. The loop of the quartic at *C* disappears, and we have a cusp at *C*.



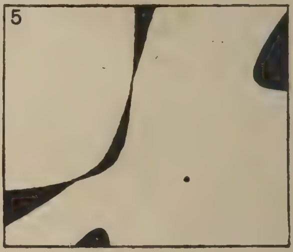
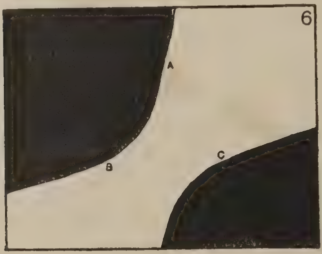


Figure 5.—As *P* continues to move in the same direction along its locus, the corresponding conic does not cut *AB*; therefore we find an acnode at *C*.

Figure 6.—When *P* coincides with *L*, the corresponding conic becomes two coincident straight lines; therefore the quartic reduces to two coincident conics.



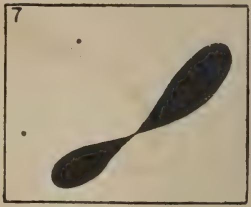
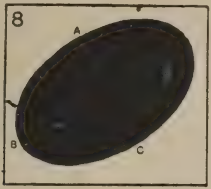


Figure 7.—As *P* continues to move in the same direction, the corresponding conic, now a hyperbola, does not cut *BC*, *CA*, or the circumscribing circle, but cuts *AB* externally. The quartic therefore has acnodes at A and B, and consists of two finite loops having a node at C, both external to the triangle.

Figure 8.—When P coincides with M, the corresponding conic becomes two coincident straight lines that do not cut the circumscribing circle. The quartic reduces to two coincident ellipses that circumscribe ABC.



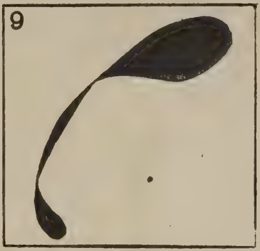


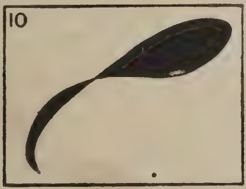
Figure 9.—When P lies between M and B, the corresponding conic may be either an ellipse or a hyperbola, for, when the locus of P cuts the circumscribing circle, the corresponding conic becomes a parabola.

This fact, however, does not affect the form of the quartic, for it only implies that it touches the circumscribing circle.

The corresponding conic cuts the sides AC and BC externally, but not the side AB.

The form of the quartic as shown by the figure is such that it has an acnode at C.

Figure 10.—When P coincides with B, the loop beyond B disappears, and is replaced by a cusp.



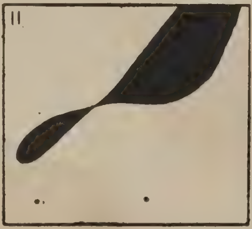
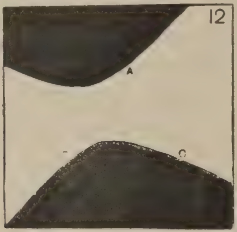
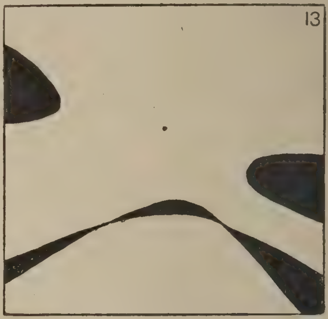
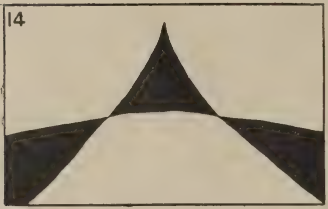


Figure 11.—When P continues to move towards N, the corresponding conic ceases to cut AC; we now find acnodes at B and C. The quartic assumes the form of Figure 7, one loop being too elongated to be entirely shown.

Figures 12, 13, 14, and 15 are similar in form to 6, 5, 4, and 3, if we interchange the angles A and C.







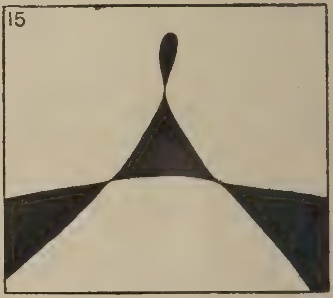
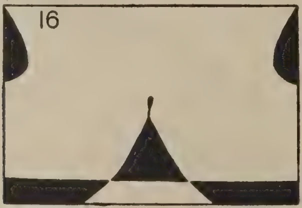


Figure 16.—Take the special case in which P moves along the line $\beta = \gamma$. At the centre of gravity, we again obtain Figure 3. Now let P move towards the angle A. The quartic reduces to the straight line $\alpha = 0$ and a cubic.

These together form a figure as given, analogous to Figure 15.

When P coincides with A, we again obtain a cusp.

When P moves up to the intersection of $\alpha + \beta = 0$ and $\alpha + \gamma = 0$, the quartic reduces to $\alpha = 0$ and $\beta + \gamma = 0$, each repeated twice.



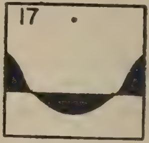


Figure 17.—We next get a figure somewhat similar in form to Figure 9.

The changes that follow can be traced from figures already given.

On the Stationary Motion of a System of Equal Elastic Spheres of Finite Diameter. By S. H. Burbury, M.A., F.R.S. Read December 10th, 1896. Received in revised form May 15th, 1897.

1. The object of this paper is to prove that in such a system in stationary motion the velocities of spheres near to one another are *correlated*. That is, that the chance that n spheres forming a group together in space shall simultaneously have component velocities

$$u_1 \dots u_1+du_1,\ \ v_1 \dots v_1+dv_1,\ \ \dots\ \ w_n \dots w_n+dw_n$$

is of the form $$A\epsilon^{-hQ}\,du_1 \dots dw_n,$$

and Q is not merely the sum of the squares, as in Maxwell's system, but a quadratic function, of the velocities, namely,

$$Q = \Sigma\,(u^2+v^2+w^2)+\Sigma\Sigma b\,(uu'+vv'+ww'),$$

the second term containing the products of every pair of u's, &c., but no products uv, uw, or vw, and b being a function of the distance at the instant between the two molecules whose velocities are u, &c., and u', &c., which becomes evanescent as that distance increases.

The sufficiency of Maxwell's distribution of velocities, according to which Q contains only the squares of the velocities, has been proved by several writers, but always on a certain fundamental assumption, which is true only for infinitely small densities—The assumption is in effect this — let $f\,dx\,dy\,dz\,du\,dv\,dw$ be the number per unit volume of molecules whose coordinates of position, x, y, z, and momenta u, v, w, are in a certain state A, or, as we may express it, $f\,dx \dots dw$ is the chance that a molecule shall be in that state. Similarly, $f'dx' \dots dw'$ is the chance that another molecule shall be in the state A'. Then it is always assumed that the chance of two molecules being in the states A, A' respectively is $ff'du \dots dw'$. That is, it is assumed that the chances are under all circumstances, *i.e.*, even when the two molecules are on the point of collision, independent.*

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* See especially Boltzmann's *Vorlesungen über Gas Theorie*, 1895, p. 22, where the assumption is made emphatically.

That assumption involves the non-existence of my coefficient b.

It will be sufficient for my present purpose to prove the necessity of the b term when the molecules are elastic spheres of finite diameter. For, if that be proved, it will be admitted that it must exist in more general cases.

Art. 15, Equations (I.) and (24), (25) show the line of argument.

I claim as an advantage of the formula that it gives a limit of density, or density and temperature, beyond which the formula cannot hold, and the motion must change type. The ordinary formula, in which Q contains squares only, is either true only in the limiting case of infinitely small density—*i.e.*, when the aggregate volume of all the spheres in unit volume is infinitely small—or it is true however great the density. There is no intermediate point at which we can with reason draw the line. Using my formula, we have such a point. For, as the density increases, b increases, until the determinant of Q, namely,

$$D = \begin{vmatrix} 2 & b_{12} & b_{13} & 1 \\ b_{12} & 2 & . & . \\ . & . & . & . \end{vmatrix}$$

or some one of its coaxial minors becomes zero, or negative. And then Q will for some values of the u's, v's and w's become negative. And the law necessarily fails. The change of type of the system's motion is represented mathematically by the vanishing of the determinant.

2. It is necessary first to treat the case where the molecules are of infinitely small dimensions, so that collisions between them will not occur. That being the case, the system will be in stationary motion if the velocities are distributed according to Maxwell's law, Q being a sum of squares only. This we may admit, whatever views we may hold as to the possibility of a distribution according to any other law. I shall then assume that distribution to exist. And it follows, from Q being a sum of squares only, that the fact of any molecule, or any group of molecules, having any given velocities affords (if the whole number of molecules considered be infinite) no presumption whatever with regard to the velocities of any other molecules.

3. Let ρ be the average number of molecules in unit of volume, the system being homogeneous in this respect. We may suppose the whole system to be comprised within an elastic boundary at an infinite distance which reverses the velocities of all molecules which strike it, so that ρ may be regarded as invariable with the time.

4. Let P be any point, r the distance of any other point from P. Let f be a function of r which satisfies the following conditions, namely,

(A) f is equal to unity at all points within a small sphere of radius a described about P as centre.

(B) f is finite positive and continuous everywhere.

(C) f is of negative degree at all points outside of the a sphere.

We may take for a the radius of a sphere which on average contains one molecule.

5. Let $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, ... $\alpha_n\beta_n\gamma_n$ be the component velocities in x, y, and z of the n molecules which at any instant are contained within a sphere of very great radius R described about P as centre. And let

$$\xi = \frac{\Sigma\alpha f}{\Sigma f}, \quad \eta = \frac{\Sigma\beta f}{\Sigma f}, \quad \zeta = \frac{\Sigma\gamma f}{\Sigma f}, \tag{1}$$

the summation including all the n molecules, and f having for each molecule the value proper to its position, as f_1 for the molecule whose velocities are α_1, β_1, γ_1, &c. ξ, η, ζ are then, R being very great, continuous functions of x, y, and z.

6. Now the chance that the n molecules shall at any instant have, for their component velocities in x, α_1 ... $\alpha_1 + d\alpha$, &c., is

$$A\epsilon^{-hQ}\, d\alpha_1 \dots d\alpha_n,$$

and in the system now under consideration

$$Q = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2.$$

But, from (1), $$\alpha_n f_n = \xi\Sigma f - \alpha_1 f_1 - \dots - \alpha_{n-1} f_{n-1}.$$

Further, we may suppose the molecule to which the suffix n belongs to be very near P, whence, from condition (A) of Art. 4,

$$f_n = 1,$$

and $$\alpha_n = \xi\Sigma f - \alpha_1 f_1 - \dots - \alpha_{n-1} f_{n-1}.$$

Substituting this value for α_n in Q, we obtain

$$\left.\begin{aligned}Q = \xi^2(\Sigma f)^2 + (1+f_1^2)\,\alpha_1^2 + (1+f_2^2)\,\alpha_2^2 + \&c.\\ -2f_1\alpha_1\xi\Sigma f - 2f_2\alpha_2\xi\Sigma f - \&c.\\ +2f_1f_2\alpha_1\alpha_2 + 2f_1f_3\alpha_1\alpha_3 + 2f_2f_3\alpha_2\alpha_3 + \&c.\end{aligned}\right\} \qquad (2)$$

7. If we now integrate ϵ^{-hQ} for $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{n-1}$ successively between limits $\pm\infty$, we get the result next stated. Let D be the determinant of Q, that is,

$$D = \begin{vmatrix} 2(\Sigma f)^2 & -2f_1\Sigma f & -2f_2\Sigma f & \ldots \\ -2f_1\Sigma f & (2+2f_1^2) & 2f_1f_2 & \ldots \\ -2f_2\Sigma f & 2f_1f_2 & (2+2f_2^2) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

in which every constituent contains the factor 2. And let D_{pq} be the minor formed by omitting the p^{th} row and q^{th} column. Similarly, let D_{pqrs} be the minor formed by omitting rows p and r and columns q and s.

Then, by integrating for $\alpha_1 \ldots \alpha_{n-1}$, Q is reduced to

$$\frac{D}{D_{11}},$$

whence

$$\bar{\xi}^2 = \frac{D_{11}}{hD}. \qquad (3)$$

Also by integrating for $\alpha_2 \ldots \alpha_{n-1}$ only, we reduce Q to

$$Q' = \tfrac{1}{2}\frac{D_{11}}{D_{1221}}\alpha_1^2 + \frac{D_{12}}{D_{1221}}\alpha_1\xi + \tfrac{1}{2}\frac{D_{22}}{D_{1221}}\xi^2.$$

8. Again, we find

$$\left.\begin{aligned} D_{11} &= 2^{n-1}(1+f_1^2+\ldots+f_{n-1}^2)\\ &= 2^{n-1}\Sigma f^2, \quad \text{since } f_n^2 = 1,\\ D_{1221} &= 2^{n-2}(\Sigma f^2 - f_1^2),\\ D_{1123} &= 2^{n-2}f_1f_2,\\ D_{22} &= 2^{n-1}(\Sigma f)^2,\\ D_{12} &= -2^{n-1}f_1\Sigma f.\end{aligned}\right\} \qquad (4)$$

9. We have then for the chance that a_1 and ξ shall lie within assigned limits, whatever $a_2 \dots a_{n-1}$ may be, the expression

$$A\epsilon^{-hQ'}\, da_1\, d\xi,$$

and
$$Q' = \frac{\Sigma f^2}{\Sigma f^2 - f_1^2} a_1^2 - 2\frac{f_1\,\Sigma f}{\Sigma f^2 - f_1^2} a_1 \xi_1 + \frac{(\Sigma f)^2}{\Sigma f^2 - f_1^2} \xi^2,$$

Let us write a for a_1. Then, if for brevity we put

$$\frac{\Sigma f}{\Sigma f^2 - f_1^2} = q, \quad \text{and} \quad f_1 = 1,$$

$$Q' = a^2 - 2qa\xi + q\Sigma f \xi^2.$$

10. Now, if at P $\xi = 0$, and at P', near P, $\xi = \partial\xi$, the chance that at P' a shall lie within assigned limits is $A\epsilon^{-ha^2}(1 + 2ha\partial\xi)$. But, if at P $\xi \neq 0$, let us use, instead of a, β, γ,

$$u = a - \xi, \quad v = \beta - \eta, \quad w = \gamma - \zeta,$$

or
$$a = u + \xi, \quad \beta = v + \eta, \quad \gamma = w + \zeta,$$

and
$$\begin{aligned} Q' &= (u+\xi)^2 - 2q\,(u+\xi)\,\xi + q\Sigma f\,\xi^2 \\ &= u^2 + 2\,(1-q)\,\xi u + (1 - 2q + q\Sigma f)\,\xi^2 \\ &= \phi\,(u, \xi), \text{ suppose.} \end{aligned}$$

If ξ be the value of that function at P, $\xi + \partial\xi$ its value at P', and if $\phi_1(u, \xi)$ be the value of $\phi(u, \xi)$ at P', then

$$\phi_1(u, \xi) - \phi(u, \xi) = \partial\xi \frac{d}{d\xi}\phi(u, \xi).$$

Again, let $\phi(u)\,du$ be the chance that at P u shall have the value $u \dots u + du$, whatever ξ may be, that is,

$$\phi(u) = \int_{-\infty}^{\infty} d\xi\, \phi(u, \xi),$$

and the same chance at P', or $\phi_1(u)$ is

$$\phi_1(u) = \int_{-\infty}^{\infty} d\xi\, \phi(u, \xi) + \partial\xi \int_{-\infty}^{\infty} d\xi \frac{d}{d\xi}\phi(u, \xi).$$

11. Also $$\frac{d}{d\xi}\phi(u,\xi)=\left(\frac{d}{d\xi}\right)\phi(u,\xi)-\frac{d}{du}\phi(u,\xi),$$

if $\left(\frac{d}{d\xi}\right)$ denotes differentiation with u constant. So

$$\phi_1(u)-\phi(u)=\partial\xi\int_{-\infty}^{\infty}d\xi\left(\frac{d}{d\xi}\right)\phi(u,\xi)-\partial\xi\int_{-\infty}^{\infty}d\xi\frac{d}{du}\phi(u,\xi).$$

But $$\int_{-\infty}^{\infty}d\xi\left(\frac{d}{d\xi}\right)\phi(u,\xi)=0,$$

because $\phi(u,\xi)$ vanishes at either limit.

Therefore

$$\begin{aligned}\phi_1(u)-\phi(u)&=-\partial\xi\int_{-\infty}^{\infty}d\xi\frac{d\phi}{du}(u,\xi)\\&=\partial\xi\int_{-\infty}^{\infty}d\xi\,\phi(u,\xi)\left[2hu+2h(1-q)\xi\right],\end{aligned}$$

and this is proportional to

$$\epsilon^{-hu^2}2h'u\partial\xi, \tag{5}$$

with h' nearly equal to h.

12. Let us now conceive the whole space in which our molecules are moving to be divided into cubes whose edges are parallel to the coordinate axes respectively, and of length $2a$. We may suppose $\frac{d\xi}{dz}+\frac{d\zeta}{dx}$ to be the mean value of that function throughout any such cube. Let $\overline{uw}$ be the mean value of uw for all the molecules which at any instant are within any such cube S. Let

$$M=\Sigma\,\overline{uw}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right), \tag{6}$$

the summation being for all such cubes. (See p. 340, Note.)

There are two other functions symmetrical with M, namely,

$$\Sigma\,\overline{uv}\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right),\quad\text{and}\quad\Sigma\,\overline{vw}\left(\frac{d\eta}{dz}+\frac{d\zeta}{dy}\right).$$

It is sufficient to deal with M.

13. In stationary motion M is invariable with the time, or

$$\frac{dM}{dt} = 0 = \Sigma \overline{uw} \frac{d}{dt}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) + \Sigma \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\frac{d}{dt}\overline{uw}. \tag{7}$$

And $\overline{uw}$ now varies with the time for the cubical spaces under consideration by the passage of molecules into or out of them. That is, generally,

$$\frac{d}{dt}\overline{uw} \neq 0.$$

We proceed to prove that in the medium of material points $\frac{d}{dt}\overline{uw}$ is of the opposite sign to, and proportional to, $\frac{d\xi}{dz} + \frac{d\zeta}{dx}$, so that the second term in $\frac{dM}{dt}$ consists of terms of the form

$$-E\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2,$$

where E is a positive constant.

14. Consider the cubical space S bounded by the planes

$$x = a, \quad x = -a,$$

$$y = a, \quad y = -a,$$

$$z = a, \quad z = -a.$$

The origin is then at the centre of S. Now during the short time dt $\overline{uw}$ is not on average affected by the motion of any molecules which during the whole of that time remain within S, because, there being no collisions, α, β, γ remain unchanged with the time for every individual molecule, and ξ is as likely to be increased as diminished. The function $\overline{uw}$ can vary during dt only by the passage of molecules out of and into S.

Further, under the circumstances supposed, ξ varying only in z, and ζ only in x, uw cannot vary on average by the passage of molecules out of or into S, through the faces parallel to xz, but only through the faces parallel to xy and yz.

Let ξ be the value of that function at the centre of S. Then, a being small, the value on the plane $z = a$ is $\xi + a\frac{d\xi}{dz}$. Similarly, on the plane $x = a$, ζ becomes $\zeta + a\frac{d\zeta}{dx}$. And on the plane $z = a$, given that ξ has the above value, the number per unit of volume of molecules for which u, v, w lie within the limits $u \ldots u + du$, &c., that is, the number of the class $(u\,v\,w)$ is, by Art. 11,

$$\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(1+2hua\frac{d\xi}{dz}\right),\text{ by (5),}$$

where A is a known constant, and h is written for h' of Art. 11.

If γ be negative, they enter the space S, if γ be positive, they pass out of the space S, at a rate per unit of area and time

$$\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(1+2hua\frac{d\xi}{dz}\right)\gamma,$$

that is, on average

$$\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(1+2hua\frac{d\xi}{dz}\right)w,$$

and the rate of entry or exit for the whole face of cube S on the plane $z=a$ is

$$4a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(1+2hua\frac{d\xi}{dz}\right)w.$$

Therefore the class $(u\,v\,w)$ is increased in number per unit of time by the passage of molecules through the plane $z=a$ by the quantity

$$-4a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(w+2huwa\frac{d\xi}{dz}\right).$$

For the plane $z=-a$, we change the sign of a and w, molecules now entering S with positive w, and the corresponding expression is

$$4a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\left(w-2huwa\frac{d\xi}{dz}\right).$$

For both faces the increase in number of the same class per unit of time is

$$-8a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\,.\,2huwa\,\frac{d\xi}{dz}.$$

By symmetry between x and z the same class is increased in number per unit of time by the passage of molecules through the faces $x=a$ and $x=-a$, by

$$-8a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\,.\,2huwa\,\frac{d\zeta}{dx}.$$

If therefore $f(u\,v\,w)\,du\,dv\,dw$ be the number of the class $(u\,v\,w)$ within S,

$$\frac{d}{dt}f(u\,v\,w)=-8a^2\rho A\epsilon^{-h(u^2+v^2+w^2)}\,2huwa\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right). \qquad (8)$$

But
$$\overline{uw} = \iiint_{-\infty}^{\infty} uwf(u\,v\,w)\,du\,dv\,dw \div 8a^3\rho\,;$$
and therefore
$$\frac{d}{dt}\overline{uw} = \iiint_{-\infty}^{\infty} uw\frac{d}{dt}f\,(u\,v\,w)\,du\,dv\,dw \div 8a^3\rho$$
$$= -A\iiint_{-\infty}^{\infty} \epsilon^{-h(u^2+v^2+w^2)}\,u^2w^2\,du\,dv\,dw\,.\,2h\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)$$
$$= -\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right);$$
and therefore
$$\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)\frac{d}{dt}\overline{uw} = -\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2. \tag{9}$$

It will be observed that $\frac{d}{dt}(\overline{uw})$, thus considered, is proportional to the rate $(-w)$ at which positive momentum, denoted by u, is transferred across the plane $z = a$, in direction of negative z, or at which positive momentum in direction z is transferred across the plane $x = a$, in direction of negative x. So that, if by any change in the nature of the molecule that rate of transfer were increased, $\frac{d}{dt}\overline{uw}$ would be increased in the same proportion. See *post*, Art. 16.

15. We have then in stationary motion of the system of material points
$$\frac{dM}{dt} = 0,$$
$$= \Sigma\,\overline{uw}\,\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right) - \Sigma\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2. \tag{I.}$$

It is not necessary to evaluate the first term, but it is important for the purposes of Art. 23, *post*, to observe its form. The function $\frac{d\xi}{dz}\,\partial z$ at P is the difference between the value of ξ at P and its value at a point distant ∂z from P in direction z. Therefore
$$\Sigma f\frac{d\xi}{dz} = -\Sigma a\,\frac{df}{dz}$$
$$= -\Sigma a\,\frac{z}{r}\,\frac{df}{dr},$$

Consider a molecule with component velocities α, β, γ, and about it a sphere of radius r (a little greater than a of Art. 4). Let xyz and xyz' be two points on that sphere, and $z' = -z$. Let ξ, f relate to xyz, and ξ', f' to xyz'. Then, integrating throughout the sphere, we have

$$\iiint dx\,dy\,dz\,\frac{d\xi}{dz} = \iint dx\,dy\,(\xi - \xi'),$$

and if $\frac{d}{dt}$ denote the change due to the motion of the molecule, and Ω be the volume of the sphere,

$$\Omega\,\frac{d}{dt}\,\overline{\frac{d\xi}{dz}} = -\frac{1}{\Sigma f}\iint dx\,dy\left(\alpha\frac{d}{dx} + \beta\frac{d}{dy} + \gamma\frac{d}{dz}\right)\alpha\,(f - f'),$$

whence
$$uw\,\frac{d}{dt}\,\frac{d\xi}{dz} = -\,uw\alpha\gamma\,\frac{1}{\Sigma f}\,\frac{1}{r}\,\frac{df}{dr}$$

$$= -\,u^2w^2\,\frac{1}{\Sigma f}\,\frac{1}{r}\,\frac{df}{dr}$$

on average, and

$$\overline{uw}\,\frac{d}{dt}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) = -\,2\,\overline{u^2w^2}\,\frac{1}{\Sigma f}\,\frac{1}{r}\,\frac{df}{dr}.^{*} \qquad (9a)$$

It will be observed that $\frac{d}{dt}\,\frac{d\xi}{dz}$ thus considered is proportional to the rate (w) at which positive momentum, denoted by u, is transferred across any plane parallel to xy in direction z. And therefore, if by any change in the condition of the molecules that rate of transfer were increased, $\frac{d}{dt}\,\frac{d\xi}{dz}$, and similarly $\frac{d}{dt}\,\frac{d\zeta}{dx}$, would be increased in the same proportion. See Art. 16, *post*.

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* It may be objected that the method of Arts. 14 and 15 is only applicable to a space containing many molecules, whereas our cube S is very small. But we may, in Art. 14, consider an infinite number of such cubes for all of which $\left(\frac{d\xi}{dz} + \frac{d\zeta}{dr}\right)$ has at the instant the same mean value. The method is then applicable to the sum of such spaces, and to each space on average.

16. We now pass to the case in which our molecules, instead of being material points, are elastic spheres, each of unit mass and diameter c. One known effect of this is that the quantity of momentum transferred across any plane per unit of area and time is increased in a certain ratio, $1 : 1+\kappa$, where, in the case of elastic spheres, $\kappa = \frac{2}{3}\pi c^3\rho$. This increases both terms in (I.) in the same ratio, and therefore, so far as this property is concerned, $\frac{dM}{dt}$ remains zero. This effect arises from the usual convention concerning elastic bodies, according to which there is on every collision an instantaneous transfer of momentum through a certain space. It has nothing to do with the change of direction of the relative velocity, which also occurs on collision, and which we shall have to consider. The first effect would take place in precisely the same way, if after every collision the direction of the relative velocity were restored by Maxwell's corps of demons to what it was before collision.

17. Since every collision changes the direction of motion of the colliding spheres, the expression in Art. 14 derived from

$$\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)\frac{d\,\overline{uw}}{dt}$$

contains, in addition to the result calculated in that article, a new term derived from collisions which we have now to calculate.

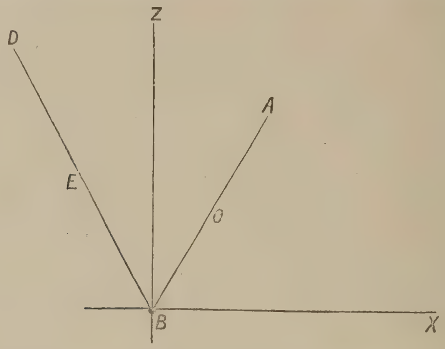
Let V be the half relative velocity of two molecules, λ, μ, ν its direction cosines before collision, λ', μ', ν' after collision. Let $\lambda V = V_x$, $\nu V = V_z$. The number of collisions per unit of volume and time for given V is $2\pi c^2\rho V$. Hence, for the change with the time of V_xV_z due to collisions with given V, we have

$$2\pi c^2\rho V.\,V^2\,(\overline{\lambda'\nu'}-\overline{\lambda\nu}),$$

the bar denoting mean values. We have now to find the values of $\overline{\lambda'\nu'}$ and $\overline{\lambda\nu}$ for all collisions, given V.

At the instant of collision let the centres of the two colliding molecules be A, B; so that AB is the line of centres. Let BD be the relative velocity. Or, if $DE = BE$, one molecule has velocity DE, and the other BE, in addition to the velocity of their common centre of inertia, whatever that may be. Let the angle $DBA = \theta$. If $AO = BO$, O is the point of contact.

Let BX, BZ be the directions of the axes of x and z, and let the



angle between the planes DBA and DBZ be φ, and the angle between the planes DBA and DBX be ϕ'.

Then, if x_0, z_0 be the x and z coordinates of A referred to O as origin, and if the direction DE be that of $\lambda\mu\nu$,

$$\left.\begin{aligned} z_0 &= -\nu\frac{c}{2}\cos\theta + \frac{c}{2}\sqrt{1-\nu^2}\sin\theta\cos\phi, \\ x_0 &= -\lambda\frac{c}{2}\cos\theta + \frac{c}{2}\sqrt{1-\lambda^2}\sin\theta\cos\phi'. \end{aligned}\right\} \qquad (10)$$

The complete definition of ϕ shall be this :—

When the plane DBA, turning round DB, contains the axis of z, $\phi = 0$ for that position of BA which makes the least angle with the axis of positive z. And ϕ' has a corresponding definition for the axis of x (see 20, *post*).

18. If ξ_0, ζ_0 be the values of ξ and ζ at O, then at A

$$\left.\begin{aligned} \xi-\xi_0 &= \frac{c}{2}\left(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi\right)\frac{d\xi}{dz}, \\ \zeta-\zeta_0 &= \frac{c}{2}\left(-\lambda\cos\theta+\sqrt{1-\lambda^2}\sin\theta\cos\phi'\right)\frac{d\zeta}{dx}. \end{aligned}\right\} \qquad (11)$$

Then, by the same reasoning as in Art. 11, the number per unit of volume of pairs of molecules whose components of half relative velocity are $V_x \ldots V_x+dV_x$, $V_y \ldots V_y+dV_y$, and $V_z \ldots V_z+dV_z$ is at A

$$C\epsilon^{-2h(V_x^2+V_y^2+V_z^2)}\,dV_x\,dV_y\,dV_z\left\{1+2h\lambda V\overline{\xi-\xi_0}+2h\nu V\overline{\zeta-\zeta_0}\right\},$$

where C is the usual constant for this case. Hence we find, for the mean value of $V_x V_z$ for all collisions, given V,

$$\overline{V_x V_z} = C\epsilon^{-2hV^2} V^2 dV_x\, dV_y\, dV_z\, \lambda\nu$$

$$\times \int_0^{\frac{1}{2}\pi} 2\sin\theta\cos\theta\, d\theta \int_0^{\pi} d\phi\, \{1 + 2h\lambda V\overline{\xi - \xi_0} + 2h\nu V\overline{\zeta - \zeta_0}\},$$

divided by $$C\epsilon^{-2hV^2} V^2 dV_x\, dV_y\, dV_z,$$

and the general mean values of functions of λ, μ, ν being taken after integration according to θ and ϕ.

That is,

$$\overline{V_x V_z} = V^2\, \overline{\lambda\nu} \int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta$$

$$+ V^3 \frac{d\xi}{dz} \int_0^{\frac{1}{2}\pi} d\theta \int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\, 2h\lambda^2\nu\, \frac{c}{2}\, (-\nu\cos\theta + \sqrt{1-\nu^2}\sin\theta\cos\phi)$$

$$+ V^2 \frac{d\zeta}{dx} \int_0^{\frac{1}{2}\pi} d\theta \int_0^{\pi} d\phi'\, 2\sin\theta\cos\theta\, 2h\nu^2\lambda\, \frac{c}{2}\, (-\lambda\cos\theta + \sqrt{1-\lambda^2}\sin\theta\cos\phi')$$

$$= -\frac{2hc}{3} V^3\, \overline{\lambda^2\nu^2} \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right), \quad \text{because} \quad \overline{\lambda\nu} = 0$$

$$= -\frac{2hc}{3} V^3 \frac{1}{3.5} \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right), \quad \text{because} \quad \overline{\lambda^2\nu^2} = \frac{1}{3.5}. \qquad (12)$$

19. We have thus proved that, on the average of all *colliding pairs* of spheres, $\overline{\lambda\nu}$ or $\overline{V_x V_z}$ is, before collision, of the opposite sign to $\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)$. This proposition does not depend on the result of collisions. It is therefore true if, instead of elastic spheres, the molecules be centres of force, or whatever be their form.

20. We have next to calculate the mean value for all collisions of $\lambda'\nu'$, or the value of $\lambda\nu$ after collision. This will depend on the form of the molecule, and we shall now deal only with elastic spheres. It will be found that $\lambda'\nu'$ is of the same sign with $\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)$.

We use the same notation as before. When the plane *DBA* turns round *DB* until it contains the axis of z, let A_z, A'_z, which are on opposite sides of *DB*, be the two positions in which that plane is cut by the circle which A describes. Let A_z be nearer to the positive axis of z than A'_z. Then, for A_z, $\phi = 0$; for A'_z, $\phi = \pi$. Similarly, when the plane turning round DB contains the axis of x, A_x, A'_x are the two positions of A in that plane, and, if A_x be nearer than A'_x to the positive axis of x, $\phi' = 0$ for A_x, $\phi' = \pi$ for A'_x.

Let ϵ be the angle between the plane of DB and z and the plane of DB and x, so that $\phi' = \phi - \epsilon$, $\phi = \phi' + \epsilon$. It will be seen that, if $\lambda\nu$ is negative, $\cos\epsilon$ is positive; if $\lambda\nu$ is positive, $\cos\epsilon$ is negative.* In either case

$$\cos\epsilon = \frac{-\lambda\nu}{\sqrt{1-\lambda^2}\sqrt{1-\nu^2}};$$

we have then, to find λ' and ν',

$$\left.\begin{aligned}\nu' &= -\nu\cos 2\theta + \sqrt{1-\nu^2}\sin 2\theta\cos\phi,\\ \lambda' &= -\lambda\cos 2\theta + \sqrt{1-\lambda^2}\sin 2\theta\cos\phi';\end{aligned}\right\} \tag{13}$$

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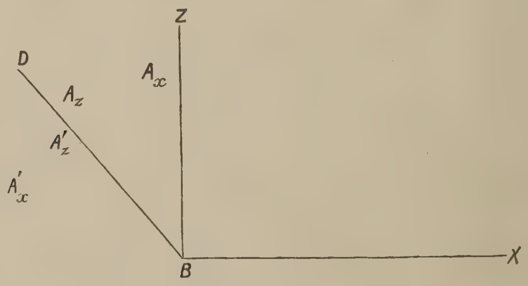
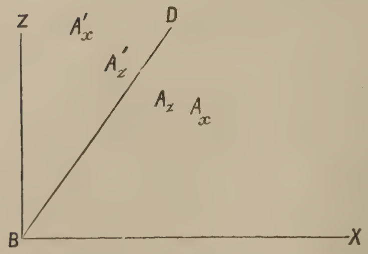
* This is most easily seen by considering the case in which V, or DB, is nearly in the plane of xz. Evidently, if $\lambda\nu$ be negative, the angle between the planes BDA_z and BDA_x is acute with the above definition of A_z and A_x. If $\lambda\nu$ be positive, it is obtuse.





whence we derive the two symmetrical systems, namely :

$$\left.\begin{aligned} \nu' &= -\nu\cos 2\theta + \sqrt{1-\nu^2}\sin 2\theta\cos\phi, \\ \lambda' &= -\lambda\cos 2\theta - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\theta\cos\phi\,; \end{aligned}\right\} \tag{13a}$$

or

$$\left.\begin{aligned} \nu' &= -\nu\cos 2\theta - \frac{\lambda\nu}{\sqrt{1-\lambda^2}}\sin 2\theta\cos\phi', \\ \lambda' &= -\lambda\cos 2\theta + \sqrt{1-\lambda^2}\sin 2\theta\cos\phi'. \end{aligned}\right\} \tag{13b}$$

The term involving $\sin\epsilon$ will disappear in the subsequent integration, and is omitted. The above systems hold for all values of λ and ν.

21. We now find that, for given V,

$$V^2\,\overline{\lambda'\nu'} = V^2\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,(1+2h\lambda V\,\overline{\xi-\xi_0}+2h\nu V\,\overline{\zeta-\zeta_0})\,\lambda'\nu'.$$

But, with the above values of λ', ν',

$$\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\; 2\sin\theta\cos\theta\;\lambda'\nu' = 0,$$

as is easily seen. Therefore

$$\begin{aligned} V^2\,\overline{\lambda'\nu'} &= V^2\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\; 2\sin\theta\cos\theta\,(2h\lambda V\,\overline{\xi-\xi_0}+2h\nu V\,\overline{\zeta-\zeta_0})\,\lambda'\nu' \\ &= \;2hcV^3\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\sin\theta\cos\theta\,\lambda(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi)\frac{d\xi}{dz}\lambda'\nu' \\ &\quad + 2hcV^3\int_0^{\pi} d\theta\int_0^{\pi} d\phi'\sin\theta\cos\theta\,\nu(-\lambda\cos\theta+\sqrt{1-\lambda^2}\sin\theta\cos\phi')\frac{d\zeta}{dx}\lambda'\nu', \end{aligned} \tag{14}$$

by (11). In these expressions we have to substitute the values above found for λ' and ν', and perform the integrations according to θ, and to ϕ or ϕ'. It is convenient in evaluating the first term to use for $\lambda'\nu'$ the expression (13*a*) above, and in evaluating the second term to use the expression (13*b*).

To effect the integration according to ϕ or ϕ', we reject from the product $\cos\phi$ and $\cos^3\phi$, $\cos\phi'$ and $\cos^3\phi'$, and for $\cos^2\phi$ or $\cos^2\phi'$ we write $\frac{1}{2}$. We have then for the first term

$$2hcV^3\int_0^{\frac{1}{2}\pi}d\theta\int_0^{\pi}d\phi\sin\theta\cos\theta\lambda(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi)\lambda'\nu'\,\frac{d\xi}{dz}$$

$$=\quad 2hcV^3\frac{d\xi}{dz}\int_0^{\frac{1}{2}\pi}d\theta\int_0^{\pi}d\phi\sin\theta\cos\theta\lambda\,(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi)$$

$$\times\left(-\nu\cos 2\theta+\sqrt{1-\nu^2}\sin 2\theta\cos\phi\right)\left(-\lambda\cos 2\theta-\frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\theta\cos\phi\right)$$

$$=-2hcV^3\frac{d\xi}{dz}\lambda^2\nu^2\int_0^{\frac{1}{2}\pi}d\theta\sin\theta\cos^2\theta\cos^2 2\theta$$

$$+2hcV^3\frac{d\xi}{dz}\lambda^2\nu^2\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}d\theta\sin\theta\cos^2\theta\sin^2 2\theta$$

$$-2hcV^3\frac{d\xi}{dz}\lambda^2(1-\nu^2)\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}d\theta\sin^2\theta\cos\theta\sin 2\theta\cos 2\theta$$

$$+2hcV^3\frac{d\xi}{dz}\lambda^2\nu^2\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}d\theta\sin^2\theta\cos\theta\sin 2\theta\cos 2\theta$$

$$=\quad 2hcV^3\frac{d\xi}{dz}\,\frac{-11+12-4}{3.5.7}\lambda^2\nu^2$$

$$+2hcV^3\frac{d\xi}{dz}\,\frac{2}{3.5.7}\lambda^2.$$

But $\overline{\lambda^2}=\frac{1}{3},\quad\overline{\lambda^2\nu^2}=\frac{1}{3.5}$;

therefore $\overline{\lambda^2}=5\overline{\lambda^2\nu^2}$,

and our expression becomes

$$2hcV^3\frac{d\xi}{dz}\,\frac{-11+12-4+10}{3.5.7}\,\overline{\lambda^2\nu^2},$$

or

$$\frac{1}{3.5}\,2hcV^3\frac{d\xi}{dz}\,\overline{\lambda^2\nu^2}.$$

By symmetry, the second term gives

$$\frac{1}{3.5}\,2hcV^3\frac{d\zeta}{dx}\,\overline{\lambda^2\nu^2};$$

and so, given V,

$$V^2\,\overline{\lambda'\nu'}=\overline{V_x'V_z'}=\frac{1}{3.5}\,2hcV^3\,\overline{\lambda^2\nu^2}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)$$

$$=\frac{1}{3.5}\,2hcV^3\,\frac{1}{3.5}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right),\qquad(15)$$

since
$$\overline{\lambda^2\nu^2} = \frac{1}{3.5};$$

and
$$V^2\left(\overline{\lambda'\nu'} - \overline{\lambda\nu}\right) = 2hcV^3\frac{1}{3.5}\left(\frac{1}{3} + \frac{1}{3.5}\right)\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right). \tag{16}$$

21*a*. To find the rate of change with the time of $\overline{V_xV_z}$, given V, we multiply the last expression by the number of collisions which take place with given V per unit of volume and time, that is, by $2\pi c^2\rho V\cdot$ The result is that, given V,

$$\frac{d}{dt}\overline{V_xV_z} = 4h\pi c^3\rho V^4\frac{1}{3.5}\left(\frac{1}{3} + \frac{1}{3.5}\right)\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right),$$

and, integrating for all values of V,

$$\frac{d}{dt}\overline{V_xV_z} = 4h\pi c^3\rho C\int_0^\infty dV\epsilon^{-2hV^2}V^6\frac{1}{3.5}\left(\frac{1}{3} + \frac{1}{3.5}\right)\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)$$
$$= \frac{1}{2h}\pi c^3\rho\left(\frac{1}{3} + \frac{1}{3.5}\right)\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right). \tag{17}$$

22. This being the value of $\frac{d}{dt}\overline{V_xV_z}$, what is the value of $\frac{d}{dt}\overline{uw}$? Let u, v, w and u', v', w' be the component velocities of two colliding molecules, and, V being their half relative velocity, let U be the velocity of their common centre of gravity. Using V_x, V_y, V_z for the components of V, and U_x, U_y, U_z for the components of U, we have

$$u = V_x + U_x, \quad u' = -V_x + U_x,$$
$$v = V_y + U_y, \quad v' = -V_y + U_y,$$
$$w = V_z + U_z, \quad w' = -V_z + U_z;$$

therefore
$$uw + u'w' = 2\left(V_xV_z + U_xU_z\right).$$

And, as the change due to collision,

$$\partial(uw) + \partial(u'w') = 2\partial V_xV_z,$$

because U_x, U_y, and U_z are not altered by collision. It follows that on average of all molecules

$$\frac{d}{dt}\overline{V_xV_z} = \frac{d}{dt}uw,$$

and we may now write, as the result of collision,

$$\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\frac{d}{dt}\overline{uw} = \pi c^3\rho\frac{1}{2h}\left(\frac{1}{3} + \frac{1}{3.5}\right)\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2. \tag{18}$$

23. We have next to consider the change in the term $uw\dfrac{d}{dt}\dfrac{d\xi}{dz}$ due to the variation of α by collisions. Consider two spheres in collision, the origin being the point of contact, and use the construction employed in Art. 15. The ordinates of their centres are z and $-z$, where

$$z = \frac{c}{2}\left(-\nu\cos\theta + \sqrt{1-\nu^2}\sin\theta\cos\phi\right).$$

For one α is increased, and for the other diminished, by $(\lambda'-\lambda)V$. Substituting for $\lambda'-\lambda$, introducing the factor $2\pi c^2\rho V$, and integrating, we find that for each of the two molecules, as the result of collisions,

$$uw\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right) = -\frac{16}{15}\pi c^3\rho uw\lambda\nu V^2\frac{1}{\Sigma f}\frac{1}{r}\frac{df}{dr}$$

$$= -\frac{4}{15}\pi c^3\rho\, 2u^2w^2\frac{1}{\Sigma f}\frac{1}{r}\frac{df}{dr}$$

(because $\lambda\nu V^2 = \tfrac{1}{2}uw$ on average)

$$= \frac{4}{3\,.\,5}\pi c^3\rho\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2, \text{ by Art. 15,} \qquad (19)$$

and, adding 19 to 18, we find for the change in $\dfrac{dM}{dt}$ due to collision

$$\frac{\partial M}{\partial t} = \tfrac{2}{3}\pi c^3\rho\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2.$$

We interpret this as follows:—*As the result of collisions, molecules moving in a given direction, say x, are on average deflected towards that side where the average motion of the spheres, i.e., the stream, is for the time being in x. And so collisions tend to increase the stream.*

24. Now, referring to equation (I.), we see that $\dfrac{dM}{dt}$, which in the medium of material points was zero, has, when the spheres have finite diameter, become positive, namely,

$$\frac{dM}{dt} = \Sigma A\iiint \epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\,.\,uw\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right) - \Sigma\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2$$

$$+\Sigma\tfrac{2}{3}\pi c^3\rho\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2, \qquad \text{(II.)}$$

of which the first two terms are together zero, as before. Therefore, if in the system of finite spheres $\left(\frac{d\xi}{dz}\right)^2$, &c., have the same mean values as in the system of material points, $\frac{dM}{dt}$ is positive. But in stationary motion $\frac{dM}{dt}$ must be zero. How can we make it zero?

25. The solution is this. We must admit that the motion is not stationary if ξ, η, ζ, and therefore also $\frac{d\xi}{dz}+\frac{d\zeta}{dx}$, have the same average values which they have in the system of material points for which equation (I.) was proved. But in the system of finite molecules ξ has become $\xi+\xi'$, &c., and $\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2$ has become

$$\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2+\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2.$$

That gives, by arranging the terms,

$$\frac{dM}{dt}=\Sigma A\iiint_{-\infty}^{+\infty}\epsilon^{-h(u^2+v^2+w^2)}\,du\,dv\,dw\,.\,uw\,\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)-\Sigma\frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2$$
$$+\Sigma\tfrac{2}{3}\pi c^2\rho\,\frac{1}{2h}\left\{\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2+\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2\right\}-\Sigma\frac{1}{2h}\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2. \tag{III.}$$

The first two are together zero by (I.)

The remaining two terms can be made zero by suitably choosing the ratio

$$\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2:\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2.$$

We have, namely,

$$\kappa\left\{\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2+\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2\right\}=\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2,$$

or, if κ^2, κ^3, &c., are neglected in comparison with κ,

$$\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)^2:\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2::\kappa:1. \tag{20}$$

If we had introduced the factor $1+\kappa$ into the first two terms of (III.), according to Art. 16, it would not affect the result, so long as κ^2, &c., are neglected.

The term $uw\frac{d}{dt}\left(\frac{d\xi'}{dz}+\frac{d\zeta'}{dx}\right)$ is zero, because, as we shall see, Art. 32, ξ', η', ζ' are independent of u, v, w.

26. It will be observed that the ratio so found is independent of the form of the functions ξ, η, ζ, that is, of the form of the function f of Art. 4. In order to confirm this result, let us consider the relation between ξ', η', ζ', and the coefficients b of the products of velocities of different molecules in the index of ϵ^{-hQ}.

27. By way of illustration let us suppose n molecules within a sphere of radius R, their velocities being distributed according to Maxwell's law ϵ^{-hQ}, where Q is the sum of squares of the velocities, and let us impart to each molecule, in addition to the velocity which it has in that motion, the small velocity X' in x.

This being done, the chance that one of these molecules shall have velocity represented by lines drawn from the origin to some point or other within an element of volume $d\Omega$, whose centre is distant p from the origin in direction making the angle α with the axis of x, is

$$A\epsilon^{-hp^2}(1+2hpX'\cos\alpha)\,d\Omega.$$

The chance that another molecule shall have velocity represented by lines drawn from the origin to some point or other with an element of volume $d\Omega'$ distant q from the origin, in direction making the angle ϕ with p, is

$$A\epsilon^{-hq^2}d\Omega'\,(1+2hqX'\cos\alpha\,\cos\phi+\sin\alpha\,\sin\phi\,\cos\epsilon),$$

where ϵ is the angle between the plane of p and q and the plane of p and x. The chance that both molecules shall have the velocities aforesaid is

$$A^2\epsilon^{-h(p^2+q^2)}d\Omega\,d\Omega'(1+2hpX'\cos\alpha)(1+2hqX'\cos\alpha\,\cos\phi+\sin\alpha\sin\phi\,\sin\epsilon).$$

Multiply this by $\sin\alpha\,d\alpha$, and integrate for all values of ϵ and α, and we find for the chance

$$A^2\epsilon^{-h(p^2+q^2)}\,d\Omega\,d\Omega'\left(1+\frac{4h^2}{3}X'^2pq\cos\phi\right),$$

expressing now the chance that the velocities p and q, of two molecules, in whatever direction either is, shall make the angle ϕ with each other. Now let u, v, w be the components of p; u', v', w' those of q. Then

$$p^2+q^2=u^2+v^2+w^2+u'^2+v'^2+w'^2,$$

$$d\Omega\,d\Omega'=du\,dv\,dw\,du'\,dv'\,dw',$$

$$pq\cos\phi=uu'+vv'+ww'.$$

Therefore, since, with these substitutions, the expression does not contain ϕ, it gives the chance that two molecules, both within the R sphere, shall respectively have velocities $u \dots u+du$, &c., in the form

$$A^2\epsilon^{-h(u^2+v^2+w^2+u'^2+v'^2+w'^2)}\,du \dots dw' \left\{1+\frac{4h}{3}X'^2\,(uu'+vv'+ww')\right\},$$

that is, assuming X'^2, &c., to be small,

$$A^2\epsilon^{-hQ'}du \dots dw',$$

in which $Q' = u^2+v^2+w^2+u'^2+v'^2+w'^2-\frac{4h}{3}X'^2(uu'+vv'+ww')$.

Evidently, if we take Y' and Z' into account, we shall have to substitute $X'^2+Y'^2+Z'^2$ for X'^2 in the last expression.

We see then that in the case now treated the coefficient of correlation has the form

$$b = -\frac{4h}{3}\,(X'^2+Y'^2+Z'^2), \qquad (21)$$

or, since

$$\overline{X^2}+\overline{Y^2}+\overline{Z^2} = \frac{3}{2nh},$$

X, Y, Z denoting the velocities of the centre of inertia of the n molecules within the R sphere,

$$b = -\frac{2}{n}\,(X'^2+Y'^2+Z'^2)/(\overline{X^2}+\overline{Y^2}+\overline{Z^2}).$$

28. Let us now employ the converse method, namely, to find ξ'^2, &c., in terms of the coefficients b, supposed given. We assumed, in Art. 4,

$$\xi = \frac{\Sigma af}{\Sigma f},$$

whence

$$\frac{d\xi}{dz} = \Sigma a\,\frac{1}{\Sigma f}\,\frac{df}{dz}.$$

Let us form the determinant

$$D = \begin{vmatrix} 2\,(\Sigma f)^2 & -2f_1\Sigma f & -2f_2\Sigma f & \dots \\ -2f_1\Sigma f & (2+2f_1^2) & 2f_1f_2 & \dots \\ -2f_2\Sigma f & 2f_1f_2 & (2+2f_2^2) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

Then, as we found in Art. 7,

$$\overline{\xi^2} = \frac{D_{11}}{hD}.$$

29. In the system of finite spheres, if we assume b to have the same value for every pair of molecules considered (which we have to do if their relative positions are unknown), we find that every constituent of D_{11} is, owing to the introduction of b, multiplied by $\frac{2-b}{2}$. And therefore, b being very small, D_{11} becomes

$$D_{11} - \frac{n-1}{2} bD_{11}.$$

And so $$\overline{\xi^2} + \overline{\xi'^2} = D_{11}/hD - \frac{n-1}{2} bD_{11}/hD$$

$$= \xi^2 - \frac{n-1}{2} bD_{11}/hD.$$

Therefore $$\overline{\xi'^2} = -\frac{n-1}{2} bD_{11}/hD$$

$$= \frac{n-1}{4n} b,$$

whence $$\overline{\xi'^2} + \overline{\eta'^2} + \overline{\zeta'^2} = -\frac{3}{4h} b,$$

n being large, or $$b = -\frac{4h}{3} (\overline{\xi'^2} + \overline{\eta'^2} + \overline{\zeta'^2}), \tag{22}$$

which agrees with Art. 27.

30. Next, let us assume b, the coefficient of correlation between the velocities of two molecules, to be a function of the distance r between the molecules to which it relates. And let it be required to find, on this hypothesis, the ratio

$$\overline{\xi'^2} : \overline{\xi^2}, \quad \text{or} \quad \left(\frac{d\xi'}{dz}\right)^2 : \left(\frac{d\xi}{dz}\right)^2.$$

For the system of finite spheres we shall have

$$Q = \Sigma a^2 + \Sigma\Sigma b_{pq} a_p a_q,$$

where b_{pq} is the value of b for the two molecules whose velocities are

α_p, α_q, &c., whence, substituting for α_n as in Art. 7,

$$Q = (1+f_1^2-b_{1n}f_1)\,\alpha_1^2+(1+f_2^2-b_{2n}f_2)\,\alpha_2^2+\&c.$$

$$+(2f_1f_2+b_{12}-b_{1n}f_2-b_{2n}f_1)\,\alpha\alpha_2+\&c. \tag{23}$$

Then $$\overline{\xi^2}+\overline{\xi'^2} = \frac{D_{11}+\partial D_{11}}{hD} = \xi^2+\Sigma b\,\frac{dD_{11}}{db}\Big/hD,$$

or $$\overline{\xi'^2} = \Sigma b\,\frac{dD_{11}}{db}\Big/hD,$$

b being very small, so that its square and higher powers may be neglected, and, for the same reason, the variation of D may be neglected, as will be easily found.

Now, for the coefficient b_{pq},

$$\frac{dD_{11}}{db} = 2D_{11pq} = 2^{n-1}f_pf_q \text{ (by Art. 8),}$$

and so $$\xi'^2 = \frac{1}{hD}\,2^{n-1}\,\Sigma\Sigma b_{pq}f_pf_q$$

$$= \frac{1}{hD}\,2^{n-1}\,\Sigma f_p\,\Sigma b_{pq}f_q, \tag{24}$$

the summation being for every molecule of the system with every other.

31. Now let us assume

$$b_{pq} = -\tfrac{2}{3}\,\frac{c^5}{r^5}, \tag{25}$$

r being the distance between the two molecules p and q. Then $\Sigma f_p\Sigma b_{pq}f_q$ is represented by the integral

$$\tfrac{2}{3}\rho c^5\iiint dx\,dy\,dz f_p\iiint \frac{f_q}{r^5}\,dx\,dy\,dz$$

$$= \tfrac{2}{3}\rho c^5\iiint dx\,dy\,dz\,f_p\int_c^\infty 2\pi r^2\,dr\,f/r^5,$$

2π being used for 4π because each pair is to be counted once.

It follows that, for all functions satisfying the conditions of Art. 4,

$$\Sigma\Sigma b_{pq} f_p f_q = 2^{n-1}\,\tfrac{2}{3}\pi c^3 \rho\, \Sigma f^2 = \kappa D_{11}, \tag{26}$$

$$\xi'^2 : \xi^2 :: \left(\overline{\frac{d\xi'}{dz}}\right)^2 : \left(\overline{\frac{d\xi}{dz}}\right)^2 :: \kappa : 1, \text{ as in (20)},$$

which is independent of the form of the function f, provided that that function satisfies the conditions of Art. 4.

32. We may therefore now construct an auxiliary system containing the same number of molecules as the given system, each one of the former corresponding to one of the latter, and in which each molecule has at a given instant the same position which in the original system at that instant the corresponding molecule has. Attribute to the molecules in the auxiliary system so placed velocities at haphazard in such way that the mean square of the velocities in x, also in y and z respectively, shall be $\dfrac{1}{2h'}$, where the ratio $\dfrac{h'}{h}$ has to be determined. But the velocities attributed to any molecule in the auxiliary system shall be independent of the velocities which the corresponding molecule has at the given instant in the original system. Further, ξ', η', ζ' shall have for the auxiliary system the same meaning which ξ, η, ζ have for the original system, so that $\xi\xi' = 0$ on average, &c.

Then, since the form of ξ', η', ζ' is indifferent so long as they satisfy the conditions of Art. 4, we may take for ξ', η', ζ' the values of those functions derived from the auxiliary system. And, whatever be their forms, the ratios

$$\overline{\xi'^2}/\overline{\xi^2} = h/h' = \left(\overline{\frac{d\xi'}{dz}}\right)^2 \Big/ \left(\overline{\frac{d\xi}{dz}}\right)^2,$$

and are the same for all functions ξ which satisfy the conditions.

Further, as a limiting case, we might take for ξ, η, ζ the component velocities of the common centre of inertia of any group of n molecules contained within a sphere of radius R: let them be X, Y, Z.

33. This, then, is the solution of the problem of stationary motion when the molecules are elastic spheres with finite diameter. Any group of contiguous spheres at any instant have greater energy of

motion of their common centre of inertia, and correspondingly less of relative motion, than they would have in the Maxwellian system as mere material particles, the entire kinetic energy being unchanged. The ratio $\xi'^2 : \xi^2$ or $X'^2 : X^2$ is, as we have seen, independent of n, the number of molecules in the group chosen.

34. Now, if Q be a sum of squares only, the energy of the motion of the common centre of inertia of the n spheres can on average have no other value than $\frac{1}{2}\,\frac{3}{2h}$, that is,

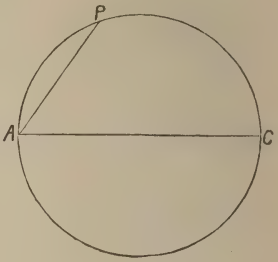
$$X^2+Y^2+Z^2 = \frac{1}{n}\,\frac{3}{2h}.$$

But we have seen that, in the system of finite spheres, $X^2+Y^3+Z^2$ has become $X^2+X'^2+Y^2+Y'^2+Z^2+Z'^2$, and is therefore greater than $\frac{1}{n}\,\frac{3}{2h}$. Therefore, in the system of finite spheres, Q cannot be a sum of squares only—to prove which was my main object.

35. In the proof above given (Art. 31) I assumed $b = -\frac{2}{3}\,\frac{c^5}{r^5}$. For that purpose, however, $\frac{c^4}{r^4}$ would answer equally well. There is, however, another test to which we may subject the form of b chosen. It appears, namely, that, for a group of n molecules within a sphere of radius R, b, if it have the same value for every pair of molecules, varies as $\frac{1}{n}$, by Art. 29. Now the number of pairs that can be formed of n molecules is $\frac{n.\overline{n-1}}{2}$. It follows that the sum of the coefficients b for every pair of molecules within the R sphere ought to vary as $n-1$, that is, n being very great, as n, that is, to vary as R^3, and therefore, if S be that sum, $\frac{dS}{dR} \propto R^2$. But now, if R becomes $R+dR$, S is altered only as follows, namely: for every molecule in the shell $4\pi R^2\,dR$, we form a pair with every molecule within the sphere. The number of molecules within the shell varies as $4\pi R^2\,dR$.

The sum of the number of pairs formed between any one of these and the molecules within the sphere is represented by the integral

$$2\pi\int_c^{2R} r^2 \operatorname{versin} PAC\,dr,$$



if $AP = r, \quad AC = 2R,$

that is, $$\int_c^{2R} dr\, r^2 \left(1 - \frac{r}{2R}\right).$$

Hence the sum of the coefficients b for all such pairs is, if $b = f(r)$,

$$\int_c^{2R} dr\, r^2 f(r) \left(1 - \frac{r}{2R}\right).$$

And therefore $$\frac{dS}{dR} \propto 4\pi R^2 \int_c^{2R} dr\, r^2 f(r) \left(1 - \frac{r}{2R}\right),$$

but it ought to vary as $4\pi R^2$. Therefore b, or $f(r)$, must be so chosen that

$$\int_c^{2R} dr\, r^2 f(r) \left(1 - \frac{r}{2R}\right)$$

shall be independent of R. Now, if

$$f(r) = \frac{c^4}{r^4},$$

the integral contains $\log R$.

But, if $$f(r) = -\tfrac{2}{3}\frac{c^5}{r^5},$$

the integral is proportional to

$$\tfrac{1}{2}c^3 \left(1 - \frac{c}{R}\right),$$

which for large values of R will answer the purpose.

36. As stated in the beginning of this paper, the law of distribution of velocities denoted by ϵ^{-hQ} can hold good only so long as Q is necessarily positive, that is, incapable of being made negative, whatever values are attributed to the velocities u, v, w; that is, so long as the determinant

$$D = \begin{matrix} 2 & b_{12} & b_{13} \\ b_{12} & 2 & b_{23} \\ \cdots & \cdots & \cdots \end{matrix}$$

and each of its coaxial minors is positive. The limit is when $D = 0$. Now, in case of elastic spheres, b, and therefore D, appears as a function of the density only. That is because, in the case of elastic spheres, the change of direction of the relative velocity due to collision, and therefore $\lambda'\nu'-\lambda\nu$ of Art. 20, depends only on θ and ϕ, and not on the magnitude of V.

In the case of the molecules being centres of repulsive force, let c be the distance at which when two molecules approach each other that force first becomes sensible, and call c the diameter of a molecule, and define θ, ϕ as in Art. 17. It is then easy to see that the change of direction of V between the beginning and end of an encounter depends on V as well as on θ and ϕ; whence we may infer that, in this and in all cases where finite forces act between molecules, b and D are functions of h, *i.e.*, of the temperature, as well as of the density. It seems reasonable to expect that $D = 0$, the point at which the mathematical formula ceases to be applicable, would, if we could solve the problem, express that relation between density and temperature at which the gas changes its form and becomes liquid.

Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holohedrically Isomorphic with the Symmetric and the Alternating Substitution-Groups on k Letters. By ELIAKIM HASTINGS MOORE, of Chicago.

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Introduction.

Every abstract group may be defined by a system of generators which are compounded in accordance with a table of generational relations. This fundamental theorem is due to Cayley.

So far as I know, no such abstract generational definitions of the groups abstractly equivalent respectively to the symmetric and the alternating substitution-groups on k letters have been given. Of course from any one such definition many other such definitions may easily be deduced.

In this paper I prove the following theorems:—

Theorem A.—The abstract group $G(k)$ $(k \geq 2)$ defined by the $k-1$ generators

(1) $$B_d \qquad (d = 1, 2, \ldots, k-1),$$

with the generational relations

(2) $$B_d^2 = 1 \qquad (d = 1, 2, \ldots, k-1),$$

(3) $$(B_d B_{d+1})^3 = 1 \qquad (d = 1, 2 \ldots, k-2),$$

(4) $$(B_d B_e)^2 = 1 \qquad \left(\begin{matrix} d = 1, 2, \ldots\ldots\ldots\ldots, k-3 \\ e = d+2, d+3, \ldots, k-1 \end{matrix}\right)$$

[whence follow at once

(5) $$(B_{d+1} B_d)^3 = 1, \quad B_d B_{d+1} B_d = B_{d+1} B_d B_{d+1},$$

(6) $$(B_e B_d)^2 = 1, \qquad B_d B_e = B_e B_d],$$

has the order $O(k) = k!$, and is holohedrically isomorphic with the symmetric substitution-group on k letters.

Theorem A'.—The abstract group $G'(k)$ $(k \geqq 3)$ defined by the two generators

(7) $$B, C,$$

with the generational relations

(8) $$B^2 = 1, \quad C^k = 1, \quad (BC)^{k-1} = 1,$$

(9) $$(BC^{\mp 1} B C^{\pm 1})^3 = 1,$$

(10) $$(BC^{\mp l} B C^{\pm l})^2 = 1 \qquad (l = 2, 3, \ldots, k-2)$$

[where, however, every relation with lower sign is a consequence of the corresponding relation with upper sign], is the group $G(k)$ of Theorem A.

Theorem B.—The abstract group $G\{k\}$ $(k \geqq 3)$ defined by the $k-2$ generators

(11) $$E_d \qquad (d = 1, 2, \ldots, k-2),$$

with the generational relations

(12) $$E_1^3 = 1, \; E_d^2 = 1 \quad (d = 2, 3, \ldots, k-2),$$

(13) $$(E_d E_{d+1})^3 = 1 \quad (d = 1, 2, \ldots, k-3),$$

(14) $$(E_d E_e)^2 = 1 \quad \left(\begin{matrix} d = 1, 2, \ldots\ldots\ldots\ldots, k-4 \\ e = d+2, d+3, \ldots, k-2 \end{matrix}\right),$$

has the order $O\{k\} = \frac{1}{2}k!$, and is holohedrically isomorphic with the alternating substitution-group on k letters.

[*Addition, June 28th*, 1897.—If two groups G, G' are generated respectively by the systems of generators $\{A_1, \dots, A_h\}$, $\{A'_1, \dots, A'_h\}$, and if the generators A' satisfy by the correspondence

$$A_i \sim A'_i \qquad (i = 1, 2, \dots, h)$$

all the generational relations of the group G, then this correspondence determines an isomorphism between the two groups G, G'. In this isomorphism every element of G corresponds to one element of G'. The identity-element of G' corresponds to the elements of a self-conjugate sub-group H of G. The group G' is holohedrically isomorphic to the quotient-group G/H.

Whence, on the basis of known theorems of the theory of substitution-groups:

Theorem C.—If the generators of a group G of order greater than $\begin{matrix}2;\ 6\\ 1;\ 3\end{matrix}$ satisfy at least the generational relations of any generational determination of the abstract $\begin{matrix}\text{symmetric}\\ \text{alternating}\end{matrix}$ group of degree k for $\begin{matrix}k \neq 4;\ k = 4\\ k \neq 4;\ k = 4\end{matrix}$, then G is a form of the abstract $\begin{matrix}\text{symmetric}\\ \text{alternating}\end{matrix}$ group, and its generators satisfy no further generational relations.]

1. *Four Lemmas—Theorems in the General Theory of Abstract Groups.*

Lemma A.—A group G contains two sub-groups K_m and L_n. K_m of order m with the m elements $k_1, \dots, k_m$ is generated by the g generators $\kappa_1, \dots, \kappa_g$. L_n of order n with the n elements $l_1, \dots, l_n$ is generated by the h generators $\lambda_1, \dots, \lambda_h$. If now every product $l_s \kappa_p \begin{pmatrix} s = 1, \dots, n \\ p = 1, \dots, g \end{pmatrix}$ can be written as a product $k_i l_j$, or, if every product $\lambda_q k_r \begin{pmatrix} q = 1, \dots, h \\ r = 1, \dots, m \end{pmatrix}$ can be so written, then the sub-groups K_m and L_m of G determine as sub-group H of lowest order containing them, the group H *of order at most* mn, whose elements are the distinct ones of the mn products $k_i l_j \begin{pmatrix} i = 1, \dots, m \\ j = 1, \dots, n \end{pmatrix}$.

Lemma B.—If the group G_n of order n with the n elements $g_1, \dots, g_n$ has an involutoric holohedric isomorphism with itself, viz.,

$$(g_1, \dots, g_i, \dots, g_n) \sim (g_{s_1}, \dots, g_{s_i}, \dots, g_{s_n})$$

$$(s_{s_i} = i,\ i = 1, \dots, n),$$

where $(s_1, \ldots, s_i, \ldots, s_n)$ is a certain permutation of the indices $(1, \ldots, i, \ldots, n)$, then, by the adjunction of a new generator ϵ with the generational relations

$$\epsilon^2 = 1, \quad g_i \epsilon = \epsilon g_{s_i} \qquad (i = 1, \ldots, n),$$

the group G_n of order n is extended to a group G'_{2n} of order $2n$ with the elements

$$g_i, \; g_i \epsilon \qquad (i = 1, \ldots, n).$$

Lemma C.—The group G_n of order n is generated by the f generators $\gamma_1, \ldots, \gamma_f$, which are subject to the generational relations Σ. Every relation Σ is taken in the convenient form

$$\gamma_{i_1}^{e_1} \gamma_{i_2}^{e_2} \cdots \gamma_{i_t}^{e_t} = 1.$$

If the relations Σ' obtained from the relations Σ by changing in every relation the sign of every exponent $e_1, \ldots, e_t$ are group-theoretic consequences of the relations Σ, then the correspondence of generators $(\gamma_1, \ldots, \gamma_f) \sim (\gamma_1^{-1}, \ldots, \gamma_f^{-1})$ defines an involutoric holohedric isomorphism of the group G_n with itself.

Lemma D.—If the generational relations Σ of the group G_n of Lemma C are all of the form

$$\gamma_i^{c_i} = 1, \quad (\gamma_i^a \gamma_j^b)^{c_{i,j;\,a,b}} = 1,$$

then the relations Σ' follow from the relations Σ.

These lemmas may be easily proved. A is generally known. B is the simplest case of a general theorem given by Mr. Hölder for use in the construction of composite groups ("Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4," *Mathematische Annalen*, Vol. XLIII., pp. 301–412, 1893. See p. 329).

2. *Proof of Theorem* A.

We consider the group $G(k)$ generated by the $k-1$ generators $B_1, \ldots, B_{k-1}$ (1) under the fundamental relations (2, 3, 4), and the derivative relations (5, 6).

The symmetric substitution-group $G_{k!}^k$ on k letters $l_1, \ldots, l_k$ is generated by the $k-1$ transpositional substitutions

(15) $$S_d = (l_d l_{d+1}) \qquad (d = 1, 2, \ldots, k-1).$$

These generators S_d of the substitution-group $G_{k!}^k$ satisfy the relations (2, 3, 4) prescribed for the generators B_d of our abstract group

$G(k)$ (and conceivably, though not actually, other relations not therefrom derivable). Hence, the order $O(k)$ of $G(k)$ is at least $k!$, and, further, if the order $O(k)$ is exactly $k!$, then $G(k)$ is holohedrically isomorphic with $G^k_{k!}$.

That $G(k)$ has the order $O(k) = k!$ is proved by a one-based induction; it is obviously true for $k = 2$; we prove that, if it is true for $k = m$, then it is true for $k = m+1$.

The group $G(m+1)$ with the generators $B_1, \dots, B_{m-1}, B_m$ has the order $O(m+1) \geqq (m+1)!$. $G(m+1)$ contains the sub-group

$$K_{m!} = G(m),$$

with the generators $B_1, \dots, B_{m-1}$, and the $m!$ elements $k_1, \dots, k_{m!}$ ($m!$, since its order is by hypothesis $m!$).

Now the substitution-group $G^m_{m!}$ on m letters $l_1, \dots, l_m$ is most conveniently extended to the substitution-group $G^{m+1}_{(m+1)!}$ on $m+1$ letters $l_1, \dots, l_m, l_{m+1}$ by the cyclic substitution

$$T = (l_1 l_2 \dots l_m l_{m+1}),$$

which has the period $m+1$, and which is expressed in terms of the generators $S_1, \dots, S_m$ (15) by the formula

$$T = S_m S_{m-1} \dots S_2 S_1. \tag{16}$$

We extend our abstract group $K_{m!} = G(m)$ to $G(m+1)$ by the extending generator B_m, or, what amounts to the same thing, by the extender

$$C = B_m B_{m-1} \dots B_2 B_1. \tag{17}$$

The relations of C to the m generators of $G(m+1)$ and the period of C (which is surely $\geqq m+1$) must be investigated.

Noticing at once that

$$C^{-1} = B_1 B_2 \dots B_{m-1} B_m, \tag{18}$$

and defining B_{m+1} by the formula

$$B_{m+1} = C^{-1} B_m C, \tag{19}$$

we have the transformation-relations

$$B_{d+1} = C^{-1} B_d C \qquad (d = 1, 2, \dots, m, m+1), \tag{20}$$

where, as in the sequel, *the indices of the* B_d *are to be reduced modulo*

$m+1$ *to indices of the series* $1, 2, \ldots, m, m+1$. $(20)_{d=m}$ is true by (19). $(20)_{d<m}$ is equivalent to

$$B_d C = C B_{d+1},$$

and, by (17, 6), this is equivalent to

$$B_m B_{m-1} \ldots B_{d+2} \,.\, B_d B_{d+1} B_d \,.\, B_{d-1} B_{d-2} \ldots B_2 B_1$$
$$= B_m B_{m-1} \ldots B_{d+2} \,.\, B_{d+1} B_d B_{d+1} \,.\, B_{d-1} B_{d-2} \ldots B_2 B_1,$$

and, by (5), this is true. ∴ (20) is then true for $d \leqq m$. Now

$$C^{-1} \,.\, C \,.\, C = C;$$

hence, by (17), and $(20)_{d \leqq m}$,

$$C^{-1} \,.\, B_m B_{m-1} \ldots B_2 B_1 \,.\, C = B_{m+1} B_m \ldots B_3 B_2 = C;$$

that is, $$B_{m+1} C = C B_1.$$

Thus (20) is true also for $d = m+1$.

Hence, more generally, from (20, 2, 5, 6, 17)

(21) $$C^{-j} B_d C^j = B_{d+j}, \quad B_d C^j = C^j B_{d+j} \quad \begin{pmatrix} d = 1, 2, \ldots, m+1 \\ j = 0, \pm 1, \pm 2, \ldots \end{pmatrix},$$

(22) $$B_d^2 = 1, \quad B_d B_{d+1} B_d = B_{d+1} B_d B_{d+1}, \quad B_d B_e = B_e B_d \quad \begin{pmatrix} d, e = 1, 2, \ldots, m+1 \\ e \neq d-1, d, d+1 \end{pmatrix},$$

(23) $$C = B_m B_{m-1} \ldots B_2 B_1 = B_{m+1} B_m \ldots B_3 B_2$$
$$= B_1 B_{m+1} B_m \ldots B_4 B_3 = \ldots = B_{m-1} B_{m-2} \ldots B_2 B_1 B_{m+1},$$

or (24) $$C = B_l B_{l-1} \ldots B_{l-m+2} B_{l-m+1} \quad (l = 1, 2, \ldots, m+1).$$

Further, the period of C is $m+1$,

(25) $$C^{m+1} = 1.$$

This is proved by a one-based induction. Denoting the C of (17) by C_{m+1}, we have

$$C_2^2 = B_1^2 = 1,$$

and we suppose that

$$C_m^m = (B_{m-1} B_{m-2} \ldots B_2 B_1)^m = 1.$$

Now $$C_m = B_m C_{m+1}.$$

Hence, by $(24)_{l=m-1}$, we have

$$(26) \qquad (B_m B_{m-1} \dots B_2 B_1 B_{m+1})^m = 1,$$

or (27) $(B_m \dots B_1)(B_{m+1} \dots B_2)(B_1 \dots B_3)(B_2 \dots B_4) \dots$

$$\dots (B_{m-2} \dots B_m)(B_{m-1} \dots B_{m+1}) = 1,$$

that is, by (24), $C_{m+1}^{m+1} = 1.$

[The specialization of C to T shows that period of C is $\geqq m+1$, and this result $C^{m+1} = 1$ shows that it is exactly $m+1$.] We notice (for use in § 3) the relations

$$(28) \quad C_{m+1} B_1 = C_{m+1}^{-1} C_m C_{m+1}, \quad (C_{m+1} B_1)^m = 1, \quad (B_1 C_{m+1})^m = 1.$$

Now we consider in the group

$$G(m+1) = G(m+1 : B_1, \dots, B_m),$$

the two sub-groups

$$K_{m!} = G(m) = G(m : B_1, \dots, B_{m-1}),$$

with the $m!$ elements $k_1, \dots, k_{m!}$, and the cyclic L_{m+1} generated by C with the $m+1$ elements C^j $(j = 1, 2, \dots, m+1)$. The sub-group H of lowest order containing $K_{m!}$ and L_{m+1} is $G(m+1)$ itself with the order $O(m+1) \geqq (m+1)!$. But Lemma A (§ 1) applies, and so $O(m+1) \leqq (m+1)!$, and hence, indeed, $O(m+1) = (m+1)!$, and our Theorem A is proved. In fact, we have, (21),

$$(29) \qquad C^s B_p = B_{p-s} C^s \qquad (p, s = 1, 2, \dots, m+1).$$

For the cases $p - s \equiv 0, -1 \pmod{m+1}$, writing these relations in the more desirable forms, by the use of (17),

$$(30) \qquad C^{p+1} B_p = B_{m-1} B_{m-2} \dots B_2 B_1 \, . \, C^p,$$

$$C^p B_p = B_1 B_2 \dots B_{m-2} B_{m-1} \, . \, C^{p+1},$$

we have in the relations (29, 30), $(p, s = 1, 2, \dots, m)$ the relations needed to make Lemma A apply.

The $G(k)$ is then the abstract symmetric group $G_{k!}^k$.

It is obvious from the substitution-group form $G_{k!}^k$ of the abstract $G(k)$ that all the $k-1$ generators B_d are necessary; if one is omitted, a sub-group is generated.

3. *Proof of Theorem* A′.

The group $G'(k)$ $(k \geq 3)$ has the generators B, C with the generational relations

$$B^2=1, \quad C^k=1, \quad (BC)^{k-1}=1, \quad (BC^{-1}BC)^3=1, \quad (BC^{-l}BC^l)^2=1$$
$$(l=2, 3, \ldots, k-2).$$

Defining

$$(31) \qquad B_{i+1}=C^{-i}BC^i, \quad B_1=B, \qquad (i=0, 1, \ldots, k-1),$$

we have, taking indices of B, modulo k, from the indices 1, 2, ..., k,

$$(32) \qquad B_{d+j}=C^{-j}B_dC^j \qquad (d, j=1, 2, \ldots, k).$$

The relations

$$(33) \qquad B_1^2=1, \ (B_1B_2)^3=1, \ (B_1B_e)^2=1 \ (e=3, 4, \ldots, k-1)$$

yield

$$(34) \qquad B_d^2=1, \quad (B_dB_{d+1})^3=1, \quad (B_dB_e)^2=1$$
$$\left(\begin{matrix} d, e=1, 2, \ldots, k \\ e \neq d-1, d, d+1 \end{matrix}\right).$$

Further, since $$(BC)^{k-1}=1,$$

and so $$(CB)^{k-1}=1,$$

$$(35) \ C_k=B_{k-1}B_{k-2}\ldots B_2B_1=C^{-(k-2)}(BC)^{k-2}B=C(CB)^{k-1}=C.$$

The relations 31, 32, 33, 34, 35 show that the group $G'(k:B, C)$ is the group $G(k:B_1, \ldots, B_{k-1})$, perhaps reduced by additional generational relations. The relations (2, 3, 4, 21, 25, 28) of §§ 1, 2 show that $G(k:B_1, \ldots, B_{k-1})$ is the $G'(k:B, C)$, perhaps reduced by additional generational relations. Hence the $G'(k:B, C)$ and the $G(k:B_1, \ldots, B_{k-1})$ are identical.

It should be added that, for $k=3$, the three relations

$$B^2=1, \quad C^3=1, \quad (BC)^2=1$$

involve the fourth $$(BC^{-1}BC)^3=1,$$

and thus define the (dihedron) group G_6.

4. *Proof of Theorem* B.

The generational relations of the abstract symmetric group $G(k)$ involve the generators B_d evenly. Hence, the different expressions for an element of the group in terms of the generators involve all either an even or an odd number of generators. We may then speak of even and of odd elements.

There are $\frac{1}{2}k!$ even elements. They constitute the abstract alternating group $G_{\frac{1}{2}k!}$. This group $G_{\frac{1}{2}k!}$ is *within the* $G(k)$ generated by the $k-2$ generators

(36) $$E_d = B_{d+1}B_1 \qquad (d = 1, 2, \dots, k-2),$$

for we have

(37) $$B_cB_d = B_cB_1 \,.\, B_1B_d = E_{c-1}E_{d-1}^{-1} \qquad (c, d = 1, \dots, k-1),$$

where we have written $E_0 = B_1B_1 = 1$.

These generators E_d satisfy the relations

(38) $$E_1^3 = 1, \quad E_d^2 = 1 \qquad (d = 2, 3, \dots, k-2),$$

(39) $$(E_dE_{d+1})^3 = 1 \qquad (d = 1, 2, \dots, k-3),$$

(40) $$(E_dE_e)^2 = 1 \qquad \left(\begin{aligned} d &= 1, 2, \dots\dots\dots, k-4 \\ e &= d+2, d+3, \dots, k-2 \end{aligned}\right)$$

(and conceivably, but not in fact, others not derivable from them), and are connected with the generator B_1, which extends the $G_{\frac{1}{2}k!}$ to the $G_{k!} = G(k)$ by the relations

(41) $$B_1^2 = 1, \quad B_1E_d = E_d^{-1}B_1 \quad (d = 1, 2, \dots, k-2).$$

Now the abstract group $G\{k\}$ of Theorem B has the $k-2$ generators E_d subject *only* to the generational relations (38, 39, 40). In accordance with Lemmas D and C (§ 1), the correspondence

(42) $$(E_1, \dots, E_{k-2}) \sim (E_1^{-1}, \dots, E_{k-2}^{-1})$$

defines an involutoric holohedric isomorphism of $G\{k\}$ with itself. Then (Lemma B, § 1) the group $G\{k\}$ of order $O\{k\}$ is extended by the adjunction of a new generator B_1 with the generational relations

(43) $$B_1^2 = 1, \quad B_1E_d = E_d^{-1}B_1 \quad (d = 1, 2, \dots, k-2),$$

to a group of order $2O\{k\}$. This new group is generated by the $k-1$ generators $B_1, \ldots, B_{k-1}$, where we set

$$B_{d+1} = E_d B_1 \qquad (d = 1, 2, \ldots, k-2).$$

These generators satisfy the relations prescribed for the generators B_a of the $G(k)$, and conceivably others not derivable from them.

Thus we have

$$\tfrac{1}{2}k! \leqq O\{k\} \quad \text{and} \quad 2O\{k\} \leqq k!.$$

Hence $$\tfrac{1}{2}k! = O\{k\},$$

and the abstract group $G\{k\}$ is indeed the abstract alternating group $G^k_{\frac{1}{2}k!}$.

If, for $k \geqq 5$, we replace the generator E_1 by the generator

$$E_1' = E_1 E_3,$$

we have a system of generators $E_1', E_2, E_3, \ldots, E_{k-2}$, all of which are of period 2. But the generational relations for this system are more complex.

Another system of generators suggests itself, viz., those corresponding to the cyclic substitutions of period 3,

$$(l_1 l_2 l_3), \quad (l_2 l_3 l_4), \quad (l_3 l_4 l_5), \quad \ldots, \quad (l_{k-2} l_{k-1} l_k).$$

The generational relations for this system are still more complex.

Thursday, January 14*th*, 1897.

Prof. E. B. ELLIOTT, F.R.S., President, in the Chair.

Present—fourteen members and a visitor.

Mr. William Henry Blythe, M.A., late Scholar of Jesus College, Cambridge, and Prof. Eliakim Hastings Moore, B.A. and Ph.D. Yale University, Professor of Mathematics, University of Chicago, were elected members.

Prof. Sylvester spoke at some length "On the Partition of an Even Number into Two Primes," and answered questions put to him by the members present.

Mr. J. J. Walker gave a solution of a certain quadratic vector equation.

The following papers were communicated :—

Supplementary Note on Matrices : Mr. J. Brill.
Some Properties of Bessel's Functions : Dr. Hobson.

Mr. T. I. Dewar exhibited a large number of stereoscopic pictures of the Algebraic Catenary.

The following presents to the Library were received :—

"Nautical Almanac for 1900," 8vo ; London, from the Lords Commissioners of the Admiralty, 1897.

"Proceedings of the Royal Society," Vol. LX., Nos. 363, 364.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xx., St. 11 ; Leipzig, 1896.

"Nyt Tidsskrift for Matematik," A. Aargang 7, Nos: 5, 6, 7 ; B. Aargang 7, No. 3 ; Copenhagen, 1896.

"Proceedings of the Physical Society," Vol. XIV., Pt. 12 ; December, 1896.

Lorenz, L.—"Œuvres Scientifiques," revues et annotées, par H. Valentiner, Tome I., Fasc. 1, 8vo ; Copenhague, 1896.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 6 ; Coimbra, 1896.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Kl., 1896, Heft 3 ; Geschäftliche Mittheilungen, Heft 2, 1896.

"Bulletin des Sciences Mathématiques," Tome xx., Nov., Dec., 1896 ; Paris.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. III., No. 3, December, 1896 ; New York.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. II., Fasc. 11 ; Napoli, 1896.

"Rendiconti del Circolo Matematico di Palermo," Tomo x., Fasc. 6 ; Nov., Dec., 1896.

Hartmann, J.—"Die Beobachtung der Mondfinsternisse," No. 5, roy. 8vo ; Leipzig, 1896.

Fischer, O.—"Beitrage zur Muskelstatik," No. 4, roy. 8vo ; Leipzig, 1896.

"Zur fünfzigjährigen Jubelfeier der K. Sächs. Gesells. der Wissenschaften zu Leipzig," roy. 8vo ; Leipzig, 1896.

"Educational Times," January, 1897.

"Annals of Mathematics," Vol. XI., No. 1 ; Sept., 1896.

"Atti della Reale Accademia dei Lincei," Sem. 2, Vol. v., Fasc. 1, Dec. 1896 ; Roma.

"Indian Engineering," Vol. xx., Nos. 20–25 (wanting 22), Nov. 28–Dec. 19, 1896.

"Astronomical, Magnetic, and Meteorological Observations made during 1890 at the U. S. Observatory," 4to ; Washington, 1895.

Cayley, A.—"Collected Mathematical Papers," Vol. XI., 4to ; Cambridge, 1896.

Supplementary Note on Matrices. By J. Brill, M.A.

Received December 12th, 1896. Read January 14th, 1897.

1. The present communication is intended to be supplementary to the "Note on Matrices" printed in the last volume of the *Proceedings* of the Society.[*] In that paper I shewed that the most general form of the differential of a matrix, such that it is commutative with the matrix itself, is given by the formula

$$dm = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}\, d\lambda_1.$$

This formula is sufficient, as stated in the paragraph appended to the note, as a test whether the differential of a matrix, due to the infinitesimal variation of some scalar element or elements contained in its expression, be commutative with the matrix itself. When, however, we attempt to replace it by an integral formula, we find that it gives rise to n conditions. These are obtained from the n equations

$$(dm-d\lambda_1)(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n) = 0, \qquad (1)$$

$$(dm-d\lambda_2)(m-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n) = 0, \qquad (2)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2)\dots(m-\lambda_{n-1}) = 0, \qquad (n)$$

made use of in proving the above theorem. That the single equation is equivalent to the set of n equations is evident, since they may all be obtained from it with the aid of the characteristic equation of the matrix. My present object is to obtain the n integral conditions which constitute the equivalent of the single differential one.

Taking equation (2), we have

$$(dm-d\lambda_2)(m-\lambda_2+\lambda_2-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n) = 0;$$

and, therefore,

$$(\lambda_1-\lambda_2)(dm-d\lambda_2)(m-\lambda_3)\dots(m-\lambda_n) \\ -(dm-d\lambda_2)(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n) = 0.$$

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[*] Vol. xxvii., pp. 35–38.

Making use of equation (1), this reduces to

$$(dm-d\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\frac{d\lambda_1-d\lambda_2}{\lambda_1-\lambda_2}=0.$$

Similarly, we should obtain

$$(dm-d\lambda_3)(m-\lambda_2)(m-\lambda_4)\dots(m-\lambda_n)$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\frac{d\lambda_1-d\lambda_3}{\lambda_1-\lambda_3}=0,$$

and so on. Adding together the $n-1$ equations so obtained, we have

$$d\{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\}$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\left\{\frac{d\lambda_1-d\lambda_2}{\lambda_1-\lambda_2}+\frac{d\lambda_1-d\lambda_3}{\lambda_1-\lambda_3}+\&c.\right\}=0.$$

This may be written

$$\frac{d\{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\sum_s\frac{d\lambda_1-d\lambda_s}{(\lambda_1-\lambda_2)\dots(\lambda_1-\lambda_s)^2\dots(\lambda_1-\lambda_n)}=0,$$

which immediately integrates in the form

$$\frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}=\text{const.}$$

Similarly, we should obtain

$$\frac{(m-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\dots(\lambda_2-\lambda_n)}=\text{const.},$$

and so on. Writing $e_1, e_2, \dots e_n$ for the n constants so obtained, we find that they obey the laws

$$e_r^2=e_r,\quad e_re_s=e_se_r=0,^*$$

and also that we have

$$e_1+e_2+\dots+e_n=1.$$

Thus we see that the differential of the matrix is expressed in the form

$$dm=e_1d\lambda_1+e_2d\lambda_2+\dots+e_nd\lambda_n.$$

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* This has been shown by Study. See "Ueber Systeme von complexen Zahlen." *Gött. Nach.*, 1889, pp. 237-268. Also "Ueber Systeme complexer Zahlen, &c.," *Monatshefte für Math. u. Phys.*, I. (1890), pp. 283-355.

In fact, since we have

$$m = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}\lambda_1,$$

we see that the matrix is expressible linearly in terms of its latent roots as follows,

$$m = e_1\lambda_1+e_2\lambda_2+\dots+e_n\lambda_n,$$

and the above expression for the differential follows immediately from this.

In the paper* referred to in the postscript to my former note, I was dealing with binary matrices, and took the equation

$$\frac{2m-\lambda_1-\lambda_2}{\lambda_1-\lambda_2} = \text{const.}$$

as the required condition. This, of course, is a consequence of the two conditions that should hold, but it is not sufficient in itself. It will, however, be seen that, in the application made, the two conditions are separately satisfied, so that the work is quite valid.

Note on some Properties of Bessel's Functions. By E. W. Hobson.

Received and communicated January 14th, 1897.

An integral theorem involving Bessel's functions is here given, which contains some special cases of interest. It is well known that, for integral values of m, there are an odd number of positive roots of the equation

$$J_{m+1}(x) = 0$$

lying between consecutive positive roots of the equation

$$J_m(x) = 0,$$

and this is easily seen to be true for fractional values of m; it does not, however, appear to have been strictly proved that this odd number must in all cases be unity;[*] the truth of this theorem is here deduced from the integral theorem. The roots of the equation

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* "On the Application of the Theory of Matrices to the Discussion of Linear Differential Equations with Constant Coefficients," *Proc. Camb. Phil. Soc.*, Vol. VIII., pp. 201–210.

$$x^{-m} J_m(x) = 0$$

for unrestricted real values of m have been discussed[†] by Hurwitz, who has shown that, if $m > -1$, the roots are all real; he has also shown that, if m is between -1 and -2, all the roots are real except one pair of purely imaginary roots, and also that, if m lies between the two negative integers $-2k$, $-2k-2$, the equation has k conjugate pairs of complex roots as well as an infinite number of real roots, and also one pair of purely imaginary roots in case m lies between $-2k-1$ and $-2k-2$.

1. Let u_m be any solution of Bessel's equation of order m, and v_n any solution of the equation of order n; it can then be shown that $\frac{u_m v_n}{x}$ is a perfect differential of a simple function involving u_m, v_n and their differential coefficients.

We have
$$\frac{d^2u_m}{dx^2} + \frac{1}{x}\frac{du_m}{dx} + \left(1 - \frac{m^2}{x^2}\right) u_m = 0,$$
$$\frac{d^2v_n}{dx^2} + \frac{1}{x}\frac{dv_n}{dx} + \left(1 - \frac{n^2}{x^2}\right) v_n = 0;$$

hence
$$x\left(u_m \frac{d^2v_n}{dx^2} - v_n \frac{d^2u_m}{dx^2}\right) + \left(u_m \frac{dv_n}{dx} - v_n \frac{du_m}{dx}\right) = (n^2 - m^2)\frac{u_m v_n}{x},$$

or
$$(n^2 - m^2)\frac{u_m v_n}{x} = \frac{d}{dx}\left\{x\left(u_m \frac{dv_n}{dx} - v_n \frac{du_m}{dx}\right)\right\};$$

hence
$$(n^2 - m^2)\int_\alpha^\beta \frac{u_m v_n}{x}\,dx$$
$$= \left[x\left(u_m \frac{dv_n}{dx} - v_n \frac{du_m}{dx}\right)\right]_{x=\beta} - \left[x\left(u_m \frac{dv_n}{dx} - v_n \frac{du_m}{dx}\right)\right]_{x=\alpha}, \quad (1)$$

where α, β are any quantities such that the integral is convergent at each limit.

Let
$$u_m = J_m(x), \quad v_n = J_n(x);$$

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[*] See Gray and Mathews, *Treatise on Bessel Functions*, p. 50.

[†] See *Mathematische Annalen*, Vol. XXXIII. (1889).

then, if $m+n$ is positive, we have

$$(n^2-m^2)\int_0^a \frac{J_m(x)\,J_n(x)}{x}\,dx = a\left[J_m(a)\frac{dJ_n(a)}{da} - J_n(a)\frac{dJ_m(a)}{da}\right],$$

or, using the relation

$$\frac{dJ_m}{da} = \frac{m}{a}J_m - J_{m+1},$$

this becomes

$$(n^2-m^2)\int_0^a \frac{J_m(x)\,J_n(x)}{x}\,dx$$

$$= (n-m)\,J_m(a)\,J_n(a) + a\left\{J_n(a)\,J_{m+1}(a) - J_m(a)\,J_{n+1}(a)\right\}. \qquad (2)$$

This theorem seems an obvious one, but I have not been able to find it in any writings on the subject.*

Whatever m and n are, we have

$$\begin{aligned}(n^2-m^2)\int_\alpha^\beta \frac{J_m(x)\,J_n(x)}{x}\,dx = (n-m)&\left\{J_m(\beta)\,J_n(\beta) - J_m(\alpha)\,J_n(\alpha)\right\}\\ +\beta&\left\{J_n(\beta)\,J_{m+1}(\beta) - J_m(\beta)\,J_{n+1}(\beta)\right\}\\ -\alpha&\left\{J_n(\alpha)\,J_{m+1}(\alpha) - J_m(\alpha)\,J_{n+1}(\alpha)\right\}.\end{aligned}$$

Let α, β be roots of the equation

$$x^{-n}J_n(\alpha) = 0;$$

this then becomes

$$(n^2-m^2)\int_\alpha^\beta \frac{J_m(x)\,J_n(x)}{x}\,dx = \alpha J_m(\alpha)\,J_{n+1}(\alpha) - \beta J_m(\beta)\,J_{n+1}(\beta).$$

In particular, let $n = m+1$, then use the equation

$$J_{m+2} = \frac{2m+2}{x}J_{m+1} - J_m,$$

and we have, since

$$J_{m+1}(\alpha) = 0, \quad J_{m+1}(\beta) = 0,$$

$$(2m+1)\int_\alpha^\beta \frac{J_m(x)\,J_{m+1}(x)}{x}\,dx = \beta J_m^2(\beta) - \alpha J_n^2(\alpha), \qquad (3)$$

where α, β are roots of the equation

$$x^{-m-1}J_{m+1}(x) = 0.$$

3. We shall now apply the theorem (3) to show that there can only be one positive zero of the function $x^{-m-1}J_{m+1}(x)$ between two

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[*] A referee has pointed out that the theorem (1) has been given by Sonine, *Mathematische Annalen*, Vol. XVI.

consecutive positive zeros of the function $x^{-m}J_m(x)$. The relation

$$x^{-m-1}J_{m+1}(x) = -2x\frac{d}{d(x^2)}\{x^{-m}J_m(x)\}$$

shows that the number of zeros must be odd, and that they are at the maxima and minima of $x^{-m}J_m(x)$. If possible, let there be three or more zeros $\alpha, \beta, \gamma, \ldots$ in ascending order and taken positive, all lying between consecutive zeros of $x^{-m}J_m(x)$. We have then

$$(2m+1)\int_{\alpha}^{\beta}\frac{J_m(x)J_{m+1}(x)}{x}\,dx = \beta J_m^2(\beta) - \alpha J_m^2(\alpha),$$

$$(2m+1)\int_{\beta}^{\gamma}\frac{J_m(x)J_{m+1}(x)}{x}\,dx = \gamma J_m^2(\gamma) - \beta J_m^2(\beta).$$

Now $J_m(x)$ is of invariable sign within the interval α to γ, and $J_{m+1}(x)$ is of opposite signs in the two intervals α to β and β to γ. Hence, if $2m+1$ is positive, each side of that one of these equations is negative for which $J_m(x)$, $J_{m+1}(x)$ have opposite signs; if $2m+1$ is negative, each side of that equation is negative for which $J_m(x)$. $J_{m+1}(x)$ have the same signs.

Also
$$\frac{d}{dx}\{xJ_m^2(x)\} = J_m(x)\left\{J_m(x) + 2x\frac{dJ_m}{dx}\right\}$$
$$= J_m(x)\{(2m+1)J_m(x) - 2xJ_{m+1}(x)\}.$$

Hence, if $2m+1$ is positive, and $J_m(x)$, $J_{m+1}(x)$ have opposite signs, $xJ_m^2(x)$ increases as x increases; this is contrary to what has been shown above, that the one of the expressions

$$\beta J_m^2(\beta) - \alpha J_m^2(\alpha), \quad \gamma J_m^2(\gamma) - \beta J_m^2(\beta)$$

is negative for which in the integral $J_m(x)$, $J_{m+1}(x)$ have opposite signs. Similarly, if $2m+1$ is negative, $xJ_m^2(x)$ decreases as x increases, if $J_m(x)$, $J_{m+1}(x)$ have the same sign; this is also contrary to what has been shown above.

It has thus been shown that there cannot be more than one positive zero of $x^{-m-1}J_{m+1}(x)$ between two consecutive positive zeros of $x^{-m}J_m(x)$.

4. If, in (1), we let $u_m = Y_m(x)$, $u_n = Y_n(x)$ where Y denotes the function of the second kind, we have

$$(n^2-m^2)\int_{a}^{\infty}\frac{Y_m(x)Y_n(x)}{x}\,dx = -(n-m)Y_m(a)Y_n(a)$$
$$+ a\{Y_m(a)Y_{n+1}(a) - Y_n(a)Y_{m+1}(a)\}, \quad (4)$$

where $a > 0$, the integral being convergent at the upper limit.

We have also, as a special case of (1),

$$(n^2-m^2)\int_a^\infty \frac{Y_n(x)\,J_m(x)}{x}\,dx$$
$$= -(n-m)\,Y_n(a)\,J_m(a) + a\left\{J_m(a)\,Y_{n+1}(a) - Y_n(a)\,J_{m+1}(a)\right\}. \quad (5)$$

For the function $K_m(x)$, which is a Bessel's function of argument ix, and is so chosen that $K_m(\infty) = 0$, we have

$$(n^2-m^2)\int_a^\infty \frac{K_m(x)\,K_n(x)}{x}\,dx$$

expressed in a similar form.

We have, from (2),

$$(n^2-m^2)\int_0^\infty \frac{J_m(x)\,J_n(x)}{x} = \underset{a=\infty}{L}\, a\left\{J_n(a)\,J_{m+1}(a) - J_m(a)\,J_{n+1}(a)\right\}.$$

Substituting on the right-hand side the asymptotic values of the functions, the expression becomes

$$\frac{2}{\pi}\left\{\cos\left(\frac{2n+1}{4}\pi - a\right)\cos\left(\frac{2m+3}{4}\pi - a\right)\right.$$
$$\left. -\cos\left(\frac{2m+1}{4}\pi - a\right)\cos\left(\frac{2n+3}{4}\pi - a\right)\right\},$$

or
$$\frac{1}{\pi}\left\{\cos\frac{n-m-1}{2}\pi - \cos\frac{n-m+1}{2}\pi\right\},$$

which is
$$\frac{2}{\pi}\sin\frac{n-m}{2}\pi;$$

thus
$$\int_0^\infty \frac{J_m(x)\,J_n(x)}{x}\,dx = \frac{2}{n^2-m^2}\,\frac{1}{\pi}\sin\frac{n-m}{2}\pi. \quad (6)$$

If $n-m$ is an even integer,

$$\int_0^\infty \frac{J_m(x)\,J_n(x)}{x}\,dx = 0; \quad (7)$$

and, if $n = m$, and m is positive,

$$\int_0^\infty \frac{J_m^2(x)}{x}\,dx = \frac{1}{2m}. \quad (8)$$

The theorems (7) and (8) have been obtained otherwise by Heine.*

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* See *Kugelfunctionen*, Vol. I., p. 255.

5. In (2), let $n = m+\delta m$, and suppose a to be a root of the equation $x^{-m} J_m(x) = 0$; we have then

$$2m\,\delta m \int_0^a \frac{J_m^2(x)}{x}\,dx = a\left\{-J_{m+1}(a)\,\frac{dJ_m(a)}{da}\,\delta a\right\},$$

where $a+\delta a$ is such that $J_{m+\delta m}(a+\delta a) = 0$, and the squares of δm, δa are omitted.

Hence, since $$\frac{dJ_m(a)}{da} = -J_{m+1}(a),$$

when $J_m(a) = 0$, we have

$$\frac{da}{dm} = \frac{2m}{a\,\{J_{m+1}(a)\}^2}\int_0^a \frac{\{J_m(x)\}^2}{x}\,dx, \tag{9}$$

a theorem which has been given without proof by Schläfli.*

The theorem (9) shows that a root a, considered as a function of the order m, increases as the order increases.

P.S.—Since writing the above I have received from the author, a copy of a paper just published in *The American Journal of Mathematics*† which contains a proof of the theorem proved above, that the positive roots of the equations $J_m(x) = 0$ and $J_{m+1}(x) = 0$ occur alternately. As the method given above is an entirely different one, I have not thought it desirable to suppress the note, especially as it contains other results of some interest.

Thursday, February 11th, 1897.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Ten members present.

Mr. Macaulay read a paper "On a Theorem in non-Euclidean Geometry." An animated discussion followed, in which Messrs. Kempe, Cunningham, Love, and the President, joined with the author.

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* *Mathematische Annalen*, Vol. x.

† "On the Roots of Bessel and *P* Functions," by E. B. Van Vleck, Vol. XIX.

An impromptu communication was made by Mr. Kempe in connexion with Prof. Sylvester's communication at the January meeting. The President and Major MacMahon spoke on the subject.

The President (Major MacMahon, Vice-President, in the Chair) gave a short account of Mr. Segar's theorem that the product of the differences of n unequal numbers is divisible by the product of the differences of 0, 1, 2, ... $(n-1)$, and showed also that the product of the differences of n unequal square numbers is divisible by the product of the differences of 0^2, 1^2, 2^2, ... $(n-1)^2$.*

Lt.-Col. Cunningham brought forward some high primes.

A paper by Mr. H. M. Taylor "On the Degeneration of a Cubic Curve" was communicated by reading its title.

The following presents were made to the Library :—

"Catalogue of the Michigan Mining School, 1894–6," 8vo; Houghton, Michigan, 1896.

"Proceedings of the Royal Society," Vol. LX., No. 365.

"Proceedings of the Royal Irish Academy," Vol. IV., No. 1; Dublin, 1896.

Emmens, S.—"The Argentaurum Papers, No. 1, Some Remarks concerning Gravitation."

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. II., No. 181; London, January, 1897.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 12, 1890, and Bd. XXI., St. 1; Leipzig, 1897.

"Proceedings of the Physical Society of London," Vol. XV., Pt. 1, No. 76; January, 1897.

"Revue Semestrielle des Publications Mathématiques," Tome V., Partie 1^e; April–October, 1896.

"Monatshefte für Mathematik und Physik," Jahrgang VIII., 1897, Pt. I.; Wien.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXX., Liv. 4; Harlem, 1896.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. II., Fasc. 12, December, 1896; Napoli.

"Bulletin de la Société Mathématique de France," Tome XXIV., No. 8 et dernier; Paris.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. III., No. 4; January, 1897; New York.

"Bulletin des Sciences Mathématiques," Tome XXI., January, 1897; Paris.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. V., Fasc. 12, 1896, Sem. 1, Vol. VI., Fasc. 1, 2, 1897; Roma.

"Educational Times," February, 1897.

"La Naturaleza," Tome VIII., Núm. 1; Madrid.

Kantor, S.—"Theorie der periodischen cubischen Transformationen im Raume R_3,"

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* *Cf.*, *Messenger of Mathematics*, CCCXIII., pp. 12–15.

4to pamphlet (from "American Journal," Vol. XIX., No. 1); "Ueber Collineationsgruppen an Kummer'schen Flächen," 4to pamphlet.

"Indian Engineering," Vol. XX., No. 26, Vol. XXI., Nos. 1, 2, 3; December 26, 1896–January 10, 1897.

The following is the note communicated by Lt.-Col. Cunningham:—

The 43 high primes (p) on the list below are all beyond the present factor-tables, *i.e.*, are all > 9 million. They are all of form

$$p = 10\varpi + 1,$$

and were found as divisors of numbers of form

$$N = (x^5 + y^5) \div (x + y),$$

where $$y = \pm 1;$$

their *prime* character was recognised from this property by aid of a MS. Table, prepared by the author, giving the base x of *all* numbers of form N with prime factors p of form $(10\varpi+1) \not> 5060$. The Table below shows the values of x, y giving the number N from which they were derived.

The fact of being factors of such numbers (N) facilitates the discovery* of the *residuacity* of the base 2 to such primes: this is shown in the Table by the letters q, n.

q denotes that 2 is a *quintic residue* of p; *i.e.*, $2^{\frac{1}{5}(p-1)} \equiv 1 \pmod{p}$,

n denotes that 2 is a *quintic non-residue* of p; *i.e.*, $2^{\frac{1}{5}(p-1} \not\equiv 1 \pmod{p}$.

Three of these primes were previously† known, as factors of numbers of form

$$N = 2^{\xi} - 1,$$

viz., $p =$ 10,567,201; 13,334,701; 18,837,001;

$\xi =$ 75; 300; 90.

The author's present research confirms the fact that they are *primes*: the remaining forty are believed to be *new primes*. In three cases, it has been found possible to determine the *minimum* exponent (ξ), for which $2^{\xi} \equiv 1 \pmod{p}$, and therewith the *maximum* residue-index

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* *See* Mr. Bickmore's paper "On the Numerical Factors of $(a^n - 1)$," in the *Messenger of Mathematics*, Vol. XXVI., Art. 21, *et seq.*

† *See* Ed. Lucas's paper *Sur la Série Récurrente de Fermat*, pp. 8–10, Rome, 1879. The mode of determining the *prime* character is not stated.

$(p-1) \div \xi$, which gives the residuacity of the base 2, viz.,

$$p = 9{,}792{,}191\,; \quad 13{,}133{,}291\,; \quad 14{,}106{,}971\,;$$

$$\xi = 5\,.\,979{,}219\,; \quad 2\,.\,131{,}329\,; \quad 2\,.\,1{,}410{,}697\,;$$

$$(p-1) \div \xi = \quad 2\,; \qquad 5\,; \qquad 5.$$

HIGH PRIMES.

p	x	y		p	x	y	
9,170,881	139	$\bar{1}$	*n*	16,007,041	63	$\bar{1}$	*n*
9,367,291	101	$\bar{1}$	*n*	16,039,531	115	$\bar{1}$	*n*
9,384,251	3125	$\bar{1}$	*n*	16,776,481	411	$\bar{1}$	*n*
9,498,581	380	$\bar{1}$	*n*	17,501,251	188	1	*n*
9,792,191	189	$\bar{1}$	*n*	17,562,031	153	1	*n*
10,332,211	103	$\bar{1}$	*n*	17,994,481	309	1	*n*
10,487,921	262	$\bar{1}$	*n*	18,304,511	4205	$\bar{1}$	*n*
10,567,201	32^3	$\bar{1}$	*n*	18,837,001	512	1	*n*
11,006,851	146	1	*q*	19,019,801	99	$\bar{1}$	*q*
11,106,421	241	$\bar{1}$	*n*	19,622,651	181	$\bar{1}$	*n*
11,249,741	287	1	*q*	20,411,341	310	1	*n*
11,295,311	231	1	*q*	20,640,071	123	1	*q*
11,498,111	311	1	*n*	21,700,501	68	$\bar{1}$	*n*
11,501,761	163	1	*n*	22,199,431	199	$\bar{1}$	*n*
12,483,671	108	$\bar{1}$	*q*	22,284,781	295	$\bar{1}$	*q*
12,827,531	341	1	?	22,996,651	243	1	*n*
13,133,291	244	$\bar{1}$	*q*	24,253,721	233	1	*n*
13,334,701	1024^3	1	*n*	25,058,741	71	1	*q*
14,106,971	321	$\bar{1}$	*q*	25,410,401	5120	$\bar{1}$	*q*
14,715,341	230	$\bar{1}$	*n*	25,447,421	269	1	*n*
14,961,091	147	1	*n*	25,621,901	194	1	*n*
15,018,571	62	$\bar{1}$	*n*				

Thursday, March 11*th*, 1897.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Eleven members present.

The following gentlemen were elected members of the Society: Paul Jerome Kirkby, M.A. Hertford College, Oxford, Lecturer in

Mathematics at St. David's College, Lampeter; Frederick William Lawrence, B.A., Scholar of Trinity College, Cambridge; and Alfred Young, B.A., Scholar of Clare College, Cambridge, of. Ridley Hall, Cambridge.

The President referred to a letter received from the President of the Royal Society with reference to the Victoria Research Fund, which it is proposed to institute in commemoration of Her Majesty's long reign, and commended the fund to the generous consideration of members. He then spoke briefly on the loss the mathematical world has sustained by the recent death of Prof. Weierstrass.[*]

Mr. M. Jenkins, Vice-President, having taken the Chair, the President communicated a paper by Mr. J. E. Campbell, "On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups."

The President, having resumed the Chair, read some "Notes on Symmetric Functions," by Mr. W. H. Metzler.

The senior Secretary briefly communicated a "Note on a System of Circles associated with a Triangle," by Prof. J. E. A. Steggall.

Lt.-Col. Cunningham announced (at this and the subsequent meeting) the eight[†] following high *primes*, and gave a sketch of the methods he had employed in determining them :—

$$47{,}763{,}361 = \frac{(3^{30}+1)(3^2+1)}{(3^{10}+1)(3^6+1)}\,;\ 85{,}280{,}581, \text{ the large factor of } (5^{35}-1)\,;$$

$$234{,}750{,}601 = (5^{14}+1) \div (5^2+1)\,;$$

$$\frac{5^{13}-1}{5-1} = 305{,}175{,}781\,;\quad \frac{1}{409}\,\frac{5^{17}-1}{5-1} = 466{,}344{,}409\,;$$

$$550{,}554{,}229 = \text{the large factor of } (3^{63}+1)\,;$$

$$632{,}133{,}361 = \frac{1}{241}\,\frac{5^{20}+1}{5^4+1}\,;\quad 2{,}413{,}941{,}289 = \frac{3^{33}-1}{3^{11}-1}\cdot\frac{3^1-1}{3^3-1}\,;$$

also the *completion* of the resolution of $(3^{105}+1) \div (3^{35}+1)$ previously given by Mr. C. E. Bickmore[‡] as

$$81{,}365{,}467{,}681\,.\,(211\,.\,1051\,.\,3{,}454{,}081),$$

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[*] See *Nature* (March 11th, 1897, p. 443).

[†] Three (the 2nd, 3rd, and 8th) were announced at this meeting, and the rest at the meeting of 8th April; but it has been thought most convenient to collect them all together here.

[‡] In *Messenger of Mathematics*, Vol. xxv., paper "On the Numerical Factors of (a^n-1)," foot of Art. 7.

into its *prime* factors, now found to be

$$(24151 \, . \, 3{,}369{,}031)(211 \, . \, 1051 \, . \, 3{,}454{,}081).$$

This number $(3^{105}+1)$ is interesting as being one of the largest ever completely resolved into *prime* factors.

The following presents were made to the Library :—

"Jahrbuch über die Fortschritte der Mathematik," Bd. xxv., Heft 3, Jahrgang 1893 and 1894 ; Berlin, 1897.

"Proceedings of the Royal Society," Vol. LX., No. 366.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 2 ; Leipzig, 1897.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 4 (Michaelmas Term, 1896) ; 1897.

Smyth, A. H.—"Memoir of Henry Phillips, Jr.," pamphlet, 8vo ; 1896 (American Philosophical Society, November, 1896).

"Proceedings of the Physical Society," Vol. XV., Pt. 2 (February, 1897), Pt. 3 (March, 1897) ; London.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLI., Pt. 11, 1897.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig" (Mathematisch-Physische Klasse, 1896), IV. ; 1897.

"Bulletin des Sciences Mathématiques," Tome XXI., Fev., 1897 ; Paris.

Kantor, S.—"Über n Momente von R_i Complexen im R_r," pamphlet, 8vo ; München, 1897.

"Nyt Tidsskrift for Matematik," A. Aargang 7, Nr. 8, 1896 ; B. Aargang 7, Nr. 4, Aargang 8, Nr. 1, 1897 ; Copenhagen.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. III., No. 5, February, 1897 ; New York.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," 1896, Heft 4.

"Rendiconti del Circolo Matematico di Palermo," Tomo XI., Fasc. 1 and 2 ; 1897.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. III., Fasc. I. ; Napoli, 1897.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VI., Fasc. 3 ; Roma, 1897.

Wessel, C.—"Essai sur la Représentation analytique de la Direction," 4to ; Copenhague, 1897.

"Educational Times," March, 1897.

"Annals of Mathematics," Vol. XI., No. 2.

"Journal für die reine und angewandte Mathematik," Bd. CXVII., Heft 3 ; Berlin, 1897.

"Indian Engineering," Vol. XXI., Nos. 4–7, Jan. 23–Feb. 13, 1897.

On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups. *By* J. E. Campbell.

Received and read March 11th, 1897.

Let y and x denote any operators which obey the distributive and associative laws, but not necessarily the commutative.

Let y_1 denote the operation $yx-xy$,
y_2 " " y_1x-xy_1,
and y_r " " $y_{r-1}x-xy_{r-1}$.

Let $[yx^r]$ denote the sum of the operations

$$yx^r+xyx^{r-1}+x^2yx^{r-2}+\dots+x^ry.$$

The theorem to be proved is the following:—If

$$a_1,\ a_2,\ a_3,\ a_4,\ \dots$$

is a series of numerical constants, of which a_1 is $\frac{1}{2}$, a_2 is $\frac{1}{12}$, a_3 is zero, $a_4=-\frac{1}{720}$, and of which the law of formation is

$$(m+1)\,a_m=a_{m-1}-(a_1a_{m-1}+a_2a_{m-2}+\dots+a_{m-1}a_1),$$

then $$\frac{yx^r}{r!}=\left[y\,\frac{x^r}{(r+1)!}\right]+a_1\left[y_1\,\frac{x^{r-1}}{r!}\right]+a_2\left[y_2\,\frac{x^{r-2}}{(r-1)!}\right]+\dots$$
$$\dots+a_{r-1}\left[y_{r-1}\,\frac{x}{2!}\right]+a_ry_r.$$

From the fact that a_3 is zero, the law of formation at once shows that a_5, a_7, ..., the series of constants with odd suffixes, are all zero.

[This series of numbers was, I believe, discovered by Schur, in his investigation of the same problem which led me to consider them. A sketch of part of his work is given in Lie's *Transformationsgruppen*, III., § 144. I have consulted his writings in *Math. Annal.*, Bd. XXXV., § 161, which bear most closely on my results, but have not been able to consult the references given in Lie to *Leipz. Ber.*, 1889, § 229, and 1890, § 1. I had independently arrived at the law of their formation, but had not noticed that the odd numbers were all zero except a_1.]

Two lemmas are required for the proof, the first,

$$x^r y_m + [y_m x^{r-1}]\, x = [y_m x_r],$$

is obvious from the definition of the symbol [ ].

The second, $$y_m x^r = x^r y_m + [y_{m+1} x^{r-1}],$$

may be proved by induction.

When $r = 1$,

$$(1) \quad y_m x \;= x y_m + y_{m+1};$$

therefore $$y_m x^2 = x y_m x + y_{m+1}\, x;$$

therefore $$\begin{aligned} y_m x^2 &= x\,(x y_m + y_{m+1}) + y_{m+1}\, x \\ &= x^2 y_m + [y_{m+1} x]. \end{aligned}$$

Thus the result holds for $r = 2$.

Assume it holds for r;

$$y_m x^r \;= x^r y_m + [y_{m+1} x^{r-1}];$$

therefore $$y_m x^{r+1} = x^r y_m x + [y_{m+1} x^{r-1}]\, x;$$

therefore, from (1), $$\begin{aligned} &= x^r\,(x y_m + y_{m+1}) + [y_{m+1} x^{r-1}] x \\ &= x^{r+1} y_m + [y_{m+1} x^r], \end{aligned}$$

by Lemma I., so that Lemma II. is established also.

Assume that for all integral values, up to and including r, it has been established that

$$y\,\frac{x^r}{r!} = \left[\frac{y x^r}{(r+1)!}\right] + a_1 \left[y_1\,\frac{x^{r-1}}{r!}\right] + \ldots + a_r y_r.$$

[It would, of course, follow that we have also

$$y_1\,\frac{x^r}{r!} = \left[y_1\,\frac{x^r}{(r+1)!}\right] + a_1 \left[y_2\,\frac{x^{r-1}}{r!}\right] + \ldots + a_r y_{r+1},$$

$$y_2\,\frac{x^r}{r!} = \left[y_2\,\frac{x^r}{(r+1)!}\right] + a_1 \left[y_3\,\frac{x^{r-1}}{r!}\right] + \ldots + a_r y_{r+2},$$

&c.],

then it will be proved that the theorem holds also for the value $r+1$.

It is obvious that the theorem holds for $r = 1$. [I have also verified it for the cases $r = 2$, $r = 3$, and $r = 4$.]

Since

$$y\frac{x^r}{r!} = \left[y\frac{x^r}{(r+1)!}\right] + a_1\left[y_1\frac{x^{r-1}}{r!}\right] + \dots,$$

therefore

$$y\frac{x^{r+1}}{r!} = \left[y\frac{x^r}{(r+1)!}\right]x + a_1\left[y_1\frac{x^{r-1}}{r!}\right]x + \dots$$

$$= \left[y\frac{x^{r+1}}{(r+1)!}\right] + a_1\left[y_1\frac{x^r}{r!}\right] + \dots - \frac{x^{r+1}}{(r+1)!}y - a_1\frac{x^r}{r!}y_1 - \dots,$$

by Lemma I.; therefore

$$y\frac{x^{r+1}}{r!} + y\frac{x^{r+1}}{(r+1)!} + a_1y_1\frac{x^r}{r!} + a_2y_2\frac{x^{r-1}}{(r-1)!} + \dots + a_ry_rx$$

$$\equiv \left[y\frac{x^{r+1}}{(r+1)!}\right] + a_1\left[y_1\frac{x^r}{r!}\right] + \dots + a_r\left[y_rx\right]$$

$$+ \left[y_1\frac{x^r}{(r+1)!}\right] + a_1\left[y_2\frac{x^{r-1}}{r!}\right] + \dots + a_ry_{r+1},$$

by Lemma II.

Expanding all the terms

$$y_1\frac{x^r}{r!},\quad y_2\frac{x^{r-1}}{(r-1)!},\quad \dots\quad y_rx,$$

by the theorem which has been assumed to hold for all integral values up to and including r, and subtracting from the two sides of the equation, we see that

$$y\frac{x^{r+1}}{r!} + y\frac{x^{r+1}}{(r+1)!} \equiv \left[y\frac{x^{r+1}}{(r+1)!}\right] + \sum_{m=1}^{m=r+1} b_m\left[y_m\frac{x^{r-m+1}}{(r-m+2)!}\right];$$

where

$$b_m = a_{m-1} + (r-m+2)\,a_m - \left[a_1a_{m-1} + a_2a_{m-2} + \dots + a_{m-1}a_1\right] - a_m$$

$$= (r+2)\,a_m,$$

by the law of formation of the coefficients.

Dividing each side of the identity by $r+2$, we see that

$$y\frac{x^{r+1}}{(r+1)!} = \left[y\frac{x^{r+1}}{(r+2)!}\right] + a_1\left[y_1\frac{x^r}{(r+1)!}\right] + \dots + a_r\left[y_r\frac{x}{2!}\right] + a_{r+1}y_{r+1};$$

that is, the theorem also holds for $r+1$, and therefore holds universally, since it obviously holds for $r=1$.

Since y and x are symbols obeying the distributive and associative law,

$$(x+\mu y)^r = x^r + \mu\left[yx^{r-1}\right] + \text{terms involving higher powers of } \mu;$$

so that, if we take μ a constant so small that its square and higher powers may be neglected,

$$(x+\mu y)^r = x^r + \mu\left[yx^{r-1}\right].$$

Let $$z \equiv y + a_1y_1 + a_2y_2 + \dots;$$

then, as above, $$(x+\mu z)^r = x^r + \mu\left[zx^{r-1}\right] + \dots.$$

From the theorem we have established, we have the following equations:—

$$\begin{aligned}
y &= y, \\
yx &= \tfrac{1}{2}[yx] + a_1y_1, \\
y\frac{x^2}{2!} &= \left[y\frac{x^2}{3!}\right] + a_1\left[y_1\frac{x}{2!}\right] + a_2y_2, \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
y\frac{x^r}{r!} &= \left[y\frac{x^r}{(r+1)!}\right] + a_1\left[y_1\frac{x^{r-1}}{r!}\right] + \dots, \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Adding these expressions, we get

$$y\epsilon^x = z + \left[z\frac{x}{2!}\right] + \left[z\frac{x^2}{3!}\right] + \dots \text{ to infinity};$$

therefore

$$(1+\mu y)\,\epsilon^x = 1 + x + \mu z + \frac{x^2}{2!} + \mu\left[z\frac{x}{2!}\right] + \dots + \frac{x^r}{r!} + \mu\left[z\frac{x^{r-1}}{(r-1)!}\right] + \dots;$$

therefore

$$(1+\mu y)\,\epsilon^x = 1 + x + \mu z + \frac{(x+\mu z)^2}{2!} + \dots + \frac{(x+\mu z)^r}{r!} + \dots,$$

if μ is a small constant whose square and higher powers may be neglected.

[It might be objected that we are dealing with an infinite series of operations, and that, for instance, the coefficient of μ^2 is an operation the result of which R when applied to any function might not be a convergent series, and hence μ^2R could not be neglected in comparison with μ.

The limitation to be placed upon the subject of the operation is that, when operated upon by

$$1+(\lambda x+\mu y)+\frac{1}{2!}(\lambda x+\mu y)^2+\dots \text{ to infinity}$$

(where λ and μ are constants), it will give a convergent series.

The coefficients of the different powers and products of λ, μ will then give convergent series.

The limitation has been implicitly assumed in the proof of Lie's theorem for

$$x_i' = 1+(\lambda_1 X_1+\dots)\, x_i + \frac{1}{2!}(\lambda_1 X_1+\dots)^2 x_i+\dots,$$

and it is assumed that x_i' is finite and definite.]

Let X denote the linear operator

$$\sum_{i=1}^{i=n} \xi_i\,(x_1,\, x_2,\, \dots\, x_n)\,\frac{\partial}{\partial x_i};$$

and Y the operator $\displaystyle\sum_{i=1}^{i=n} \eta_i\,(x_1,\, x_2,\, \dots\, x_n)\,\frac{\partial}{\partial x_i}$;

let Y' denote the linear operator

$$\sum_{i=1}^{i=n} \eta_i\,(x_1',\, x_2',\, \dots\, x_n')\,\frac{\partial}{\partial x_i'},$$

obtained from Y by writing for x_i, x_i', where

$$x_i' = \left(1+tX+t^2\frac{X^2}{2!}+\dots\right) x_i,$$

t being a constant; then it will be proved that

$$Y' \equiv Y - tY_1 + \frac{t^2}{2!}Y_2 - \frac{t^3}{3!}Y_3 + \dots,$$

where

$$\begin{aligned} Y_1 &\equiv YX - XY, \\ Y_2 &\equiv Y_1X - XY_1, \\ Y_r &\equiv Y_{r-1}X - XY_{r-1}, \end{aligned}$$

all of the Y's being linear operators.

[This is the completion of the theorem given by Lie, *Transformationsgruppen*, I., p. 141 where he neglects powers of t above the first.]

Now $\quad Y' \equiv Y'(x_1)\,\dfrac{\partial}{\partial x_1} + Y'(x_2)\,\dfrac{\partial}{\partial x_2}+\dots,$

this being a general property of all linear differential operators;

and $$x_i \equiv \left(1 - tX' + \frac{t^2}{2!} X'^2 - \dots\right) x_i' \quad (a) \text{ [Lie, I., p. 53 (7a)]};$$

therefore $$Y'(x_i) \equiv Y' \left(1 - tX' + \frac{t^2}{2!} X'^2 - \dots\right) x_i';$$

therefore since the right-hand member is now a function $x_1', x_2', \dots$, we have [Lie, I., p. 52 (7)]

$$(\beta) \quad Y'(x_i) \equiv \left(1 + tX + \frac{t^2}{2!} X^2 + \dots\right) Y \left(1 - tX + \frac{t^2}{2!} X^2 - \dots\right) x_i.$$

[For convenience of reference, I give a proof of these theorems only slightly modified from Lie's proof.

Since $$x_i' = \left(1 + tX + t^2 \frac{X^2}{2!} + \dots\right) x_i,$$

therefore $$\frac{\partial x_i'}{\partial t} = X \left(1 + tX + t^2 \frac{X^2}{2!} + \dots\right) x_i$$

$$= Xx_i';$$

therefore $$\frac{\partial f(x_1', x_2', \dots x_n')}{\partial t} \equiv \frac{\partial f}{\partial x_1'} Xx_1' + \dots$$

$$\equiv Xf(x_1' \dots x_n').$$

Similarly, $$\frac{\partial^r f'}{\partial t^r} \equiv X^r f',$$

writing f' for $f(x_1' \dots x_n')$.

Now, by Taylor's theorem,

$$f' = (f')_{t=0} + t \left(\frac{\partial f'}{\partial t}\right)_{t=0} + \dots.$$

Now, $$(f')_{t=0} \equiv f \quad \text{and} \quad (X^r f')_{t=0} \equiv X^r f;$$

therefore $$f' = f + tXf + \frac{t^2 X^2}{2!} f + \dots.$$

This is the theorem of which (β) is a particular case. Again, $X'^r f'$ is a function of $x' \dots x_n'$, and therefore, by the above theorem,

$$X'^r f' \equiv \left(1 + tX + \frac{t^2 X^2}{2!} + \dots\right) X^r f;$$

therefore $\left(1-tX'+\frac{t^2X'^2}{2!}-\dots\right)f'$

$$\equiv\left(1+tX+\frac{t^2X^2}{2!}+\dots\right)\left(1-tX+\frac{t^2X^2}{2!}-\dots\right)f\equiv f$$

(proof exactly the same as proof that $\epsilon^x\epsilon^{-x}=1$).

This is the theorem of which (α) is a particular case.]

Expanding the right-hand member of β, we see that it is equal to

$$\{Y+t(XY-YX)+\dots\}\,x_i.$$

It is at once seen that the coefficient of t^r in the bracket is

$$\frac{X^rY}{r!}-\frac{X^{r-1}Y}{(r-1)!}\frac{X}{1!}+\frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!}-\dots;$$

and it is to be shown that this is

$$(-1)^r\frac{Y_r}{r!}.$$

Assume $(-1)^{r-1}\dfrac{Y_{r-1}}{(r-1)!}=\dfrac{X^{r-1}Y}{(r-1)!}-\dfrac{X^{r-2}Y}{(r-2)!}\dfrac{X}{1!}+\dots;$

then $(-1)^{r-1}\dfrac{\{Y_{r-1}X-XY_{r-1}\}}{(r-1)!}$

$$=-\frac{X^rY}{(r-1)!}+r\frac{X^{r-1}Y}{(r-1)!}\frac{X}{1!}-r\frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!}+\dots;$$

therefore $\dfrac{(-1)^rY_r}{r!}=\dfrac{X^rY}{r!}-\dfrac{X^{r-1}Y}{(r-1)!}\dfrac{X}{1!}+\dots,$

so that the theorem, being true when r is 1, is true universally.

Therefore $Y'(x_i)\equiv\left(Y-tY_1+\dfrac{t^2}{2!}Y_2-\dots\right)x_i,$

but $Y'\equiv Y'(x_1)\dfrac{\partial}{\partial x_1}+Y'(x_2)\dfrac{\partial}{\partial x_2}+\dots,$

$$Y_1\equiv Y_1(x_1)\frac{\partial}{\partial x_1}+Y_1(x_2)\frac{\partial}{\partial x_2}+\dots,$$

with similar expressions for Y_2, Y_3, ... ;

therefore $Y'\equiv Y-tY_1+\dfrac{t^2}{2!}Y_2-\dfrac{t^3}{3!}Y_3+\dots.$

I propose to employ these results to prove the theorem given

(*Transformationsgruppen*, I., p. 158) by Lie, and forming the foundation of his theory.

That theorem might be thus stated:—If $x_1', x_2', \dots x_n'$ is a point obtained from the point $x_1, x_2, \dots x_n$ by the operation

$$1+X+\frac{X^2}{2!}+\dots;$$

and $x_1'', x_2'', \dots x_n''$ is a point obtained from the point $x_1', x_2', \dots x_n'$ by the operation

$$1+Y'+\frac{Y'^2}{2!}+\dots,$$

where $$X\equiv\lambda_1X_1+\dots\lambda_rX_r,$$

and $$Y\equiv\mu_1X_1+\dots\mu_rX_r,$$

X_k denoting the linear operator

$$\sum_{i=1}^{i=n}\xi_{ki}(x_1, x_2, \dots x_n)\frac{\partial}{\partial x_i},$$

then $x_1'', x_2'', \dots x_n''$ can be directly derived from the point $x_1, x_2, \dots x_n$ by the operation

$$1+Z+\frac{Z^2}{2!}+\dots,$$

where $$Z\equiv\nu_1X_1+\dots\nu_rX_r,$$

provided that, for all values of k, j,

$$X_kX_j-X_jX_k\equiv\sum_{s=1}^{s=r}c_{kjs}X_s.$$

In Lie's theorem the sets λ, μ, ν and c are all constants; I shall prove that the same result holds if they are any functions of the variables.

It has been proved [Lie I., § 13] that* every transformation of the simple group

$$1+tY+\frac{t^2}{2!}Y^2+\dots$$

can be obtained through repeated operations with the infinitesimal transformation $1+\partial tY$; it will therefore be sufficient to prove the

---

* [Just as in ordinary algebra, we see that

$$\left(1+\frac{tY}{n}\right)^n\equiv1+tY+\frac{t^2}{2!}Y^2+\dots,$$

when n is taken a very large integer.]

theorem for the case when $x_1'', x_2'', \dots x_n''$ is indefinitely near to $x_1', x_2', \dots x_n'$.

We have to prove therefore that, $\mu_1, \mu_2, \dots \mu_r$ being so small that their squares may be neglected,

$$(1+\mu_1X_1'+\mu_2X_2'+\mu_rX_r')\,x_i' = \left(1+Z+\frac{Z^2}{2!}+\frac{Z^3}{3!}+\dots\right)x_i$$

X' denoting the result of substituting x' for x in X.

Now we have proved that

$$Y' \equiv Y - tY_1 + \frac{t^2}{2!}Y_2 - \dots;$$

and, by our hypothesis, Y_1 must belong to the family

$$\rho_1X_1+\rho_2X_2+\dots+\rho_rX_r,$$

where $\rho_1, \rho_2, \dots$ are some functions of the variables $x_1, x_2, \dots x_n$; and, since Y_1 belongs to the family, so also must Y_2, and, by parity of reasoning, $Y_3, Y_4, \dots$. That is, Y' belongs to the family

$$\rho_1X_1+\rho_2X_2+\dots+\rho_rX_r;$$

the theorem required will then be proved if we can prove that

$$(1+k_1X_1+k_2X_2+\dots+k_rX_r)\,x_i' \equiv \left(1+Z+\frac{Z^2}{2!}+\dots\right)x_i,$$

where k_1, k_2, k_r are small; or, remembering that

$$x_i' = \left(1+X+\frac{X^2}{2!}+\frac{X^3}{3!}+\dots\right)x_i,$$

we have to prove that

$$(1+kY)\left(1+X+\frac{X^2}{2!}+\dots\right) \equiv \left(1+Z+\frac{Z^2}{2!}+\dots\right),$$

where Y and X belong to the family

$$\rho_1X_1+\dots+\rho_rX_r,$$

and when k is now a constant so small that its square may be neglected.

Now, we have already proved that

$$(1+kY)\,\epsilon^X \equiv 1+(X+k\overline{Y})+\frac{(X+k\overline{Y})^2}{2!}+\dots+\frac{(X+k\overline{Y})^p}{p!}+\dots,$$

where $$\overline{Y} \equiv Y+a_1Y_1+a_2Y_2+\dots;$$

Y, Y_1, Y_2 being each members of the family, and therefore $\overline{Y}$ also being a member; that is,

$$(1+kY)\,\epsilon^X \equiv \epsilon^Z,$$

where Z is a member of the family. This proves the generalization of Lie's theorem.

It might appear that, in taking k a constant, the proof of the generalization was vitiated, but this is not so; the variables come in through k_1, k_2, ..., and k is merely introduced to make kY small, *e.g.*, we might take $k = \frac{1}{m}$ where m is any large integer.

Some Notes on Symmetric Functions. By William H. Metzler.

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1. In this paper I wish to state three laws by means of which certain symmetric functions are immediately obtained from those already known.

Let $g_1, g_2, g_3, \dots g_n$ represent the n roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0, \tag{1}$$

and let $a_1, a_2, a_3, \dots a_r$ and $b_1, b_2, b_3, \dots b_r$ represent the

$$\lambda = n_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

products of those roots r and $n-r$ at a time respectively.

The coefficient p_i may be said to be complementary to p_{n-i} with respect to n.

The first law in question may be stated as follows:—

If we have given the value, in terms of the coefficients, of the symmetric functions

$$\Sigma g_1^{\alpha} g_2^{\beta} g_3^{\gamma} \dots g_n^{\kappa}, \tag{a}$$

where $\alpha \geqq \beta \geqq \gamma \geqq \dots \geqq \kappa \geqq 0$ and $\alpha+\beta+\gamma+\dots+\kappa = n$,

we can immediately write down all those terms, involving the coefficients $(p_1, p_2, \dots p_n)$ only in the value of

$$\Sigma g_1^{\alpha'-\kappa} \dots g_{n-2}^{\alpha'-\gamma} g_{n-1}^{\alpha'-\beta} g_n^{\alpha'-\alpha}, \tag{b}$$

by replacing every p in the value of (a) by its complementary with respect to n, and multiplying every term by such a power of p_n as will make all the terms of the same weight $n(\alpha'-1)$, where $\alpha' \geqq \alpha$.

Let $h_1, h_2, h_3, \ldots h_n$ be the products of the roots of equation (1) $(n-1)$ together, and let

$$x^n+q_1x^{n-1}+q_2x^{n-2}+\ldots+q_n=0 \tag{2}$$

be the equation having $h_1, h_2, \ldots h_n$ for roots.

Let $f(p_{n-1}, p_{n-2}, \ldots p_1, 1)$ denote the function obtained on writing for every p in $f(p_1, p_2, \ldots p_n)$ its complementary with respect to n.

The value of (a) is evidently isobaric of weight n, and may be denoted by $f(p_1, p_2, \ldots p_n)$.

The value of
$$\Sigma h_1^{\alpha} h_2^{\beta} h_3 \ldots h_n^{\kappa} \tag{c}$$
is then $f(q_1, q_2, \ldots q\ )$.

But $\quad (c) = \Sigma g_1^{n-} \ldots g_{n-2}^{n-\gamma} g_{n-1}^{n-\beta} g_n^{n-\alpha} = p_n^{n-\alpha'} \Sigma g_1^{\alpha'-\kappa} \ldots g_{n-2}^{\alpha'-\gamma} g_{n-1}^{\alpha'-\beta} g_n^{\alpha'-\alpha}$

and $\qquad q_1 = p_{n-1}, \quad q_2 = p_n p_{n-2}, \quad \ldots, \quad q_n = p_n^{n-1};$

therefore $\quad \Sigma g_1^{\alpha'-\kappa} \ldots g_{n-2}^{\alpha'-\gamma} g_{n-1}^{\alpha'-\beta} g_n^{\alpha'-\alpha} = p_n^{\alpha'-n} f(p_{n-1}, p_n p_{n-2}, \ldots p_n^{n-1})$

$$= p_n^{\beta} f(p_{n-1}, p_{n-2}, \ldots p_1, 1),$$

where β may vary from term to term, which proves the law.

In this law we have, for greater definiteness, supposed that the degree of the symmetric function (a) was n, but it is evident that the principle of the law, though not exactly as worded, is true, no matter what the degree of the function may be. If $\alpha+\beta+\gamma+\ldots+\kappa=m$ and $m<n$, it is necessary, in the application of the law, since the value of (a) is isobaric of weight m, and therefore can contain no coefficient whose suffix is greater than m, to replace every p by its complementary with respect to m, and to multiply every term by such a power of p_m as will make all the terms of the weight $(n\alpha'-m)$. Similarly, if $m>n$.

As an example, we find from the tables that

$$\Sigma g_1^3 g_2^2 g_3^1 = p_1p_2p_3 - 3p_3^2 - 3p_1^2p_4 + 4p_2p_4 + 7p_1p_5 - 12p_6.$$

If $n=5$ and $\alpha'=3$, then

$$\Sigma g_1^3 g_2^3 g_3^2 g_4^1 = p_2p_3p_4 - 3p_2^2p_5 - 3p_4^2p_1 + 4p_1p_3p_5 + 7p_4p_5$$
$$+\text{terms involving } p_6,\ p_7, \text{ \&c.}$$

If $n=4$ and $\alpha'=4$, then

$$\Sigma g_1^4 g_2^3 g_3^2 g_4 = p_1p_2p_3p_4 - 3p_1^2p_4^2 - 3p_3^2p_4 + 4p_2p_4^2 + \text{terms involving } p_5, p_6, \text{ \&c.}$$

2. The other two laws may, for convenience, be stated together as follows:—

If we have given the value of $\Sigma a_1 a_2 \dots a_s$ in terms of the fundamental symmetric functions $(p_1, p_2, \dots p_n)$, then we may immediately write down the value of

$$\begin{Bmatrix} \Sigma b_1 b_2 \dots b_s \\ \Sigma a_1 a_2 \dots a_{\lambda-s} \end{Bmatrix}$$

in terms of the fundamental symmetric functions by replacing every p in the value of $\Sigma a_1 a_2 \dots a_s$ by its complementary with respect to n, and multiplying every term by such a power of p_n as will make all the terms of the weight

$$\begin{Bmatrix} (n-r)s \\ (\lambda-s)r \end{Bmatrix}.$$

Stating the first of these two laws symbolically, we have,

if $$\Sigma a_1 a_2 \dots a_s = f(p_1, p_2, \dots p_n),$$

then $$\Sigma b_1 b_2 \dots b_s = p_n^{\beta}(p_{n-1}, p_{n-2}, \dots p_1, 1),$$

where β may vary from term to term.

Let $d_1, d_2, \dots d_\lambda$ denote the products of the roots $h_1, h_2, \dots h_n$ r together; then

$$\Sigma d_1 d_2 \dots d_s = f(q_1, q_2, \dots q_n).$$

But $$d_1 = b_1 p_n^{r-1}, \quad d_2 = b_2 p_n^{r-1}, \quad \text{\&c.},$$

and $$q_1 = p_{n-1}, \qquad q_2 = p_n p_{n-2}, \quad \text{\&c.};$$

therefore $$\Sigma d_1 d_2 \dots d_s = p_n^{s(r-1)} \Sigma b_1 b_2 \dots b_s,$$

$$f(q_1, q_2, \dots q_n) = f(p_{n-1}, p_{n-2}, \dots p_n^{n-1}),$$

and therefore $$\Sigma b' b_2 \dots b_s = p_n^{s(1-r)} f(p_{n-1}, p_n p_{n-2}, \dots p_n^{n-1})$$

$$= p_n^{\beta} f(p_{n-1}, p_{n-2}, \dots p_1, 1).$$

The proof of this law may be made to depend upon the proof of the law in § 1, by observing that for every term in $\Sigma a_1 a_2 \dots a_s$, when expressed in terms of the roots $g_1 g_2 \dots g_n$, of the form $g_1^{\alpha} g_2^{\beta} g_3^{\gamma} \dots g_n^{\kappa}$, there is a term in $\Sigma b_1 b_2 \dots b_s$ of the form $g_1^{a-\kappa} \dots g_{n-1}^{a-\beta} g_n^{a-\alpha}$.

The second of the two laws follows immediately on observing that

$$b_1 b_2 \dots b_s = p_n^{(sn-\lambda r)/n} a_1 a_2 \dots a_{\lambda-s},$$

and therefore $$\Sigma b_1 b_2 \dots b_s = p_n^{(sn-\lambda r)/n} \Sigma a_1 a_2 \dots a_{\lambda-s}.$$

Ex.— $n=5$, $r=2$, $s=3$, and therefore $\lambda=10$ and $\lambda-s=7$.

Given that $\Sigma a_1 a_2 a_3 = p_1^2 p_4 + p_3^2 - 2p_2 p_4 - p_1 p_5$,

the laws give $\Sigma b_1 b_2 b_3 = p_4^2 p_1 + p_2^2 p_5 - 2p_1 p_3 p_5 - p_4 p_5$,

$$\Sigma a_1 a_2 \dots a_7 = p_5 \{p_4^2 p_1 + p_2^2 p_5 - 2p_1 p_3 p_5 - p_4 p_5\}$$

and $$\Sigma b_1 b_2 \dots b_7 = p_5^3 \{p_1^2 p_4 + p_3^2 - 2p_2 p_4 - p_1 p_5\}.$$

It is evident now that, if we have given the value of

$$\Sigma a_1^{\alpha} a_2^{\beta} \dots a_s^{\kappa},$$

where $$\alpha \overset{=}{>} \beta \overset{=}{>} \gamma \overset{=}{>} \dots \overset{=}{>} \kappa,$$

we can write down the values of

$$\Sigma b_1^{\alpha} b_2^{\beta} \dots b_s^{\kappa},$$

and $$\Sigma a_1^{\alpha} a_2^{\alpha} \dots a_{\lambda-\kappa}^{\alpha} a_{\lambda-\kappa+1}^{\alpha-\kappa} \dots a_{n-1}^{\alpha-\beta} a_n^{\alpha-\alpha}.$$

Thursday, April 8th, 1897.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Seventeen members present.

The following gentlemen were elected members: — Samuel Hawksley Burbury, M.A., F.R.S., late Fellow of St. John's College, Cambridge; Rev. Andrew Hollingworth Frost, M.A., St. John's College, Cambridge, Teacher of Marathi in the University; and Edmund Taylor Whittaker, B.A., Fellow and Lecturer, Trinity College, Cambridge. Messrs. R. Bryant, F. W. Lawrence, A. Young, and the Rev. F. H. Jackson were admitted into the Society.

The President dwelt upon the loss the Society and the mathematical world generally had sustained by the death of Prof. Sylvester,* the second President of the Society, and stated that, at the wish of the Council, he would write a message of sympathy to the deceased Professor's nearest relative.

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* *Cf. Nature* (March 25, 1897, pp. 492–494).

The Rev. F. H. Jackson read a paper on "An Extension of the Theorem $F(\alpha, \beta, \gamma) = \dfrac{\Pi(\gamma-\alpha-\beta-1) \,.\, \Pi(\gamma-1)}{\Pi(\gamma-\alpha-1) \,.\, \Pi(\gamma-\beta-1)}$."

Mr. Macaulay gave a sketch of a "Note on the Deformation of a Closed Polygon so that a certain Function remains Constant."

The President communicated a paper by Herr Sommerfeld, entitled "Über verzweigte Potentiale im Raum."

Mr. S. Roberts then took the Chair, and the President read the following papers:—"On the Potentials of Rings," by Mr. A. L. Dixon, and "Certain Concomitant Determinants," by Mr. J. W. Russell.

Mr. Love communicated the abstract of a paper by Prof. Sampson, entitled "A Continuation of Gauss's Dioptrische Untersuchungen."

Mr. Hargreaves and Lt.-Col. Cunningham made impromptu communications.

The following presents were made to the Library:—

"Smithsonian Report, 1894, 8vo; Washington, 1896.

"Proceedings of the Royal Society," Vol. LX., Nos. 367, 369.

Gentry, Ruth.—"On the Forms of Plane Quartic Curves," 8vo; New York, 1896. (Dissertation for degree of Doctor of Philosophy.)

"Bulletin des Sciences Mathématiques," Tome XXI., Mars 1897; Paris.

"Bulletin de la Société Mathématique de France," Tome XXV., No. 1; Paris, 1897.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. III., No. 6, March, 1897; New York.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. III., Fasc. 2; Napoli, 1897.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VI., Fasc. 4, 5; Roma, 1897.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Hefte XL.–LIII.

"Annali di Matematica," Tomo XXV., Fasc. 1, 2; Milano, 1897.

"Educational Times," April, 1897.

"Journal für die reine und angewandte Mathematik," Bd. CXVI., Heft 4; Berlin, 1896.

"Indian Engineering," Vol. XXI., Nos. 8–11; Feb. 20–March 13, 1897.

Neovius, E. R.—"Minimalflächenstücke."

Stenberg, E. A.—"Zur Theorie der Linearen Homogenen Differentialgleichungen mit doppeltperiodischen Coefficienten."

Extracted from the "Acta Societatis Scientiarum Fennicæ," Tom XIX. and XX., and presented by the Meteorological Office, London:—

Neovius, A.—"Tafeln zum Gebrauch bei Stereometrischen Wägungen."

Lindelöf, E.—"Sur l'Integration de l'Equation différentielle de Kummer."

Levänen, Dr. S.—"En Symmetrisk Lësning af Likheter af 2-dra, 3-dje, och 4-de Graden."

Tallqvist, H. S.—"Ueber Specielle Integrationen bei denen die Oberfläche eines Ungleichaxigen Ellipsoids das Integrationsgebiet Bildet."

Lindelöf, E.—"Sur le Mouvement d'un Corps de Révolution roulant sur un plan horizontal."

Mellin, H. J.—"Om definita Integraler."

Lindelöf, E. —"Sur les Systèmes complets et le Calcul des Invariants différentiels des Groupes continus finis."

Hällström, C. P.—"Triangelmätning ifrån Åbo öfver Åland Till Stacksten."

Varsovie: "Comptes Rendus de la Section de Physique et de Chimie," 1889-1890, 1894-5; "Comptes Rendus des Assemblées Générales," 1891-2, 1894-5; "Comptes Rendus de la Séance Annuelle," 1891-3; "Travaux de la Société des Naturalistes," Tome I.-III. (16 Nos. in Russian.)

Über verzweigte Potentiale im Raum. *Von* A. SOMMERFELD, *in* Göttingen.

Received April 5th, 1897. Read April 8th, 1897.

INHALTSÜBERSICHT.

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§ 1. *Die Thomson'sche Spiegelmethode und ihre Erweiterung mittelst verzweigter Potentiale.*

Die *Methode der Thomson'schen Spiegelbilder* besteht im Grunde darin, dass man eine Randwertaufgabe, d. h. eine für ein begrenztes Gebiet gestellte Potentialaufgabe, zurückführt auf eine Aufgabe für den unbegrenzten und unverzweigten Raum. Zu dem Zwecke hat man sich vor Allem die analytische Fortsetzung der gesuchten Function über das vorgelegte Gebiet hinaus klar zu machen, was bei ebenflächiger oder sphärischer Begrenzung einfach dadurch gelingt, dass man das Ausgangsgebiet durch elementare oder durch sphärische Spiegelung an den Grenzflächen reproducirt. *In Folge dessen ist es klar, dass die Thomson'sche Methode dann und nur dann zum Ziele führen wird, wenn das betrachtete (eben- oder kugelflächig begrenzte) Gebiet bei der fortgesetzten symmetrischen Wiederholung an seinen Grenzflächen den Raum lückenlos und einfach ausfüllt.*

Bekanntlich kann man die Potentialaufgaben dadurch übersichtlicher gestalten, dass man sich principiell auf die Herstellung der *Green'schen Function* beschränkt, d. h. einer Function, welche im Innern des Gebietes an *einer* Stelle einen "einfachen Polen" (eine Unendlichkeitsstelle von der Ordnung $1/R$) besitzt und auf der Begrenzung etwa verschwindet. Der Pol im Innern des Gebietes findet sich in jedem Spiegelbilde des Ausgangsraumes je einmal wieder vor.

Wir mögen zunächst einige Bezeichnungen der gewöhnlichen Functionentheorie (zweidimensionalen Potentialtheorie) auf die räumlichen Potentiale übertragen. Wir wollen ein Potential ein *rationales* nennen, wenn es im ganzen Raume eindeutig definirt ist und nur eine endliche Anzahl von Polen besitzt. Andrerseits werden wir ein Potential *transcendent* nennen, wenn die Pole in unendlicher Anzahl vorhanden sind. Dann können wir sagen: Die Thomson'sche Methode führt, je nachdem sich bei der symmetrischen Wiederholung des Ausgangsgebietes das Verfahren nach einer endlichen Anzahl von Schritten schliesst oder nicht, auf rationale oder auf eindeutige, im ganzen Raum definirte transcendente Potentiale. Auf ein rationales Potential kommen wir z. B. bei einem Ausgangsgebiete, welches von zwei unter einem Winkel π/m sich schneidenden Ebenen, oder, um ein complicirteres Beispiel zu nennen, von vier zu einander orthogonalen Kugeln begrenzt wird. Ein bekanntes Beispiel eines im obigen Sinne transcendenten Potentiales liefert die unendliche planparallele Platte oder das Äussere von zwei sich nicht schneidenden Kugeln.

Es liegt nahe, auch den Begriff der *algebraischen*, oder allgemeiner der *verzweigten Functionen* auf den Raum zu übertragen. Wir können ein räumliches Potential ein *algebraisches* nennen, wenn es im ganzen Raume definirt ist, nur eine endliche Anzahl von Polen und im Übrigen beliebige Verzweigungen von endlicher Multiplicität besitzt. Die Bezeichnung rechtfertigt sich dadurch, dass die analog definirten zweidimensionalen Potentiale — im Falle $p=0$ stets, im Falle $p>0$ unter Umständen — reelle Teile von gewöhnlichen algebraischen Functionen sind. Natürlich haben wir im Raume nicht

Verzweigungspunkte, sondern Verzweigungslinien zu erwarten, die wir als stetige und etwa als stückweise analytische Curven voraussetzen werden. Um die verschiedenen Zweige der Function bequem von einander zu sondern, wird man als Analogon zu der gewöhnlichen Riemann'schen Fläche einen "*Riemann'schen Raum*" betrachten, in welchem die Function eindeutig und stetig wird. Einen solchen Riemann'schen Raum stellen wir uns folgendermassen her. Wenn unser algebraisches Potential in dem schlichten Raume n-deutig ist, so nehmen wir n-Exemplare des gewöhnlichen Raumes her und markiren in ihnen die Verzweigungslinien. Darauf spannen wir zwischen den Verzweigungslinien Membranen von beliebiger Gestalt aus und schneiden jedes Raumexemplar längs dieser Membranen auf. Die so entstehenden rechten und linken Seiten der Schnittflächen fügen wir derart aneinander, wie es durch die Werteverteilung des Potentials angezeigt ist. Jeder ebene Schnitt durch einen Riemann'schen Raum liefert eine gewöhnliche Riemann'sche Fläche; die Durchstossungspunkte unserer Schnittebene mit den Verzweigungslinien entsprechen den Verzweigungspunkten, die Durchdringungslinien mit den Membranen geben Verzweigungsschnitte der Riemann'schen Fläche im gewöhnlichen Sinne.

Auch bei der Herstellung der algebraischen Potentiale wird man sich passend auf die *Green'sche Function des betr. Riemann'schen Raumes* beschränken können, d. h. einer Function, welche in dem Raume nur einen einfachen Pol besitzt. (Wegen der genaueren Definition der Green'schen Function vergl. den folgenden §.) Die Green'sche Function des schlichten Raumes ist das Newton'sche Potential $1/R$, umgekehrt ist die Green'sche Function eines beliebigen Riemann'schen Raumes das genaue Analogon zu dem Newton'schen Potentiale in diesem.

Die Kenntnis dieser Green'schen Functionen setzt uns in den Stand, das Anwendungsgebiet der Thomson'schen Spiegelmethode erheblich auszudehnen. *In der That werden wir aus der Green'schen Function für den unbegrenzten Riemann'schen Raum die Green'sche Function für ein solches (eben- oder kugelflächig begrenztes) Gebiet des gewöhnlichen Raumes synthetisch zusammensetzen können, welches bei fortgesetzter symmetrischer Wiederholung den betr. Riemann'schen Raum einfach und lückenlos ausfüllt.* Das Gebiet muss also so beschaffen sein, dass seine symmetrischen Wiederholungen den schlichten Raum n-fach überdecken, und dass die successiven Spiegelbilder sich gerade längs der Verzweigungslinien unseres Riemann'schen Raumes aneinander legen.

Das einfachste Beispiel liefert ein von zwei Ebenen (oder Kugeln) begrenztes Gebiet, welche unter dem Winkel $\frac{n\pi}{m}$ an einander stossen. Die Green'sche Function dieses Gebietes wird, analytisch fortgesetzt, in einem Riemann'schen Raume von der Exemplarenzahl n mit einer einzigen, geradlinigen (oder kreisförmigen) Verzweigungslinie eindeutig und besitzt in diesem Raume $2m$ Pole. Nimmt man speciell $m=1$, $n=2$, so verwandelt sich das Gebiet in den unendlichen Raum, die Begrenzung des Gebietes in einen ebenen, geradlinig begrenzten, unendlich dünnen Schirm (oder in eine Kugelkalotte).

Den entsprechenden Ansatz wird man offenbar bei einem beliebig berandeten, ebenen, unendlich dünnen Schirm zu machen haben. Man wird zunächst nach der Green'schen Function eines Doppelraumes fragen, welcher den Rand des Schirmes zur einfachen Verzweigungslinie hat. Die Fläche des Schirmes kann dabei als "Verzweigungsmembran" angesehen werden, welche die beiden Exemplare des Raumes trennt. Auf diesen Riemann'schen Raum wird man nämlich geführt, wenn man den gewöhnlichen unendlichen Raum an der Ebene des Schirmes spiegelt. Die Green'sche Function für das ganze Äussere des Schirmes mit dem Pole P wird alsdann dargestellt durch die Differenz zweier Green'scher Functionen unseres Riemann'schen Raumes,

$$u_P - u_{P'},$$

von denen die eine in dem vorgeschriebenen Punkte P, die andere in dem dem zweiten Raumexemplare angehörigen, zu P hinsichtlich des Schirmes spiegelbildlich gelegenen Punkte P' einen einfachen Pol besitzt.

Wir haben damit einen Ansatz gewonnen, der eine "*theory of electric screens*," wie sie Lord Kelvin gelegentlich* postulirt, oder eine *Theorie der elektrischen Schattenwirkung metallischer Schirme*, wie ich im Anschluss an die entsprechenden Probleme der Optik sage, zu liefern im Stande ist.

§ 2. *Bedingungen für die eindeutige Bestimmtheit verzweigter Potentiale.*

Wir müssen uns zunächst klar machen, welche Bedingungen wir unsern verzweigten Potentialen auferlegen müssen, um sie eindeutig festzulegen. Wir werden zeigen, dass folgende Bedingungen eine Function u eindeutig festlegen, welche wir die *Green'sche Function unseres Riemann'schen Raumes* nennen.

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* Papers on "Electrostatics and Magnetism," Art. xv., Sect. 248, p. 191.

(i.) *Im ganzen Riemann'schen Raum incl. der Verzweigungslinien soll u eindeutig definirt, endlich und stetig sein, ausser in einem Punkte P desselben, wo u unendlich wird, wie $1/R$ für $R = 0$, falls R den Abstand eines beliebigen Punktes Q des Raumes von P bedeutet.*

(ii.) *u soll der Differentialgleichung $\Delta u = 0$ im ganzen Riemann'schen Raume genügen, ausser im Punkte P und in den Verzweigungslinien. In dieser Bedingung ist enthalten, dass u sonst überall endliche erste und zweite Differentialquotienten haben soll.*

(iii.) *u soll im Unendlichen verschwinden.*

Dass diese Bedingungen zur eindeutigen Festlegung der Green'schen Function *hinreichen*, dass also *nur* eine Green'sche Function existirt, zeigen wir auf Grund des Green'schen Satzes. Dass die Bedingungen aber mit einander verträglich sind, dass also *wirklich* eine Green'sche Function existirt, werden wir im Allgemeinen hier nicht beweisen, sondern später nur in speciellen Fällen darthun. Offenbar erfordert die Beantwortung der Existenzfrage schwierigere Betrachtungen im Sinne der bekannten *Schwarz-Neumann'schen Methoden.*

Wir nehmen an, es gäbe zwei verschiedene Functionen, v_1 und v_2, welche den Bedingungen (i.), (ii.) und (iii.) genügen. Dann ist

$$u = v_1 - v_2$$

eine Function, welche den Bedingungen (ii.) und (iii.) genügt, und überdies im ganzen Riemann'schen Raume ausnahmslos endlich und stetig ist. Es wird zu zeigen sein, dass diese Function notwendig identisch verschwindet.

Zunächst können wir voraussetzen, dass das Unendliche unverzweigt ist; es genügt nämlich von irgend einem Punkte aus, welcher keiner Verzweigungslinie angehört, eine Inversion zu machen, um alle Verzweigungslinien aus dem Unendlichen fortzuschaffen. Sodann wollen wir um jede Verzweigungslinie herum eine Röhrenfläche construiren, indem wir eine Kugel vom Radius ϵ mit ihrem Mittelpunkte längs der Verzweigungslinie entlang schieben. Die Enveloppe der so bewegten Kugel bilde unsere Röhrenfläche. Denken wir uns den Riemann'schen Raum durch die zwischen den Verzweigungslinien ausgespannten Membranen, wie früher, in n-Raumexemplare geschieden, so haben wir in jedem Exemplare um jede Verzweigungslinie herum eine Röhrenfläche. Endlich wollen wir um einen beliebigen Punkt P des Raumes eine Kugel geschlagen denken, deren Radius r so gross gewählt werde, dass alle Verzweigungslinien im Innern der

Kugel verlaufen, was nach Obigem stets möglich ist. Entsprechend den n-Raumexemplaren, haben wir wieder n solcher Kugeln zu unterscheiden.

Jedenfalls können wir den Green'schen Satz auf denjenigen Teil eines jeden unserer n-Raumexemplare anwenden, welcher begrenzt ist einerseits durch die Kugel K, andrerseits durch die Röhrenflächen und die beiden Seiten der Membranen, soweit sie aus den Röhren herausragen. Wir benutzen folgende Form des Green'schen Satzes:

$$(1)\qquad \int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]d\tau+\int u\,\Delta u\,d\tau=\int u\frac{\partial u}{\partial n}\,do.$$

Hier sind die beiden Integrale links über das ganze Innere des soeben näher beschriebenen Gebietes, das Integral rechts über seine Oberfläche, d. h. die Kugel K, die Röhrenfläche und die beiden Seiten der Membran zu erstrecken. ∂n bedeutet ein Element der von dem Gebiete fortgerichteten Oberflächennormalen.

Für jedes Raumexemplar haben wir eine Gleichung von der Form (1); wir wollen alle diese Gleichungen addiren. Dann sind fernerhin die Integrale links über den ganzen Riemann'schen Raum, mit Ausschluss des Innern der Röhren und des Äussern der Kugeln, zu erstrecken. Das Integral rechts ist zu erstrecken (i.) $2n$-mal über die einzelne Membran, soweit sie aus der Röhrenfläche herausragt; (ii.) n-mal über die einzelne Röhrenfläche; (iii.) n-mal über die Kugel K. Wir wollen zeigen, dass diese Integrale sämtlich verschwinden.

I. *Es ist leicht zu sehen, dass die Integrale* (i.) *sich in der Summe gegenseitig zerstören.* Bezeichnen wir nämlich die beiden Seiten einer unserer Membranen irgendwie als rechte und linke, und ein Raumexemplar als erstes, ferner dasjenige, welches mit dem ersten längs der rechten Seite der Membran zusammenhängt als zweites, so wird

$$\int_{R_1} u\frac{\partial u}{\partial n}\,do=-\int_{L_2} u\frac{\partial u}{\partial n}\,do,$$

d. h. das Integral über die rechte Seite der Membran im ersten Exemplar wird gleich dem negativen Werte des Integrals über die linke Seite im zweiten. In der That haben wir

$$u_{R_1}=u_{L_2},\quad \left(\frac{\partial u}{\partial n}\right)_{R_1}=-\left(\frac{\partial u}{\partial n}\right)_{L_2},$$

weil die Normale im Green'schen Satz immer positiv nach aussen zu rechnen ist. Wenn einige Exemplare durch die Membran hindurch schlicht verlaufen, so ändert dieses nichts. Ist nämlich das k^{te} Exemplar ein solches schlicht verlaufendes, so gilt einfach

$$u_{R_k} = u_{L_k}, \quad \left(\frac{\partial u}{\partial n}\right)_{R_k} = -\left(\frac{\partial u}{\partial n}\right)_{L_k}.$$

II. *Die n-Integrale* (ii.) *über die Oberflächen unserer Röhren werden einzeln gleich Null, wenn wir den Radius* ϵ *der die Röhre beschreibenden Kugel in geeigneter Weise zu Null abnehmen lassen.* Um dies zu zeigen, müssen wir mit einiger Vorsicht verfahren, weil die Differentialquotienten von u in den Punkten der Verzweigungslinien unseren Voraussetzungen nach nicht endlich zu sein brauchen, und es in der That auch nicht sind, wie sich später zeigen wird. Wir führen auf der Röhrenfläche zwei Variable, ϕ und s, ein, von denen ϕ ein Azimuth ist, welches innerhalb des einzelnen Raumexemplares um die Verzweigungslinie herum von 0 bis 2π wächst, und s die Bogenlänge der Verzweigungslinie von einem beliebigen Punkte aus gerechnet bedeutet. Das Oberflächenelement der Röhrenfläche lautet dann $\epsilon\, d\phi\, ds$, und unser Integral (ii.) wird

$$\epsilon \iint d\phi\, ds\, u \frac{\partial u}{\partial n} = \frac{\epsilon}{2} \frac{d}{d\epsilon} \iint d\phi\, ds\, u^2.$$

Wir betrachten die Function

$$F(\eta) = \iint d\phi\, ds\, u^2,$$

wo die Integration über die sämtlichen Röhrenflächen zu erstrecken ist, und die diese erzeugende Kugel den Radius η haben soll; für $\eta \geqq 0$ ist diese Function, ebenso wie u selbst nach Voraussetzung, endlich und stetig. Mithin ist nach dem Mittelwertsatze der Differentialrechnung

$$F(\eta) - F(0) = \eta F'(\epsilon),$$

oder

$$\frac{\epsilon}{\eta}\left[F(\eta) - F(0)\right] = \epsilon \frac{dF(\epsilon)}{d\epsilon},$$

wo ϵ ein zwischen 0 und η passend gewählter Wert ist. Lassen wir η zu Null abnehmen, so wird die linke Seite gleich Null, weil $\epsilon < \eta$ und $F(\eta)$ stetig ist. Wir können also eine Reihe abnehmender Werte, ϵ_1, ϵ_2, ..., so bestimmen, dass

$$\lim_{\eta = 0} \epsilon \frac{dF(\epsilon)}{d\epsilon} = 0$$

wird.

Gerade in dieser Weise wollen wir den Radius ϵ unserer Röhrenflächen wählen und zu Null abnehmen lassen. Alsdann verschwindet das Integral (ii.), wie behauptet wurde.

III. *Wir haben schliesslich zu zeigen, dass unser Integral* (iii.) *über jede der Kugeln K in den n-Raumexemplaren erstreckt, gleichfalls verschwindet, wenn wir den Radius r der Kugel unendlich wachsen lassen.* Wir stellen zu dem Zwecke u im ersten Raumexemplar durch ein Oberflächenintegral dar, indem wir uns auf eine andere Form des Green'schen Satzes stützen. Man hat bekanntlich allgemein

$$(2) \qquad 4\pi u_Q = \int \left(\frac{1}{R} \frac{\partial u}{\partial n} - \frac{\partial 1/R}{\partial n} u \right) do,$$

wo R den Abstand des Punktes Q von dem Oberflächenelement do bedeutet, und wo das Integral in unserem Falle einerseits über beide Seiten der Membranen, soweit sie aus den Röhrenflächen herausragen, und über die Röhrenflächen, soweit sie dem ersten Raumexemplar angehören, und andrerseits über eine Kugel K' zu führen ist, welche um den Punkt Q herum mit unendlich wachsendem Radius beschrieben ist. Dass letzteres Integral verschwindet, folgt in bekannter Weise daraus, dass u im Unendlichen verschwindet und dass

$$\int \frac{\partial u}{\partial n} do$$

über K' geführt einen vom Radius der Kugel K' unabhängigen endlichen Wert hat.

Durch Gleichung (2) ist also u_Q als Potential einer einfachen Oberflächenbelegung und einer Doppelschicht dargestellt, wobei es jetzt keineswegs nötig ist, den Radius ϵ der Röhrenfläche zu Null abnehmen zu lassen.

Aus der Darstellung (2) schliesst man nun in üblicher Weise, dass u_Q im Unendlichen wie $1/r$, und $\frac{\partial u_Q}{\partial h}$, d. h. ein in beliebiger Richtung genommener Differentialquotient von u_Q wie $1/r^2$, verschwindet, wo r den Abstand des Punktes Q von irgend einem endlichen Punkte bedeutet.

Mithin verschwindet auf der Kugel K

$$u \frac{\partial u}{\partial n} \text{ von der Ordnung } 1/r^3,$$

und $$\int u \frac{\partial u}{\partial n} do \quad \text{,,} \quad \text{,,} \quad 1/r,$$

wenn man den Radius der Kugel r in's Unendliche wachsen lässt. Also kann auch das oben unter III. aufgeführte Integral (durch Wahl von r) zum Verschwinden gebracht werden.

Die Gleichung (1) reducirt sich nun, wenn man noch berücksichtigt, dass $\Delta u = 0$ ist, auf

$$(3) \qquad \int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} d\tau = 0;$$

hier ist das Integral auf unsern ganzen Riemann'schen Raum zu erstrecken, mit Ausschluss des Innern der Röhren und des Äusseren der Kugeln, welche Gebiete aber beide beim Grenzübergange $\epsilon = 0$, $r = \infty$ in Fortfall kommen. Aus (3) schliessen wir sofort

$$u = \text{const},$$

und, da im Unendlichen u verschwinden soll,

$$u = 0, \quad \text{q. e. d.}$$

Unsere Green'sche Function ist also durch die Bedingungen (i.), (ii.), (iii.) *eindeutig festgelegt.*

Wir fügen noch einige weitere Bemerkungen über die allgemeinen Eigenschaften der verzweigten Potentiale hinzu, deren Beweis leicht durch die Green'schen Methoden erbracht werden kann.

1. *Der Green'sche Satz kann direkt von dem schlichten Raum auf einen beliebigen Riemann'schen Raum übertragen werden.* Bedeuten nämlich U und V irgend zwei Functionen, welche in demselben Riemann'schen Raume eindeutig sind, so gilt die Gleichung

$$(4) \quad \int \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right) d\tau = -\int U \Delta V d\tau + \int U \frac{\partial V}{\partial n} do$$
$$= -\int V \Delta U d\tau + \int V \frac{\partial U}{\partial n} do,$$

die Raumintegrale ($d\tau$) sind über ein beliebiges zusammenhängendes Stück T des Riemann'schen Raumes, die Oberflächenintegrale (do) über die Begrenzung von T zu erstrecken. Sind die Functionen U und V, oder ihre ersten Differentialquotienten in dem Gebiete T an einzelnen Stellen unstetig, so müssen diese Stellen in Flächen eingeschlossen werden, welche der Begrenzung von T hinzuzurechnen sind. Reicht T bis an die Verzweigungslinien heran, so sind diese Linien durch Röhrenflächen (vgl. oben) auszuschliessen; dehnt sich T in's Unendliche aus, so haben wir T im Unendlichen etwa durch eine Kugel abzugrenzen.

Die Richtigkeit der Gleichung (4) ergiebt sich aus dem Green'schen Satz für den schlichten Raum mit Rücksicht darauf, dass die Integrale

$$\int U \frac{\partial V}{\partial n} do$$

über die beiden Seiten unserer Membranen, oder über Stücke dieser Membranen, geführt, sich gegenseitig zerstören. (Vgl. die oben unter I. durchgeführte Betrachtung.) Wenn speciell U und V sich beim Übergange auf die Verzweigungslinien endlich und stetig verhalten, und das Gebiet T bis an die Verzweigungslinien heranreicht, so kommen die Oberflächenintegrale über die Röhrenflächen in Fortfall (wie oben unter II. gezeigt). Wenn ferner U und V im Unendlichen verschwinden, und das Gebiet T sich in's Unendliche ausdehnt, so verschwinden auch die Oberflächenintegrale über die unendlich ferne Kugel (wie oben unter III.).

2. *Wenn v eine im ganzen Riemann'schen Raume eindeutige Function ist, welche im Unendlichen verschwindet und im Endlichen sich überall regulär verhält und der Differentialgleichung* $\Delta v = 0$ *genügt, ausser an den Stellen* $P_1, P_2, \ldots P_m$, *wo v, wie*

$$\frac{a_1}{\overline{QP_1}}, \quad \frac{a_2}{\overline{QP_2}}, \quad \ldots \frac{a_m}{\overline{QP_m}},$$

für $Q = P_i$ *unendlich wird, so lässt sich v folgendermassen darstellen:*

$$(5) \qquad v = a_1 u_{P_1} + a_2 u_{P_2} + \ldots a_m u_{P_m}.$$

Hier bedeutet u_P, oder ausführlicher geschrieben $u_P(Q)$, die Green'sche Function des betr. Riemann'schen Raumes mit dem Pole P und dem variablen "Aufpunkte" Q. Dieser Satz folgt aus Gleichung (4), wenn wir dort $U = u_Q(P)$ und $V = v$ setzen.

3. *Wenn v eine im Riemann'schen Raume eindeutige, im Unendlichen verschwindende Function ist, welche der Differentialgleichung* $\Delta v = 0$ *genügt, ausser in einem Gebiete* T', *wo sie die Gleichung* $\Delta v = -4\pi\rho$ *befriedigt, so hat v die folgende Form:*

$$v = \int u_Q(P)\, \rho\, d\tau,$$

das Integral über das Gebiet T' genommen.

4. *Für die Green'sche Function eines beliebigen Riemann'schen Raumes gilt der wichtige Satz, dass ihr Wert sich nicht ändert, wenn man den Pol und den Aufpunkt gegenseitig vertauscht.* Die Behauptung dieses Satzes können wir kurz folgendermassen schreiben:

$$u_P(Q) = u_Q(P).$$

Der Beweis folgt aus (4), wenn wir $U = u_P(Q)$, $V = u_Q(P)$ setzen.

5. Wir wollen schliesslich noch einen Satz aufstellen, welcher von dem Verhalten der verzweigten Potentiale in der Nähe der Verzweigungslinie handelt, dessen Beweis wir allerdings nur in den später zu betrachtenden speciellen Fällen erbringen werden.

Wenn v *ein verzweigtes Potential ist, welches in den Punkten einer Verzweigungslinie den endlichen Wert* v_0 *hat, und stetig in diesen Wert übergeht, so verschwindet* $v - v_0$ *beim Übergange auf die Verzweigungslinie wie* $r^{1/n}$ *oder wie eine Potenz dieser Grösse, falls* r *den kürzesten Abstand des gerade betrachteten Punktes von der Verzweigungslinie und* n *die Anzahl von Malen bedeutet, dass sich der Raum um die Verzweigungslinie herumwindet.* Der entsprechende Satz für das Verhalten der algebraischen Functionen in den Verzweigungspunkten einer Riemann'schen Fläche ist wohlbekannt.

§ 3. *Die Green'sche Function eines Riemann'schen Raumes mit einer einzigen, geradlinigen Verzweigungscurve.*

Wir wenden uns nun zur Aufstellung mehrwertiger Potentiale für specielle Riemann'sche Räume. Das Princip, welches wir dabei befolgen, ist allerdings ziemlich formal, aber hinreichend ergiebig, (es reicht sogar über die Potentialtheorie hinaus).

Wir gehen aus von dem Newton'schen Potentiale $1/R$, d. h. der reciproken Entfernung eines festen Punktes, des Poles,

P mit den Coordinaten x', y', z',

und eines variablen Punktes, des "Aufpunktes,"

Q mit den Coordinaten x, y, z,

so dass $$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

Offenbar erhalten wir wieder eine Lösung der Potentialgleichung, wenn wir x', y', z' als Functionen eines beliebigen Parameters α auffassen und bilden

$$(1) \quad u = \int \frac{1}{R} f(\alpha)\, d\alpha,$$

wobei das Integral auf einem beliebigen, ev. complexen Wege in der Ebene des Parameters α genommen werden kann. Durch passende Auswahl des Parameters α und der Function $f(\alpha)$ wird es uns gelingen, das eindeutige Potential $1/R$ in ein mehrdeutiges von analogen Eigenschaften zu verwandeln.

Es handle sich zunächst um einen Riemann'schen Raum von n-facher Überdeckung, welcher sich um eine einzelne, unendlich lange Gerade n-mal herumwindet.

Die Verzweigungslinie nehmen wir zur z-Axe; in der Ebene $z=0$ führen wir Polarcoordinaten ein, indem wir setzen

$$(2) \quad x+iy = e^{\rho+i\phi} = re^{i\phi}, \quad x'+iy' = e^{\rho'+i\phi'} = r'e^{i\phi'}.$$

Die Entfernung der Punkte P und Q kann dann so gechrieben werden: $R^2 = 2rr'\{\cos i(\rho-\rho') - \cos(\phi-\phi')\} + (z-z')^2$,

oder auch $R^2 = r^2 + r'^2 - 2rr' \cos(\phi-\phi') + (z-z')^2$.

Offenbar sind unsere Polarcoordinaten zur Darstellung von Functionen geeignet, welche auf der durch eine Ebene $z = \text{const}$ aus unserem Windungsraume ausgeschnittenen Riemann'schen Fläche eindeutig, in der schlichten Ebene aber mehrdeutig sind. In der That wird jede in r and ϕ eindeutige Function $F(r, \phi)$, welche in ϕ die Periode $2\pi n$ hat, auf der genannten Riemann'schen Fläche eindeutig. Dementsprechend liefert auch $e^{\frac{1}{n}(\rho+i\phi)}$ die einfachste conforme Abbildung dieser Fläche auf die schlichte Ebene. Ebenso sind die Coordinaten z, ρ und ϕ zur Darstellung von Functionen geeignet, welche in unserem Riemann'schen Raume eindeutig sind, indem nämlich jede eindeutige Function der Coordinaten $F(z, \rho, \phi)$, welche in ϕ die Periode $2\pi n$ besitzt, in jenem Raume eindeutig wird.

Zunächst nehmen wir mit dem Newton'schen Potential eine identische Umformung vor. Wir ersetzen ϕ' durch eine complexe Variable α, und bezeichnen den so modificirten Ausdruck von R mit R':

$$R'^2 = 2rr'\{\cos i(\rho-\rho') - \cos(\phi-\alpha)\} + (z-z')^2.$$

Das Vorzeichen von R' soll so bestimmt werden, dass R' für reelle α positiv wird. Sodann multipliciren wir $\frac{1}{R'}$ mit einer Function $f(\alpha)$, welche an der Stelle $\alpha = \phi'$ von der ersten Ordnung mit dem Residuum 1 unendlich wird und integriren nach α auf einem Wege, welcher diese Stelle im positiven Sinne genügend enge umschliesst. Das so entstehende Integral ist, nach dem Cauchy'schen Satze, mit $2\pi i/R$ identisch. Speciell wollen wir die Function $f(\alpha)$ so wählen, dass sie sowohl in α wie in ϕ' die Periode 2π hat. Am einfachsten bietet sich

$$f(\alpha) = \frac{ie^{i\alpha}}{e^{i\alpha} - e^{i\phi'}};$$

unsere identische Umformung lautet also

$$\frac{1}{R} = \frac{1}{2\pi}\int \frac{1}{R'} \frac{e^{i\alpha}d\alpha}{e^{i\alpha} - e^{i\phi'}}.$$

Hier wollen wir noch den Integrationsweg deformiren. Dabei haben wir auf die Singularitäten des Integranden in der α-Ebene zu achten. Die Verzweigungspunkte sind gegeben durch

$$R'^2 = 0 \quad \text{und} \quad R'^2 = \infty .$$

Nun wird $R'^2 = \infty$, wenn $\alpha = \infty$ ist. Andrerseits haben wir $R'^2 = 0$, wenn

$$\cos(\phi - \alpha) = \cos i\alpha_1, \quad \text{d. h.} \quad \alpha = \phi + 2k\pi \pm i\alpha_1,$$

wo k irgend eine ganze Zahl bedeutet und wo gesetzt ist:

$$(3) \quad \cos i\alpha_1 = \cos i(\rho - \rho') + \frac{(z-z')^2}{2rr'} = \frac{r^2 + r'^2 + (z-z')^2}{2rr'};$$

die Verzweigungspunkte des Integranden sind also die folgenden:

$$(4) \quad \begin{cases} \alpha = \infty \\ \alpha = \phi + 2k\pi \pm i\alpha_1. \end{cases}$$

Die Pole des Integranden sind gegeben durch

$$\alpha = \phi' + 2k\pi.$$

Wir markiren uns diese Punkte und schneiden die α-Ebene längs der der imaginären Axe parallelen Geraden von $\phi + 2k\pi \pm i\alpha_1$ bis $\phi + 2k\pi \pm i\infty$ auf. Wir haben also in der α-Ebene unendlich viele, aequidistante Pole und unendlich viele Paare von Verzweigungsschnitten (vgl. die Fig. 1).

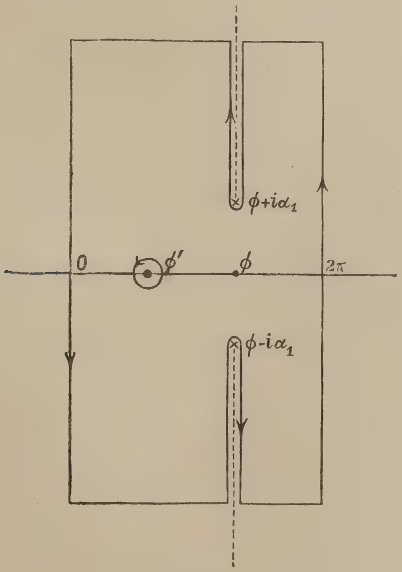

FIG. 1.

Darauf ziehen wir den ursprünglichen, den Punkt $\alpha = \phi'$ umlaufenden Integrationsweg bis an die Verzweigungsschnitte heran, so dass er aus zwei längs den beiden Ufern der Verzweigungsschnitte verlaufenden Schleifen, und im Übrigen aus der Peripherie eines Rechtecks von der Breite 2π und von beliebig zu vergrössernder Höhe besteht. Hier heben sich noch die Integrationen längs der in der Figur senkrechten Rechteckseiten wegen der Periodicität des Integranden gegenseitig auf; ferner verschwinden die Integrale längs der horizontalen Rechteckseiten, wenn wir die Höhe des Rechtecks unendlich werden lassen, weil $1/R'$ für $\alpha = \infty$ hinreichend stark verschwindet. Mithin lässt sich der Integrationsweg einfach auf die beiden (in richtigem Sinne genommenen) Umläufe um zwei Verzweigungsschnitte der α-Ebene reduciren. Wir wollen diesen Weg kurz als "Weg W" bezeichnen.

Jetzt machen wir den Übergang zu einem Potential des n-fachen Windungsraumes. Zu dem Zwecke wollen wir die oben mit $f(\alpha)$ bezeichnete Function so bestimmen, dass sie in α und ϕ' die Periode $2\pi n$ hat und übrigens an der Stelle $\alpha = \phi'$ von der ersten Ordnung mit dem Residuum 1 unendlich wird. Eine solche Function ist

$$f(\alpha) = \frac{i}{n} \frac{e^{i\alpha/n}}{e^{i\alpha/n} - e^{i\phi'/n}}.$$

Wir bilden nun, wie oben,

$$(5) \quad u = \frac{1}{2\pi i} \int \frac{1}{R'} f(\alpha)\, d\alpha = \frac{1}{2\pi n} \int \frac{1}{R'} \frac{e^{i\alpha/n} d\alpha}{e^{i\alpha/n} - e^{i\phi'/n}},$$

wobei das Integral auf dem eben definirten Wege W erstreckt werden soll. Die Verzweigungen des Integranden sind dieselben wie vorher, seine Pole liegen jetzt aber nur noch an den Stellen

$$(6) \quad \alpha = \phi' + 2kn\pi.$$

Wir behaupten, dass diese Function *die Green'sche Function unseres Riemann'schen Raumes ist*, d. h., dass sie den Bedingungen (i.), (ii.), (iii.) des vorigen § genügt.

Ad (i.). Dass u in dem vorliegenden Riemann'schen Raume eindeutig ist, erkennt man folgendermassen: Wir lassen den Punkt Q n-mal etwa auf einem Kreise um die Verzweigungslinie herumwandern, indem wir unter Festhaltung der Coordinaten r und z ϕ von ϕ_0 bis zu $\phi_0 + 2n\pi$ wandern lassen, und fragen nach den Änderungen unseres Integrales. Es ist zu zeigen, dass dieses hierbei auf seinen ursprünglichen Wert zurückkommt. Zunächst ändert sich der Integrand bei einer Änderung von ϕ um $2k\pi$ nicht, da er ja in ϕ die Periode 2π hat. Andrerseits verschiebt sich aber der Integrationsweg W, während wir Q in der angegebenen Weise wandern lassen, parallel der reellen α-Axe um $2\pi n$, weil ja nach Gl. (4) die Verzweigungspunkte der α-Ebene von φ abhängen. Es ändern sich also auch die Werte von α, welche bei der Integration zu benutzen sind. Nun hat aber der Integrand in α die Periode $2\pi n$; also hat eine Verschiebung von W um $2\pi n$ auf den Wert des Integrales thatsächlich keinen Einfluss; wir haben

$$u(\varphi+2\pi n) = u(\phi).$$

Wir haben ferner zu zeigen, dass u für $0 < r < \infty$ nur in dem einen Punkte $Q = P$ des Riemann'schen Raumes unendlich wird. Zunächst scheint ein Unendlichwerden des Integranden überhaupt nicht einzutreten, weil dieser im Unendlichen der α-Ebene (bei reellen Werten der Coordinaten von Q und P) stets in einer für die Convergenz hinreichenden Weise verschwindet, und weil wir die Pole $\alpha = \varphi' + 2kn\pi$ durch passende Deformation des Weges W scheinbar stets vermeiden können. Letzteres ist aber in Wirklichkeit nicht der Fall. Der Weg W ist nämlich zwischen die beiden Verzweigungspunkte $\phi \pm i\alpha_1$ sozusagen eingepflockt. Wenn wir also diese Punkte nach der reellen Axe hin wandern lassen, indem wir $\alpha_1 = 0$ werden lassen, so geht der Weg notwendig zweimal durch den Punkt $\alpha = \phi$ hindurch. Wenn ausserdem $\phi = \phi' + 2kn\pi$ wird, so müssen wir über einen Pol des Integranden hinüberintegriren, so dass unser Integral möglicher Weise unendlich wird.

Das Genauere ergiebt sich folgendermassen: Wir denken uns $z - z'$, $r - r'$, $\phi - \phi'$ sehr klein, so dass α_1 nahezu gleich Null ist und der Pol $\alpha = \phi'$ den Verzweigungspunkten $\alpha = \phi \pm i\alpha_1$ sehr nahe liegt. Die beiden Teile des Weges W können wir so deformiren, dass wir (vgl. die Fig. 2) einerseits einen vollen Umlauf um $\phi = \phi'$ erhalten, und andrerseits zwei der imaginären Axe parallele Gerade. Die Integrationen über die letzteren haben sicher einen endlichen Betrag, und behalten ihn auch, wenn wir $Q = P$ werden lassen. Bei dem Umlauf um $\varphi = \varphi'$ ergiebt sich, nach dem Cauchy'schen Satze, der Wert $1/R$. Lassen wir schliesslich $Q = P$, d. h. $\phi = \phi'$, $\alpha_1 = 0$ werden, so wird $1/R = \infty$.

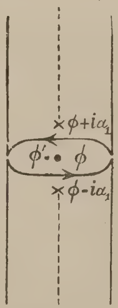

Fig. 2.

Wohlgemerkt ergiebt sich ein Unendlichwerden von u nur für $\phi = \phi'$, bez. $= \phi' + 2kn\pi$, dagegen nicht für

$\phi = \phi' + 2\pi, = \phi' + 4\pi, \ldots \phi' + 2(n-1)\pi$. Im letzteren Falle liefert bei derselben Umformung des Integrationsweges wie oben der Umlauf um den Punkt $\phi' + 2\pi$ etc., wiederum nach dem Cauchy'schen Satze, den Wert Null; der Weg W lässt sich dann direkt durch die beiden genannten parallelen Geraden ersetzen und liefert einen endlichen Wert des Integrales.

Hiermit ist dargethan, dass u für $r > 0$ nur den einen vorgeschriebenen Unendlichkeitspunkt P hat; dass u auch stetig ist, bedarf keines ausdrücklichen Beweises.

Es bleibt schliesslich noch das Verhalten von u in der Verzweigungslinie $r = 0$ zu untersuchen. Wir wollen zu dem Zwecke zunächst den Integrationsweg durch eine passende Substitution in einen reellen verwandeln. Wir setzen $\alpha = \phi \pm i\beta$, je nachdem wir uns bei der Integration oberhalb oder unterhalb der reellen Axe befinden. Lassen wir α einen der beiden Wege durchlaufen, aus denen sich W zusammensetzt, so beschreibt β beidemal denselben Weg (nämlich von ∞ bis α_1 und zurück). Ferner werden noch die Integrale längs des rechten und linken Ufers unseres Verzweigungsschnittes in der α-Ebene einander gleich. Wir können also den Integrationsweg von β durch einen einmaligen Weg von α_1 bis ∞ ersetzen, wenn wir dafür den Factor 2 hinzufügen. Schliesslich ergiebt sich

$$(7)\quad u = \frac{1}{\pi n}\frac{1}{\sqrt{2rr'}}\int_{\alpha_1}^{\infty}\frac{1}{\sqrt{\cos i\alpha_1 - \cos i\beta}}\;\frac{\sin\frac{i\beta}{n}\,d\beta}{\cos\frac{\varphi-\phi'}{n} - \cos\frac{i\beta}{n}}.$$

Für ein sehr kleines r wird nun nach (3) $\cos i\alpha_1$ und α_1 sehr gross; es bedeutet also β während der ganzen Integration eine sehr grosse Zahl. Wir setzen etwa $\cos i\beta = t$, und berücksichtigen, dass für grosse Worte von β gilt:

$$\cos\frac{i\beta}{n} = -i\sin\frac{i\beta}{n} = \tfrac{1}{2}\sqrt[n]{2\cos i\beta} = \sqrt[n]{\frac{t}{2^{n-1}}}.$$

In Folge dessen nimmt (7) die Form an

$$u = \frac{1}{\pi n}\frac{1}{\sqrt{2rr'}}\int_{\cos i\alpha_1}^{\infty}\frac{1}{\sqrt{t - \cos i\alpha_1}}\left[1 + \sqrt[n]{\frac{t}{2^{n-1}}}\cos\left(\frac{\phi-\phi'}{n}\right)\right]\frac{dt}{t}.$$

Die Integration lässt sich leicht ausführen und liefert, unter C eine von den Coordinaten von P und Q unabhängige Constante verstanden:

$$(8)\quad u = \frac{1}{n}\frac{1}{\sqrt{2rr'\cos i\alpha_1}}\left(1 + C\cos\frac{\phi-\phi'}{n}\sqrt[n]{\frac{1}{\cos i\alpha_1}}\right).$$

Hier tragen wir noch für $\cos ia_1$ seinen Wert aus Gleichung (3) ein und vernachlässigen höhere Potenzen von r. Dann erhalten wir die folgende, *für kleine Werte von r gültige Näherungsformel:*

$$(9) \quad u = \frac{1}{n}\frac{1}{R_0}\left(1 + C\cos\frac{\phi-\phi'}{n}\sqrt[n]{\frac{2rr'}{R_0^2}}\right),$$

wo R_0 den Wert bedeutet, den R für $r = 0$ annimmt. Aus unserer letzten Formel ziehen wir vor Allem den Schluss, dass u auf der Verzweigungslinie einen endlichen Wert hat und sich beim Übergange auf dieselbe stetig verhält. Der *ad* (i.) zu führende Nachweis ist hiermit vollständig erbracht.

Ad (ii.). Wir haben uns ferner zu überzeugen, dass unsere Function u der Bedingung (ii.) von Pg. 399 genügt. Dies ist aber selbstverständlich, weil nach dem vorangestellten Princip (vgl. Pg. 405) jede Function von der Form (1) bei ganz beliebiger Wahl von $f(a)$ der Differentialgleichung $\Delta u = 0$ " im Allgemeinen," d. h. mit Ausnahme solcher Stellen, wo die Function oder ihre Differentialquotienten unendlich werden, genügt. Letzteres findet in unserem Falle nur im Punkte P, wo u, und in der Verzweigungslinie, wo $\frac{\partial u}{\partial r}$ (vgl. unten) unendlich wird, statt. Diese Ausnahmen sind aber durch unsere Bedingung (ii.) ausdrücklich zugelassen.

Ad (iii.). Wir haben schliesslich zu zeigen, dass, der Bedingung (iii.) von Pg. 399 entsprechend, u für $r = \infty$ verschwindet. Wir brauchen zu dem Zweck eine Näherungsformel für u, welche bei grossen Werten von r gültig ist.

Für grosse Werte von r wird $\cos ia_1$ sehr gross, ebenso wie für kleine Werte von r. Mithin gilt jetzt wieder die Gleichung (8). Tragen wir in dieser den Wert für $\cos ia_1$ ein, so erhalten wir die gesuchte Näherungsformel:

$$(9') \quad u = \frac{1}{n}\frac{1}{\sqrt{r^2+r'^2+(z-z')^2}}\left\{1 + C\cos\frac{\phi-\phi'}{n}\sqrt[n]{\frac{2rr'}{r^2+r'^2+(z-z')^2}}\right\}.$$

Wir lesen aus ihr sofort ab, dass u mit wachsendem r verschwindet.

Der Nachweis ist jetzt vollständig erbracht, dass u die gewünschte Green'sche Function ist.

Die Näherungsformel (9) können wir auffassen als den Anfang einer Art *Puiseux'scher Entwickelung des algebraischen Potentiales u in* *der Nähe der Verzweigungslinie* $r=0$. Das erste Glied dieser Entwickelung lautet:

$$u_0 = \frac{1}{n}\frac{1}{\sqrt{r'^2+(z-z')^2}} = \frac{1}{n}\frac{1}{PQ};$$

der Wert den unser Potential u in einem Punkte Q_0 der Verzweigungslinie annimmt, ist also gerade der n^{te} Teil desjenigen Wertes, der in dem betr. Punkte bei gleicher Lage des Punktes P herrschen würde, wenn die Verzweigungslinie nicht vorhanden und der Raum ein schlichter Raum wäre.

Der zweite Term unserer Entwickelung verschwindet wie $r^{1/n}$ für $r=0$, d. h. gerade so, wie eine gewöhnliche algebraische Function in einem Windungspunkte n^{ter} Ordnung; die folgenden Terme würden den Factor $r^{2/n}$, $r^{3/n}$ etc. haben. Hieraus schliessen wir auf das Verhalten eines beliebigen Potentiales v, welches in unserem Windungsraume eindeutig ist und nur eine endliche Anzahl von Polen hat. Ein solches Potential lässt sich nach der Bemerkung (2) von Pg. 404 als lineare Combination von Green'schen Functionen u mit geeigneten Polen darstellen:

$$v = a_1 u_{P_1} + a_2 u_{P_2} + \dots\, a_m u_{P_m}.$$

Bedeutet also v_0 den Wert, den v in einem Punkte der Verzweigungslinie annimmt, so wird auch $v-v_0$ von der Ordnung $r^{1/n}$ zu Null. Im Besonderen kann es eintreten, dass der Factor von $r^{1/n}$ in dem Ausdrucke von $v-v_0$ selbst gleich Null ist; alsdann verschwindet $v-v_0$ wie das zweite Glied unserer Puiseux'schen Entwickelung etc. Wir haben damit den Pg. 405, unter 5, ausgesprochenen Satz für den Fall unseres Windungsraumes bewiesen.

Aus der Gleichung (9) folgt ferner, dass der nach r genommene Differentialquotient von u (oder von der allgemeineren Function v) in den Punkten der Windungslinie unendlich wird, jedoch nur von solcher Ordnung, dass $r\frac{\partial u}{\partial r}$ $\left(\text{bez. } r\frac{\partial v}{\partial r}\right)$ für $r=0$ verschwindet. Diese Thatsache wurde früher gelegentlich erwähnt.

Lassen wir in unserem Ausdrucke (5) für u n in's Unendliche wachsen, so erhalten wir *die Green'sche Function eines Windungsraumes von unendlicher Exemplarenzahl.* Und zwar ergiebt sich (über den Weg W integrirt):

$$(10)\quad u = \frac{1}{2\pi i}\int \frac{1}{R'}\frac{d\alpha}{\alpha-\varphi'}.$$

Diese Function wird nur in dem einen Punkte $r = r'$, $z = z'$, $\phi = \varphi'$ unendlich; dagegen bleibt sie endlich in allen Punkten $r = r'$, $z = z'$, $\phi = \varphi' + 2k\pi$ für $k \lesseqgtr 0$. Aus dem letztgenannten Potential können wir umgekehrt das frühere zusammensetzen, indem wir uns in dem unendlichfachen Windungsraume eine Function construiren, welche an allen Stellen $r = r'$, $z = z'$, $\phi = \phi' + 2kn\pi$ einen einfachen Pol besitzt.

Man bemerke noch, dass in (5) n nicht notwendig eine ganze Zahl zu sein braucht, sondern dass es auch *gebrochen oder irrational* sein kann. Immer erhalten wir durch unsere Formel (5) ein Potential, welches nach $2\pi n$-maligem Umlauf des Punktes Q um die Gerade $r = 0$ zu seinem ursprünglichen Werte zurückkehrt, und welches in dem Sektor von der Winkelöffnung $2\pi n$ nur eine Unendlichkeitsstelle P hat. Der Raum, auf welchen sich unser Potential in diesem Falle bezieht, ist allerdings dreidimensional nicht zu realisiren. Er verhält sich zu dem gewöhnlichen einfachen Raume so, wie ein Kegelmantel von der Winkelöffnung $2\pi n$ zu der einfachen Ebene.

In dem Falle $n = 2$, der für die Anwendungen am wichtigsten ist, lässt sich das Integral (5) leicht auf elementare Functionen zurückführen. Wir gehen am bequemsten von dem reellen Integrale (7) aus und setzen etwa

$$\cos\frac{i\beta}{2} = x, \qquad \cos\frac{i\alpha_1}{2} = \sigma, \qquad \cos\left(\frac{\phi-\phi'}{2}\right) = \tau.$$

Dann folgt aus (7)

$$(11) \qquad u = \frac{1}{2\pi}\,\frac{1}{\sqrt{rr'}}\int_\sigma^\infty \frac{dx}{\sqrt{x^2-\sigma^2}}\,\frac{1}{x-\tau} = \frac{1}{R}\cdot\frac{2}{\pi}\ \text{arc tang}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}}$$

wo unter arc tang der modulo $\pi/2$ genommene absolut kleinste Wert des Arcus zu verstehen ist.

Einen ganz ähnlichen Ausdruck erhält man übrigens für die Lösung des entsprechenden zweidimensionalen Potentialproblems, d. h. für die Green'sche Function einer Riemann'schen Fläche mit einem im Endlichen gelegenen einfachen Windungspunkt. Derselbe lautet:

$$u = \text{Reelle Teil von } lg\,\frac{1}{\sqrt{x+iy}-\sqrt{x'+iy'}} = \tfrac{1}{2}lg\,\frac{1}{R}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}},$$

wo $x + iy = re^{i\phi}$ den Aufpunkt Q, $x' + iy' = r'e^{i\phi'}$ den Pol P bedeutet, und wo wieder gesetzt ist:

$$\sigma = \cos\frac{i\alpha_1}{2}, \quad \tau = \cos\left(\frac{\phi-\phi'}{2}\right), \quad \text{und} \quad \cos i\alpha_1 = \frac{r^2+r'^2}{2rr'}.$$

Durch die nebenstehende Figur wird der Verlauf der Function u aus Gleichung (11) illustrirt. Als Zeichenebene haben wir die zur Verzweigungslinie senkrechte Ebene durch P gewählt. Die Zeichenebene ist dabei als Riemann'sche Fläche von der soeben genannten Beschaffenheit zu denken, für welche der Durchstossungspunkt O der Verzweigungslinie mit der Zeichenebene ein einfacher Windungspunkt ist. Als Verzweigungsschnitt, welcher das erste und zweite Blatt der Riemann'schen Fläche trennt, haben wir den zu OP entgegengesetzten Halbstrahl, welcher strich-punktirt gezeichnet ist, gewählt. Wir haben einerseits die Niveaucurven (Schnitte der Niveauflächen mit unserer Zeichenebene), und andrerseits ihre orthogonalen Trajectorien, d. h. die Kraftlinien, eingezeichnet. Soweit sie dem ersten Blatt angehören, sind diese Curven ausgezogen, im zweiten Blatte punktirt. Die Symmetrie des Problems bringt es mit sich, dass jede Kraftlinie, welche von P in einer in die Zeichenebene fallenden Richtung ausläuft, ganz in dieser Ebene enthalten ist.

Die *Niveaucurven* sind, wie man sieht, geschlossene Curven, welche zunächst bei grossen Werten von u um P in dem ersten Blatte herumlaufen. Diejenige Niveaucurve, welche an den Verzweigungspunkt anstösst, hat (nach Gl. 9) eine Spitze; der zu dieser Curve gehörige Potentialwert ist, wie wir nach Früherem wissen, $\frac{1}{2\overline{OP}}$.

Die folgenden Curven, d. h. die Curven kleineren Potentials, bestehen aus einer Doppelschleife; die Rundung der einen Schleife läuft im zweiten Blatt um O, die etwas weitere Rundung der anderen Schleife im ersten Blatt gleichzeitig um O und P herum.

Die *Kraftlinien* andrerseits, welche beim Nicht-Vorhandensein des zweiten Blattes geradlinig von P ausstrahlen würden, sind im ersten Blatte etwas nach dem Verzweigungsschnitt hin gekrümmt; ein Teil der Kraftlinien überschreitet diesen Schnitt und tritt in's zweite Blatt ein, ohne doch einen vollen Umlauf um O auszuführen. Der Halbstrahl OP im ersten, und der darunter liegende Halbstrahl im zweiten Blatte ist selbst eine Kraftlinie.*

§ 4. *Anwendungen der Green'schen Function des Windungsraumes auf Probleme der gewöhnlichen Potentialtheorie.*

Wir kommen nun zu den Anwendungen der bisher entwickelten Functionen.

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* Die Zeichnung hat Herr stud. v. Schaper, auf Grund einer genauen Tabelle über die Function u, in dankenswerter Weise entworfen.

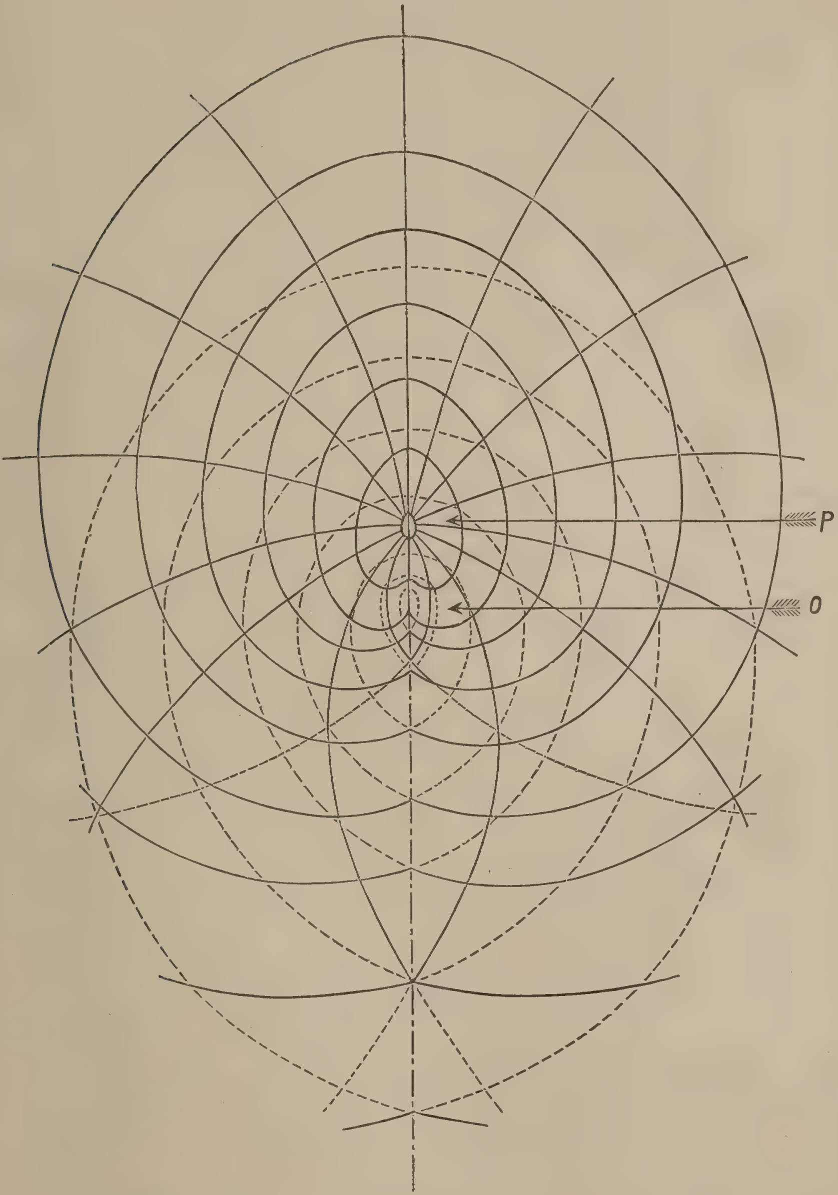

Fig. 3.

Die einfachste Anwendung besteht in der Ableitung der Green'schen Function oder allgemeiner in der Lösung der Randwertaufgabe für einen Raum, welcher durch zwei leitende Ebenen begrenzt ist. Man könnte dieses Gebiet einen "Winkelspiegel" oder einen "Keil" nennen. Nehmen wir zunächst an, dass die Ebenen einen Winkel γ einschliessen, welcher, innerhalb desjenigen Gebietes gemessen, auf welches sich die Potentialaufgabe bezieht, gleich $\frac{n\pi}{m}$ ist (m und n ganze Zahlen). Wir reproduciren das Ausgangsgebiet durch Spiegelung an den Begrenzungsebenen, und wollen die successiven Spiegelbilder abwechselnd als "schraffirte" und "nicht-schraffirte Gebiete" unterscheiden; das Ausgangsgebiet selbst sei ein schraffirtes Gebiet. Das Verfahren schliesst sich erst nach $2m$-Spiegelungen; wir erhalten im Ganzen m schraffirte und m nicht-schraffirte Gebiete, welche sich um die Schnittkante der Grenz-Ebenen herumlegen. Es handle sich um die Aufstellung der Green'schen Function v für unser Gebiet, d. h. eines Potentiales, welches innerhalb des Gebietes einen Pol P hat und auf den Begrenzungsebenen, sowie im Unendlichen, verschwindet. Zu dem Pole P construiren wir seine $2m-1$ Spiegelbilder hinzu, von denen $m-1$ in schraffirten, m in nicht-schraffirten Gebieten liegen. Die ersteren, zusammen mit dem vorgeschriebenen Pole P, bezeichnen wir als Punkte P_1, die letzteren als Punkte P_2. Darauf bilden wir, unter u die Green'sche Function des n-fachen Windungsraumes aus Gl. (5) des vorigen § verstanden,

$$(1) \quad v = \Sigma u_{P_1} - \Sigma u_{P_2},$$

wo sich die Summation über die m Punkte P_1, bez. P_2, erstreckt. Diese Function besitzt in dem Ausgangsgebiete nur den einen Pol $P_1 = P$, und verschwindet aus Symmetriegründen auf den Begrenzungsebenen des Gebietes (ebenso wie auf allen Spiegelbildern der letzteren). Dass sie auch im Unendlichen verschwindet und der Potentialgleichung genügt, ist klar. Wir haben also in (1) die gewünschte Green'sche Function hergestellt.

Haben wir z. B. $\gamma = \frac{3\pi}{2}$, besteht also das Gebiet aus dem Äusseren eines metallischen Klotzes mit einer rechtwinkligen Kante, so zerfällt unsere Function u in vier Terme, deren jeder die Form der Function u für $n = 3$ besitzt.

Einfacher ist es noch, eine Bemerkung von Pg. 413 benützend, in Gl. (5) des vorigen § n einfach gleich γ/π zu machen. Dann ergiebt sich die Green'sche Function des Keiles in der Form

$$(2) \quad v = u_P - u_{P'};$$

hier bedeutet P' den Spiegelpunkt von P in Bezug auf eine der Grenzebenen des Keiles (er fällt mit dem Spiegelpunkte in Bezug auf die andere zusammen, weil der Riemann'sche Raum, in welchem wir uns jetzt befinden, die Winkelöffnung $2n\pi = 2\gamma$ hat). Wird das Azimuth ϕ von einer der Grenzebenen aus gerechnet, und hat P die Coordinaten r', z', ϕ', so bekommt P' einfach die folgenden Coordinaten: r', z', $-\phi'$.

Am wichtigsten ist der Fall, wo der Winkel des Keiles gleich 2π geworden ist. Wir haben dann eine metallische Halbebene im unendlichen Raume. Die Green'sche Function v dieses Gebietes hat wieder die Form (2); sie lautet, wenn wir u aus Gleichung (11) von Pg. 413 entnehmen:

$$(3) \quad v = \frac{1}{R} \cdot \frac{2}{\pi} \left(\text{arc tang} \sqrt{\frac{\sigma+\tau}{\sigma-\tau}} - \text{arc tang} \sqrt{\frac{\sigma+\tau'}{\sigma-\tau'}} \right).$$

Hier ist gesetzt worden:

$$\tau = \cos\frac{\phi-\phi'}{2}, \quad \tau' = \cos\frac{\phi+\phi'}{2};$$

σ hat die frühere Bedeutung. Für $\phi = 0$ und $\phi = 2\pi$, d. h. auf beiden Seiten unseres metallischen Schirmes, verschwindet die Klammer und also v, wie es sein muss. In dem ersten Raumexemplare ($0 < \phi < 2\pi$), für welches die Potentialaufgabe gestellt ist, hat v nur einen Pol P (mit den Coordinaten r', z', ϕ'). Der Spiegelpunkt P' (mit den Coordinaten r', z', $-\phi'$) ist dadurch, dass wir ihn in das zweite Raumexemplar ($-2\pi < \phi < 0$) verlegt haben, sozusagen unschädlich gemacht.

Durch Inversion erhalten wir aus den vorgenannten Lösungen für Gebiete mit ebenflächiger Begrenzung solche für Gebiete mit kugelförmigen Grenzflächen. So verwandeln sich unsere beiden Ebenen, welche sich unter dem Winkel γ schneiden, in zwei Kugeln, welche sich unter demselben Winkel durchdringen. Haben wir z. B. $\gamma = \frac{3\pi}{2}$, und nehmen wir überdies das Inversionscentrum auf einer der beiden Grenzebenen des Keiles an, so wird das invertirte Gebiet zum Äusseren einer Halbkugel. Wir sind also in der Lage, die eigentümlichen Beobachtungen, welche Faraday* für dieses Gebiet mitteilt, mit der Theorie numerisch zu vergleichen.

Speciell ergiebt sich aus der Halbebene, je nachdem wir das

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* Vgl. *Experimental Researches*, Art. 1242.

Inversionscentrum auf der Halbebene, oder in ihrer Verlängerung, oder ausserhalb derselben annehmen, eine unendliche Ebene mit kreisförmigem Loch, oder eine ebene Kreisscheibe, oder eine Kugelkalotte, welche wir uns sämtlich ebenso wie früher die Halbebene als leitend zu denken haben. Gleichzeitig ergiebt sich aus unserem Potentiale v in Gleichung (3) nach bekannter Methode diejenige Potentialverteilung, welche durch die Anwesenheit eines beliebig gelegenen äusseren Poles P erzeugt wird, wenn jene Leiter auf dem Potentiale Null gehalten werden, d. h. die Green'sche Function für das Äussere der Leiter.

Die Kante der Kreisscheibe, des keisförmigen Loches, sowie die der Kugelkalotte entsprechen bei der Inversion der Begrenzungslinie unserer Halbebene, d. h. der Verzweigungslinie des Riemann'schen Raumes. Da nach Pg. 412 in der Verzweigungslinie $r = 0$ des Riemann'schen Doppelraumes der Differentialquotient der Green'schen Function $\frac{\partial u}{\partial r}$ unendlich wird, wie $r^{-\frac{1}{2}}$, so wird auch in den invertirten Potentialverteilungen längs der Kante der Kreisscheibe etc. der senkrecht zu dieser Kante genommene Differentialquotient des Potentiales, welcher bekanntlich im Wesentlichen die elektrische Flächendichte giebt, von der gleichen Ordnung unendlich.

Die Tendenz der statischen Elektricität, sich in etwa vorhandenen Kanten der Leiter anzusammeln, entspricht also genau unserem Satze, dass die algebraischen Potentiale in den Verzweigungslinien einen unendlich grossen Differentialquotienten besitzen.

Unsere Methode liefert uns nicht nur die Werte des Potentials in dem "physikalischen Raumexemplar," sondern auch seine analytischen Fortsetzungen. Es zeigt sich z. B. unmittelbar, dass das Potential der von aussen influencirten Kugelkalotte in dem gewöhnlichen Raume zweiwertig ist, und dass die beiden Zweige des Potentials in dem Rande der Kugelkalotte zusammenhängen. Offenbar wird es immer nützlich sein, bevor man an die specielle Aufstellung einer Function geht, sich zunächst, wie wir es hier gethan haben, ihre allgemeinen Eigenschaften und namentlich ihre analytischen Fortsetzungen klar zu machen.

Da wir die Green'sche Function v für den Keil, die sich schneidenden Kugeln, die Kreisscheibe etc. kennen, lässt sich die Lösung beliebiger, auf diese Gebiete bezüglicher *Randwertaufgaben* nach der bekannten Formel

$$V = -\frac{1}{4\pi}\int V_0 \frac{\partial v}{\partial n} do$$

hinschreiben, wo das Integral über die Oberfläche des Gebietes zu erstrecken ist, und wo V_0 die vorgeschriebenen Randwerte bedeutet. Nimmt man in's Besondere V_0 überall gleich 1, so erhält man diejenige Potentialverteilung, welche der sog. *natürlichen Belegung* entspricht. Die so erhaltenen Potentiale besitzen natürlich hinsichtlich ihrer Verzweigung denselben Charakter wie die zugehörige Green'sche Function v. In's Besondere sind also bei der Kreisscheibe und der Kugelkalotte die der natürlichen Belegung entsprechenden Potentiale in einem Riemann'schen Raum mit einer kreisförmigen Verzweigungslinie eindeutig.

Die nach der Methode dieses § zu behandelnden Probleme sind grösstenteils schon früher auf anderem, allerdings zum Teil sehr complicirtem Wege gelöst worden, ohne dass man den Zusammenhang mit den mehrwertigen Potentialen erkannt zu haben scheint.

Wir nennen zunächst eine Arbeit von Herrn Macdonald,* welcher die Green'sche Function für den Keil durch Reihenentwickelungen nach Bessel'schen Functionen gefunden hat, und nach erheblichen Umrechnungen auf einen der Gleichung (7) von Pg. 410 entsprechenden geschlossenen Ausdruck geführt ist. Im Wesentlichen dieselbe Aufgabe war schon früher (1867) von Mehler † behandelt worden. Mehler bezieht das Problem der sich schneidenden Kugeln auf gewisse, von W. Thomson eingeführte Coordinaten und erhält eine Entwickelung nach sog. Kegelfunctionen.

Die Aufgaben für die Kreisscheibe und für die Kugelkalotte sind bekanntlich von W. Thomson ‡ 1845 auf einem höchst genialen Wege gelöst. Thomson fasst die Kreisscheibe als Grenzfall eines Ellipsoides auf und erzeugt die Kugelkalotte durch Inversion aus jener. Dabei gelangt er, umgekehrt wie wir, zunächst zur Potentialverteilung der natürlichen Belegung; aus dieser leitet er die Green'sche Function der Kugelkalotte durch eine abermalige Inversion ab.

Ein von Herrn P. Appell § angegebenes zweiwertiges Potential hat, ebenso wie die zuletzt genannten Potentialverteilungen, einen Kreis zur Verzweigungslinie; es wird aber in den Punkten desselben unendlich gross, und fällt daher nicht direkt unter den hier betrachteten Typus.

§ 5. *Die Green'sche Function eines Riemann'schen Raumes mit zwei geradlinigen parallelen Verzweigungscurven und ihre Anwendungen.*

Während, wie wir sahen, die Potentiale des dritten § zu schon anderweitig bekannten Anwendungen Anlass gaben, werden wir in diesem § verzweigte Potentiale ableiten, welche einige, bisher wohl noch nicht behandelte Potentialaufgaben zu lösen gestatten.

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* *Proceedings, London Math. Soc.*, Vol. XXVI., 1895.

† *Crelle's Journal*, Bd. LXVIII.

‡ Vgl. "Papers on Electrostatics and Magnetism," Art. XV., Sect. 231–248.

§ *Math. Annalen*, Bd. XXX., 1887, Pg. 155.

Die Verzweigungslinien mögen aus zwei parallelen Geraden bestehen, um welche sich der Raum je n-mal herumwindet. Wir haben also n Raumexemplare, welche längs des von den beiden Verzweigungslinien begrenzten Streifens cyklisch zusammenhängen. Der Streifen selbst kann als Verzweigungsmembran dienen; jeder ebene, etwa senkrecht zu den Verzweigungslinien gelegte Schnitt schneidet aus dem Riemann'schen Raume eine Riemann'sche Fläche aus, welche in den beiden Durchstossungspunkten mit den Verzweigungsgeraden je einen $(n-1)$-fachen Windungspunkt besitzt und im Unendlichen unverzweigt ist. Es handelt sich darum, die Green'sche Function dieses Raumes zu finden.

Wir führen zunächst geeignete Coordinaten ein. Die in der Mitte zwischen beiden Verzweigungslinien verlaufende Gerade nehmen wir zur z-Axe. In der Ebene $z=0$ führen wir "Bipolar-Coordinaten" ein durch folgende Festsetzungen: Es sei ζ eine in der Ebene $z=0$ ausgebreitete complexe Variable, wobei die Werte 0, $+1$, -1 bez. den Durchstossungspunkten dieser Ebene mit der z-Axe und den beiden Verzweigungslinien zukommen mögen. Den Projectionen der Punkte Q und P auf die ζ-Ebene entsprechen bez. die ζ-Werte

$$\zeta = x+iy, \quad \zeta' = x'+iy'.$$

Wir setzen nun

$$(1) \qquad \frac{\zeta-1}{\zeta+1} = e^{\rho+i\phi}, \quad \frac{\zeta'-1}{\zeta'+1} = e^{\rho'+i\phi'},$$

und nennen ρ, ϕ, bez. ρ', ϕ', die Bipolar-Coordinaten der auf die Ebene $z=0$ projicirten Punkte P und Q. Offenbar sind ϕ und ϕ' nur modulo 2π bestimmt.

Bekanntlich liefern die Curven $\rho = \text{const}$ und $\phi = \text{const}$ je ein System von Kreisen. Die Kreise $\rho = \text{const}$ sind sämtlich orthogonal zu der reellen ζ-Axe (Verbindungslinie der Verzweigungspunkte ± 1); ihre Durchschnitte mit dieser Axe liegen zu den Punkten ± 1 harmonisch. Die Curven $\phi = \text{const}$ bilden das Büschel sämtlicher, durch ± 1 hindurchgehender Kreise, wobei die beiden Stücke, in welche jeder dieser Kreise durch die Punkte ± 1 zerlegt wird, zwei verschiedenen Werten von ϕ entsprechen, deren Differenz modulo 2π gleich $\pm\pi$ ist. Jedem Punkte der positiv-imaginären ζ-Halbebene entspricht ein (modulo 2π) positiver, jedem Punkt der negativ-imaginären ζ-Halbebene ein (modulo 2π) negativer Wert von ϕ. Die Verbindungslinie der Punkte ± 1 erhält, je nachdem wir sie der positiven oder negativen ζ-Halbebene hinzurechnen, die Coordinate $\varphi = +\pi$ oder $\phi = -\pi$; die Verlängerung dieser Linie über $+1$ und -1 hinaus ist durch den Wert $\phi = 0$ charakterisirt.

Die Variablen z, ρ und ϕ liefern geeignete Coordinaten zur Darstellung von Functionen, welche in unserem Riemann'schen Raume eindeutig, in dem gewöhnlichen Raume aber mehrdeutig sind. In der That wird jede Function $F(\rho, \phi)$, welche in ρ und ϕ eindeutig ist und in φ die Periode $2\pi n$ hat, auf der von der Ebene $z = 0$ aus unserem Riemann'schen Raume ausgeschnittenen Riemann'schen Fläche eindeutig. Dementsprechend liefert auch die complexe Function $e^{\frac{\rho+i\phi}{n}}$ die einfachste conforme Abbildung der genannten Riemann'schen Fläche auf die ζ-Ebene. Ferner wird jede eindeutige Function $F(z, \rho, \phi)$, welche in ϕ die Periode $2\pi n$ hat, in unserem Riemann'schen Raume eindeutig.

Nach diesen Vorbereitungen verfahren wir bei der Aufstellung der Green'schen Function ebenso wie im § 3. Wir transformiren zunächst $R = \overline{PQ}$ in die Coordinaten ρ, ϕ, z und ρ', ϕ', z'. Aus 1 ergiebt sich

$$\zeta = \frac{1+e^{\rho+i\phi}}{1-e^{\rho+i\phi}}, \quad \zeta' = \frac{1+e^{\rho'+i\phi'}}{1-e^{\rho'+i\phi'}},$$

und nach einer kleinen Umrechnung

$$|\zeta-\zeta'|^2 = (x-x')^2+(y-y')^2 = 2\,\frac{\cos i\,(\rho-\rho')-\cos\,(\phi-\phi')}{(\cos i\rho-\cos\phi)\,(\cos i\rho'-\cos\varphi')}.$$

Mithin wird

$$(2) \quad R^2 = 2\,\frac{\cos i\,(\rho-\rho')-\cos\,(\phi-\phi')}{(\cos i\rho-\cos\phi)\,(\cos i\rho'-\cos\varphi')} + (z-z')^2.$$

Darauf formen wir $1/R$ nach dem Cauchy'schen Satze um. Wir ersetzen ϕ' in $\cos\,(\phi-\phi')$ durch eine complexe Variable α, schreiben

$$R'^2 = 2\,\frac{\cos i\,(\rho-\rho')-\cos\,(\phi-\alpha)}{(\cos i\rho-\cos\phi)\,(\cos i\rho'-\cos\phi')} + (z-z')^2,$$

und haben

$$\frac{1}{R} = \frac{1}{2\pi i}\int \frac{1}{R'}\,f(\alpha)\,d\alpha,$$

wo der Integrationsweg in der Ebene der Variablen α positiv um die Stelle $\alpha = \phi'$ herumzuführen ist, und wo $f(\alpha)$ eine Function bedeutet, welche für $\alpha = \phi'$ von der ersten Ordnung mit dem Residuum 1 un endlich wird. Wir wählen die Function $f(\alpha)$ überdies so, dass sie in α und ϕ' die Periode 2π hat, nehmen also etwa wie früher

$$f(\alpha) = \frac{ie^{i\alpha}}{e^{i\alpha} - e^{i\phi'}}.$$

Die Pole des Integranden sind hiernach die sämtlichen Punkte

$$\alpha = \phi' + 2k\pi.$$

Die Verzweigungspunkte sind gegeben durch $R'^2 = 0$ und $R'^2 = \infty$. Nun wird $R'^2 = \infty$ nur, wenn $\alpha = \infty$. Andrerseits haben wir $R'^2 = 0$, wenn

$$\cos(\phi - \alpha) = \cos i\alpha_1, \quad \text{d. h.} \quad \alpha = \phi + 2k\pi \pm i\alpha_1,$$

wo gesetzt ist

$$(3) \quad \cos i\alpha_1 = \cos i(\rho - \rho') + \tfrac{1}{2}(\cos i\rho - \cos\phi)(\cos i\rho' - \cos\phi')(z - z')^2.$$

Die Verzweigungspunkte des Integranden sind also die folgenden:

$$(4) \qquad \begin{cases} \alpha = \infty, \\ \alpha = \phi + 2k\pi \pm i\alpha_1. \end{cases}$$

Darauf zerschneiden wir die α-Ebene genau so wie früher, indem wir die Punkte $\phi + 2k\pi \pm i\alpha_1$ durch Schnitte parallel der imaginären Axe mit dem Unendlichen verbinden, und deformiren unsern ursprünglichen Integrationsweg, indem wir ihn bis an die Verzweigungsschnitte heranziehen. In dem deformirten Wege (vgl. Fig. 1) heben sich die Integrationen über die der imaginären Axe parallelen Rechteckseiten gegenseitig auf, während die Integrationen über die der reellen Axe parallelen Rechteckseiten verschwinden, sofern wir die Höhe des Rechtecks unbegrenzt wachsen lassen. Schliesslich bleibt nur wieder der Weg W übrig.

Indem wir den Übergang zu den verzweigten Potentialen suchen, ändern wir die Definition von $f(\alpha)$ dahin ab, dass diese Function in α und ϕ' die Periode $2\pi n$ hat und machen

$$f(\alpha) = \frac{i}{n} \frac{e^{i\alpha/n}}{e^{i\alpha/n} - e^{i\phi'/n}}.$$

Wir betrachten also jetzt das über den Weg W geführte Integral

$$(5) \quad u = \frac{1}{2\pi n} \int \frac{1}{R'} \frac{e^{i\alpha/n}\, d\alpha}{e^{i\alpha/n} - e^{i\phi'/n}},$$

und behaupten, dass dieses die Green'sche Function unseres Riemann'schen Raumes ist, d. h. dass u den Bedingungen (i.), (ii.) und (iii.) von Pg. 399 genügt. Zum Beweise haben wir die Erörterungen von Pg. 408 bis Pg. 411 wörtlich zu wiederholen.

Eine kleine Abweichung ergiebt sich nur bei der Aufstellung der Näherungsformeln, durch welche wir u in unmittelbarer Nähe der Verzweigungen darstellen. Nach (1) wird in der Nähe der einen oder anderen Verzweigungslinie $\rho = \pm\infty$; mithin wird in der Nähe dieser Linien ρ, und nach Gleichung (3) auch $\cos i\alpha_1$, sehr gross.

Unser Integral (5) können wir nun, indem wir ebenso wie Pg. 410 $\alpha = \phi \pm i\beta$ setzen, so schreiben:

$$(6) \quad u = \frac{A}{\pi n} \int_{\alpha_1}^{\infty} \frac{1}{\sqrt{\cos i\alpha_1 - \cos i\beta}} \, \frac{\sin \frac{i\beta}{n}\, d\beta}{\cos \frac{\phi - \phi'}{n} - \cos \frac{i\beta}{n}};$$

hier bedeutet A die folgende Grösse:

$$(7) \quad A = \sqrt{\tfrac{1}{2}(\cos i\rho - \cos\phi)(\cos i\rho' - \cos\phi')}.$$

Ganz dasselbe Integral ergab sich aber Pg. 410, wobei nur als Factor des Integrals statt A die Grösse auftrat

$$A' = \sqrt{\frac{1}{2rr'}}.$$

In Folge dessen können wir die für grosse Werte von α_1 gültige Umrechnung von Pg. 410 direkt übernehmen; sie liefert

$$(8) \quad u = \frac{1}{n} \frac{A}{\sqrt{\cos i\alpha_1}} \left(1 + C \cos\frac{\phi - \phi'}{n} \sqrt[n]{\frac{1}{\cos i\alpha_1}}\right).$$

Hier tragen wir noch für A und $\cos i\alpha_1$ die Werte aus Gleichung (7) und (3) ein und berücksichtigen, dass, nach Gleichung (1),

$$e^\rho = \frac{r_1}{r_2}$$

ist, wo r_1 und r_2 die senkrechten Abstände des Punktes Q von der einen oder anderen Verzweigungslinie bedeuten, und dass für sehr grosse positive oder negative Werte von ρ, d. h. für sehr kleine Werte von r_2 oder r_1

$$\cos i\rho = \tfrac{1}{2}e^\rho = \frac{r_1}{2r_2} \quad \text{oder} \quad \cos i\rho = \tfrac{1}{2}e^{-\rho} = \frac{r_2}{2r_1}$$

wird. Dann erhalten wir aus (8) für sehr kleine Werte von r_1

$$(9) \quad u = \frac{1}{n} \frac{1}{R_0} \left(1 + C \cos\frac{\phi - \phi'}{n} \sqrt[n]{\frac{r_1}{r_2 R_0^2 (\cos i\rho' - \cos\phi')}}\right),$$

und für sehr kleine Werte von r_2

$$(9') \quad u = \frac{1}{n}\frac{1}{R_0}\left(1 + C\cos\frac{\phi-\phi'}{n}\sqrt[n]{\frac{r_2}{r_1 R_0^2(\cos i\rho' - \cos\phi')}}\right);$$

R_0 bedeutet in der ersten, bez. zweiten der vorstehenden Formeln den Wert von R für $r_1 = 0$, bez. für $r_2 = 0$, d. h. den Abstand der Punkte P und Q für den Fall, dass Q auf eine der Verzweigungslinien gerückt ist.

Wir erkennen aus (9) und (9′) zunächst wieder, dass u beim Übergang auf die Verzweigungslinien endlich und stetig bleibt, während die Differentialquotienten $\frac{\partial u}{\partial r_1}$ und $\frac{\partial u}{\partial r_2}$ für $r_1 = 0$ und $r_2 = 0$ unendlich werden. Ferner sehen wir, dass $u - u_0$ (unter u_0 den Wert von u für $r_1 = 0$ oder $r_2 = 0$ verstanden) von der Ordnung $r_1^{1/n}$, bez. $r_2^{1/n}$, verschwindet, in Übereinstimmung mit dem Pg. 405 für beliebige verzweigte Potentiale ausgesprochenen Satz.

Überhaupt können wir auch alle weiteren Bemerkungen über die Green'sche Function des § 3 auf unsere jetzige Green'sche Function übertragen, da beide Functionen, wie aus dem Vergleich der Formel (6) von Pg. 423 und (7) von Pg. 410 hervorgeht, einen ganz analogen Ausdruck haben. Wir können auch hier den Übergang zu $n = \infty$ und zu einem beliebigen irrationalen n machen, und können im Falle $n = 2$ u durch elementare Functionen darstellen. Es folgt nämlich aus (6) für $n = 2$, wenn wir noch setzen

$$\cos\frac{i\beta}{2} = x, \quad \cos\frac{i\alpha_1}{2} = \sigma, \quad \cos\left(\frac{\phi-\phi'}{2}\right) = \tau:$$

$$(10) \quad u = \frac{A}{2\pi}\int_\sigma^\infty \frac{dx}{\sqrt{x^2-\sigma^2}}\,\frac{1}{x-\tau} = \frac{1}{R}\,\frac{2}{\pi}\operatorname{arc\,tang}\sqrt{\frac{\sigma+\tau}{\sigma-\tau}},$$

wo unter arc tang der modulo $\pi/2$ genommene absolut kleinste Wert des Arcus zu verstehen ist. Dabei ist $\cos i\alpha_1$ jetzt durch die Gleichung (3) erklärt und die Bedeutung der Coordinaten ρ, ρ', ϕ, ϕ' natürlich eine andere, wie im dritten §.

Schliesslich kann man sich abermals, auf Grund der vorstehenden einfachen Formel, ein vollständiges Bild von dem Verlauf der Niveauflächen und Kraftlinien machen. Wir exemplificiren wieder auf die durch P senkrecht zu den Verzweigungslinien hindurchgelegte Ebene $z = z'$, und beschreiben den qualitativen Verlauf der Niveaucurven und Kraftlinien in dieser Ebene. Wegen der Symmetrie des Problems ist jede Kraftlinie, welche in eine in die Ebene $z = z'$ fallenden Richtung von P ausläuft, ganz in dieser Ebene enthalten. Die Ebene selbst haben wir uns als zweiblättrige Riemann'sche Fläche zu denken, für welche die Punkte $r_1 = 0$ und $r_2 = 0$, d. h. die Durchstossungspunkte der Ebene mit den Verzweigungslinien des Raumes Verzweigungspunkte sind. Derjenige von diesen Punkten, welcher von P den kleinsten Abstand hat, heisse O_1, der andere O_2; das Blatt der Riemann'schen Fläche, welches P enthält, heisse das erste, das andere das zweite Blatt. Die beiden Blätter hängen längs der Verbindungslinie der Punkte O_1 und O_2 wechselseitig zusammen.

Von dem Verlauf der *Niveaucurven* wird man sich darauf folgendes Bild zu machen haben. Für sehr grosse Werte von u laufen die Niveaucurven um den Punkt P im ersten Blatt nahezu kreisförmig herum und erweitern sich allmählich bei abnehmendem u. Für $u = \frac{1}{2} \frac{1}{\overline{O_1 P}}$ läuft die Niveaucurve [der Gleichung (9) zufolge] durch den Punkt O_1 hindurch und bildet hier eine Spitze. Für noch kleinere Werte von u besteht die Curve aus einer Doppelschleife; die eine Rundung der Doppelschleife läuft im zweiten Blatt um O_1, die etwas weitere andere im ersten gleichzeitig um O_1 und P herum. Lassen wir $u = \frac{1}{2} \frac{1}{\overline{O_2 P}}$ werden, so geht die Niveaucurve durch O_2 hindurch und besitzt hier einen Selbstberührungspunkt. Bei weiter abnehmendem u zerfällt die Doppelschleife in zwei einfache Ovale; das eine derselben läuft im zweiten Blatte um O_1 und O_2, das andere, etwas grössere, im ersten Blatte gleichzeitig um O_1, O_2 und P herum. Beide Ovale verlieren sich in's Unendliche, wenn wir u zu Null abnehmen lassen.

Die *Kraftlinien* sind als orthogonale Trajectorien des Niveaucurvensystems bestimmt. Während sie beim Nicht-Vorhandensein der Verzweigungspunkte (im Falle $n = 1$) geradlinig verlaufen würden, sind sie in unserem Falle ($n = 2$) gekrümmt und kehren ihre concave Seite der Linie $O_1 O_2$ zu. Ein Teil der Kraftlinien überschreitet diese Linie und tritt in's zweite Blatt ein.

Bemerkenswerte Anwendungen der zuletzt entwickelten Potentiale auf Probleme des gewöhnlichen Raumes ergeben sich im Falle $n = 2$, also glücklicher Weise gerade in demjenigen Falle, wo die numerische Auswertung nach Gleichung (10) keine Schwierigkeiten hat. Wir gelangen nämlich durch unser erweitertes Spiegelungsverfahren von der Green'schen Function für unsern Doppelraum sofort zu der *Green'schen Function für das Äussere eines durch zwei parallele gerade Linien begrenzten ebenen, unendlich dünnen Schirmes.*

Zu demjenigen Raumexemplar, für welches die Aufgabe gestellt ist, denken wir uns ein zweites hinzu, welches längs der Fläche des Schirmes mit jenem wechselweise zusammenhängt. Für den so entstehenden Doppelraum spielt der Schirm die Rolle einer Verzweigungsmembran. Wählen wir den Maassstab so, dass die Längeneinheit gleich der halben Breite des Schirmes wird, so können wir unsere früheren Coordinaten z, ρ und ϕ direkt anwenden. Das erste Raumexemplar wird in diesen Coordinaten durch die Ungleichungen $-\pi<\phi<+\pi$, das zweite etwa durch $+\pi<\varphi<3\pi$ charakterisirt. Zu dem Pole P des ersten Raumexemplares construiren wir ferner in Bezug auf die Ebene des Schirmes den Spiegelpunkt P' hinzu, welcher in das zweite Raumexemplar fällt. Wenn P die Coordinaten z', ρ', $\pi-\phi'$ hat, so bekommt P' die Coordinaten z', ρ', $\pi+\phi'$.

Darauf bilden wir, unter u_P die Green'sche Function unseres Doppelraumes mit dem Pole P verstanden:

$$v=u_P-u_{P'}.$$

Diese Function verschwindet aus Symmetriegründen für $\phi=\pm\pi$, d. h. auf beiden Seiten unseres Schirmes, wie man übrigens auch unmittelbar auf Grund der Gleichung (10) verificirt, und hat überhaupt alle Eigenschaften, welche für die Green'sche Function unseres Gebietes erforderlich sind. Auf die zweiwertige Function u müssen wir natürlich deshalb zurückgreifen, damit wir den Spiegelpunkt in das zweite Raumexemplar verlegen können. Wollten wir die Aufgabe unter Zugrundelegung der einwertigen Function $1/R$ mittelst des Spiegelungsprincipes lösen, so würden wir in dem ersten Raumexemplar einen Pol schaffen, welcher den Bedingungen unseres Problems widerspricht.

Ganz analog ergiebt sich die Green'sche Function für das "reciproke" Gebiet, d. h. für den Fall, wo der Schirm aus einer *unendlichen Ebene* besteht, *in welche ein unendlich langer Spalt mit parallelen, geradlinigen Rändern eingeschnitten ist.* Als Verzweigungsmembran unseres Doppelraumes wählen wir wiederum die Fläche des Schirmes, dessen beide Seiten jetzt durch die Gleichungen $\phi=0$ und $\phi=\pm 2\pi$ charakterisirt werden können. (Die Breite des Spaltes nehmen wir gleich 2 an, um unsere früheren Coordinaten direkt verwenden zu können.) Die Bezeichnungen "erstes" und "zweites Raumexemplar" sollen also jetzt so gewählt werden, dass im ersten $0<\phi<2\pi$, im zweiten etwa $-2\pi<\phi<0$ gilt. Besitzt der vorgeschriebene Pol P die Coordinaten z', ρ', ϕ', so wird sein Spiegelpunkt P' in Bezug auf die Ebene unseres jetzigen Schirmes die Coordinaten haben z', ρ', $-\phi'$.

Dieser Punkt P' fällt bei der jetzigen Lage unserer Verzweigungsmembran in das entgegengesetzte Raumexemplar wie P (bei der früheren Lage derselben würde er zu demselben Raumexemplar wie P gehören). Schliesslich erhalten wir wieder in

$$v = u_P - u_{P'},$$

die Green'sche Function unseres Gebietes.

Durch Inversion können wir neue, merkwürdige Schirmformen ableiten, für welche sich die Green'sche Function sofort hinschreiben lässt.

Aus dem streifenförmigen Schirm entsteht, wenn wir das Inversionscentrum zunächst in der Ebene des Streifens oder in ihrer Verlängerung annehmen, ein *ebener Schirm mit zwei kreisförmigen Löchern, deren Conturen sich im Inversionscentrum berühren.* Und zwar haben wir, je nachdem wir dem Inversionscentrum in der ursprünglichen Figur (v) die Lagen I, II, III oder IV geben, die durch Fig. I, II, III und IV dargestellten Schirme. Nehmen wir andrerseits das Inversionscentrum ausserhalb des ursprünglichen Schirmes, so bekommen wir allemal als Bild desselben einen *sphärischen Schirm mit zwei kreisförmigen Ausschnitten, welche sich in dem Inversionscentrum berühren.*

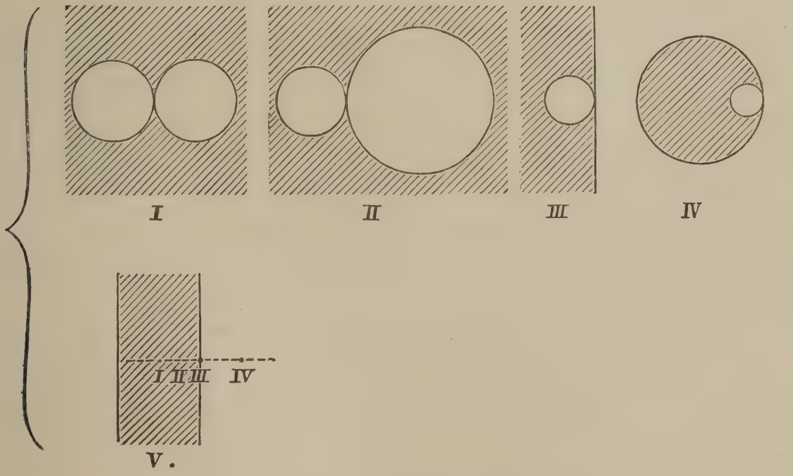

Fig. 4 — I, II, III, IV, V.

Gehen wir von dem Schirm mit spaltförmigem Einschnitt aus, so erhalten wir aus diesem, indem wir das Inversionscentrum in seine Ebene legen, eine Reihe neuer behandelbarer Schirmformen, welche zu den Figuren I bis IV "reciprok" sind. Sie werden ebenfalls durch die Figuren I bis IV dargestellt, wenn wir in ihnen die nichtschraffirten Gebiete schraffiren und umgekehrt. Nehmen wir aber das Inversionscentrum ausserhalb der Ebene des ursprünglichen Schirmes an, so entsteht eine *doppelte Kugelkalotte*, nämlich ein *Schirm, welcher aus zwei, ein und derselben Kugel angehörenden, sich berührenden Kugelsegmenten besteht.*

Will man die zuletzt genannten Aufgaben nach den *ursprünglichen Thomson'schen Methoden* behandeln, so würde man folgendermassen zu verfahren haben: Man gehe von dem Potential der natürlichen Belegung eines unendlich langen Streifens (oder des reciproken Gebietes) aus. Dieses verwandelt sich durch Inversion von einem ausserhalb des Streifens gelegenen Centrum in das Potential der natürlichen Belegung unseres sphärischen Schirms mit zwei sich berührenden kreisförmigen Löchern (oder des reciproken Gebietes). Das letztgenannte Potential vermindere man um 1 (d. h. um den constanten Potentialwert, welcher in den Punkten des Schirmes statt hat) und mache eine abermalige Inversion von einem ausserhalb der Kugel gelegenen Punkte. Der Schirm behält dabei qualitativ seine vorher genannte Beschaffenheit; gleichzeitig entsteht in dem Inversionscentrum ein einfacher Pol. Man gelangt so ebenfalls zu den oben angegebenen Green'schen Functionen.

§ 6. *Schlussbemerkungen betr. mögliche Verallgemeinerungen der Methode.*

Unser Ansatz ist offenbar einer Reihe weiterer Verallgemeinerungen fähig. Man kann sich zunächst die Aufgabe stellen, die Green'sche Function eines *Gitters* herzustellen, welches aus einer Serie von unendlich vielen, unendlich dünnen, congruenten und einer Ebene angehörigen Metallstreifen besteht. Diese Function ergiebt sich durch Spiegelung aus der Green'schen Function eines Riemann'schen Doppelraumes von unendlich vielen, aequidistanten, parallelen Verzweigungslinien. Indessen soll auf die Aufstellung dieser Function hier nicht näher eingegangen werden.

Ferner kann man wünschen, die entsprechende Aufgabe für einen *beliebig berandeten, ebenen Schirm*, oder, was auf dasselbe hinauskommt, für einen Riemann'schen Doppelraum mit einer beliebigen, ebenen Verzweigungslinie zu lösen. Der erste Schritt hierzu besteht in der Einführung geeigneter Coordinaten. Man muss neben zwei anderen krummlinigen Coordinaten ein um die Verzweigungslinie herum zu zählendes Azimuth ϕ einführen. Die Green'sche Function des einfachen Raumes $1/R$ wird in dieser Coordinate die Periode 2π haben; um aus ihr die Green'sche Function unseres zuletzt genannten Doppelraumes zu erhalten, haben wir sie in geeigneter Weise so zu modificiren, dass sie in ϕ die Periode 4π erhält.

Schliesslich aber ist der ganze Ansatz nicht auf die Potentialtheorie beschränkt; er dehnt sich sofort auf gewisse andere partielle Differentialgleichungen mit constanten Coefficienten aus, so auf die "*Schwingungsgleichung*" $\Delta u + k^2 u = 0$ und auf die *Wärmeleitungsgleichung* $\Delta u = k^2 \frac{\partial u}{\partial t}$. Man wird dabei statt von dem Newton'schen Potential $1/R$ von den für die letztgenannten Differentialgleichungen charakteristischen Lösungen

$$\frac{e^{ikR}}{R} \quad \text{bez.} \quad t^{-3/2} e^{\frac{-(kR)^2}{4t}}$$

auszugehen haben und mit diesen ähnlich verfahren, wie hier mit dem Newton'schen Potentiale geschehen. Die beiden im Vorstehenden besprochenen Riemann'schen Räume mit einer oder mit zwei parallelen, geradlinigen Verzweigungslinien lassen sich im Gebiete der Schwingungs- und der Wärmeleitungsgleichung genau ebenso behandeln, wie im Gebiete der Potentialgleichung. Ich habe dieses teilweise schon früher, wenn auch auf anderem, minder einfachem Wege ausgeführt.* Der Übergang von einer geraden zu einer kreisförmigen Verzweigungslinie, oder, was dasselbe bedeutet, der Übergang von einem ebenen zu einem kugelförmigen Schirm ist dagegen bei diesen allgemeineren Differentialgleichungen nicht ausführbar, und stellt eine besondere Eigentümlichkeit der Potentialgleichung dar.

Die verzweigten Lösungen der Differentialgleichung $\Delta u + k^2 u = 0$ dürfen ein besonderes Interesse beanspruchen, da sie die von einem ebenen Schirm hervorgerufenen Störungen beliebiger akustischer, optischer, elektro-magnetischer Wellenzüge, d. h. gewisse charakteristische Diffractionsphänomene zu berechnen gestatten. Und zwar ist die so entstehende *Theorie der Diffraction*, im Gegensatz zu der in der Physik üblichen, vom mathematischen Standpunkte aus exakt und für beliebige Grössen der Wellenlänge gültig.

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* "Zur analytischen Theorie der Wärmeleitung," *Math. Annalen*, Bd. XLV., 1894, § 3; und
"Mathematische Theorie der Diffraction," *Math. Annalen*, Bd. XLVII., 1896.

Certain Concomitant Determinants. *By* J. W. Russell, M.A.

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1. The object of this note is to give a simple proof of the invariancy of certain differential operators, viz., in the case of binary quantics of

$$\begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dy_1} \\ \frac{d}{dx_2}, & \frac{d}{dy_2} \end{vmatrix}, \qquad \begin{vmatrix} \frac{d^2}{dx_1^2}, & \frac{d^2}{dx_1 dy_1}, & \frac{d^2}{dy_1^2} \\ \frac{d^2}{dx_2^2}, & \frac{d^2}{dx_2 dy_2}, & \frac{d^2}{dy_2^2} \\ \frac{d^2}{dx_3^2}, & \frac{d^2}{dx_3 dy_3}, & \frac{d^2}{dy_3^2} \end{vmatrix}, \ \ldots$$

the constituents of any row in the general case being the several terms in the expansion of

$$\left(\frac{d}{dx}+\frac{d}{dy}+\frac{d}{dz}+\ldots\right)^n.$$

Calling these operators Δ_1, Δ_2, ..., it is shown that Δ_n^2 gives us in the form of a determinant a covariant of any quantic, which reduces in the case of a $2n$-ic to the catalecticant. It is then shown that, to some extent, we can operate on Δ_n with several lower operators of the same form, so that the result may still remain a determinant. It is hoped that, in this way, many old theorems may be proved more easily, and that possibly some new facts may be obtained which have hitherto escaped notice.

2. To show that the determinant whose successive rows are made up of the several terms in the expansions

$$\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u_1,$$

$$\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u_2,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

is a covariant of the q-ary quantics $u_1, u_2, \ldots u_r$, where r is the number of terms in each of these expansions.

For brevity, take the very simple case when $n = 2$ and $q = 2$.

Then we have to prove that

$$\begin{vmatrix} \frac{d^2u}{dx^2}, & \frac{d^2u}{dx\,dy}, & \frac{d^2u}{dy^2} \\ \frac{d^2v}{dx^2}, & \frac{d^2v}{dx\,dy}, & \frac{d^2v}{dy^2} \\ \frac{d^2w}{dx^2}, & \frac{d^2w}{dx\,dy}, & \frac{d^2w}{dy^2} \end{vmatrix}$$

is a covariant of the binary quantics u, v, and w.

First Method.—Let $\frac{d}{dx}$ operating on u be denoted by a_1, on v by a_2, and on w by a_3; so let $\frac{d}{dy}$ be denoted in these cases by b_1, b_2, b_3; then the operator

$$\begin{vmatrix} a_1^2, & a_1b_1, & b_1^2 \\ a_2^2, & a_2b_2, & b_2^2 \\ a_3^2, & a_3b_3, & b_3^2 \end{vmatrix} = -(a_1b_2-a_2b_1)(a_2b_3-a_3b_2)(a_3b_1-a_1b_3),$$

as we see by considering the identity

$$\begin{vmatrix} 1, & \alpha, & \alpha^2 \\ 1, & \beta, & \beta^2 \\ 1, & \gamma, & \gamma^2 \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

Hence the operator is invariant; and therefore produces a covariant.

Second Method.—Since

$$\frac{d}{dX} = l\,\frac{d}{dx} + l'\,\frac{d}{dy},$$

and $$\frac{d}{dY} = m\,\frac{d}{dx} + m'\frac{d}{dy};$$

therefore $A_1^2 = (la$ $\;{}_1 + l'b_1)^2 = l^2a_1^2 + 2ll'a_1b_1 + l'^2b_1^2,$

$A_1B_1 = (la_1+l'b_1)(ma$ $\;{}_1+m'b_1) = lma_1^2 + (lm'+l'm)\,a_1b_1 + l'm'b_1^2,$

and so on. Hence

$$\begin{vmatrix} A_1^2, & A_1B_1, & B_1^2 \\ A_2^2, & A_2B_2, & B_2^2 \\ A_3^2, & A_3B_3, & B_3^2 \end{vmatrix},$$

multiplying rows by rows,

$$= \begin{vmatrix} a_1^2, & a_1b_1, & b_1^2 \\ a_2^2, & a_2b_2, & b_2^2 \\ a_3^2, & a_3b_3, & b_3^2 \end{vmatrix} \times \begin{vmatrix} l^2, & 2ll', & l'^2 \\ lm, & lm'+l'm, & l'm' \\ m^2, & 2mm', & m'^2 \end{vmatrix}.$$

Hence (see Prof. Elliott's *Algebra of Quantics*, § 23) the operator is invariant; and therefore produces a covariant.

The first method applies only to binary quantics, and shows us that Δ_n is equivalent to $\overline{12}\,.\,\overline{23}\,.\,\overline{34}\ldots$, where every number from 1 to $n+1$ is taken with every other. The second method applies in all cases.

A particular case arises when all the quantics are of the n^{th} order; in this case we get a joint invariant.

Also, since $a_1, b_1, \ldots,$ are any contragredient quantities, we may replace one row by $(\xi+\eta+\ldots)^n$, or in the case of binary quantics by $(-y+x)^n$. In the latter case we may also replace $\left(\frac{d}{dx}+\frac{d}{dy}\right)^n u$ in any row by $\left(\frac{du}{dx}+\frac{du}{dy}\right)^n$.

Notice that we have indirectly proved an interesting algebraic identity, viz., that the determinant whose first row consists of the coefficients in

$$(l_1x_1+l_2x_2+\ldots+l_qx_q)^n,$$

and whose second row of those in

$$(l_1x_1+l_2x_2+\ldots)^{n-1}(m_1x_1+m_2x_2+\ldots),$$

and so on, the general row being given by

$$(l_1x_1+\ldots)^\lambda(m_1x_1+\ldots)^\mu(n_1x_1+\ldots)^\nu\ldots,$$

where

$$\lambda+\mu+\nu+\ldots=n,$$

is a power of the determinant $|\,l_1m_2n_3\ldots\,|$.

This is proved directly for the case $q=2$ in Prof. Elliott's *Algebra of Quantics*, § 16.

3. To prove that the determinant whose successive rows are made up of the several terms in the expansions

$$\left(\frac{d}{dx_1}\right)^n \left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\left(\frac{d}{dx_1}\right)^{n-1} \left(\frac{d}{dx_2}\right) \left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the multipliers $\left(\frac{d}{dx_1}\right)^n$, $\left(\frac{d}{dx_1}\right)^{n-1}\left(\frac{d}{dx_2}\right)$, ... repeat the terms of $\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots\right)^n$, is a covariant of the q-ary quantic u.

For brevity, consider the very simple case when $n = 2$ and $q = 2$. Then we have to show that

$$\begin{vmatrix} \frac{d^4u}{dx^4}, & \frac{d^4u}{dx^3\,dy}, & \frac{d^4u}{dx^2\,dy^2} \\ \frac{d^4u}{dx^3\,dy}, & \frac{d^4u}{dx^2\,dy^2}, & \frac{d^4u}{dx\,dy^3} \\ \frac{d^4u}{dx^2\,dy^2}, & \frac{d^4u}{dx\,dy^3}, & \frac{d^4u}{dy^4} \end{vmatrix}.$$

is a covariant of u.

Consider first three quantics u, v, w. Then the following operator is invariant, viz.,

$$\begin{vmatrix} a_1^2, & a_1b_1, & b_1^2 \\ a_2^2, & a_2b_2, & b_2^2 \\ a_3^2, & a_3b_3, & b_3^2 \end{vmatrix} = \Delta_2.$$

Take the first element $a_1^2 \, . \, a_2b_2 \, . \, b_3^2$ of this determinant; and multiply the first row by a_1^2, the second row by a_2b_2, and the third row by b_3^2. We get

$$a_1^2a_2b_2b_3^2\Delta_2 = \begin{vmatrix} a_1^4, & a_1^3b_1, & a_1^2b_1^2 \\ a_2^3b_2, & a_2^2b_2^2, & a_2b_2^3 \\ a_3^2b_3^2, & a_3b_3^3, & b_3^4 \end{vmatrix}.$$

Next, multiply Δ_2 in the same way by the element got by interchanging the first and second rows, viz., $a_2^2 \, . \, a_1b_1 \, . \, b_3^2$, having first interchanged the same two rows in Δ_2. We get

$$-a_2^2 \cdot a_1 b_1 \cdot b_3^2 \cdot \Delta_2 = \begin{vmatrix} a_2^4, & a_2^3 b_2, & a_2^2 b_2^2 \\ a_1^3 b_1, & a_1^2 b_1^2, & a_1 b_1^3 \\ a_3^2 b_3^2, & a_3 b_3^3, & b_3^4 \end{vmatrix}$$

Proceeding in this way, and noticing that in each case the interchanges are the same and therefore the sign of the element is correct, we finally obtain, by permuting every two rows and adding, the identity

$$\Delta_2^2 = \Sigma \begin{vmatrix} a_1^4, & a_1^3 b_1, & a_1^2 b_1^2 \\ a_2^3 b_2, & a_2^2 b_2^2, & a_2 b_2^3 \\ a_3^2 b_3^2, & a_3 b_3^3, & b_3^4 \end{vmatrix}.$$

Hence this sum, when operating on $u_1 u_2 u_3$, gives a covariant; *i.e.*,

$$\Sigma \begin{vmatrix} \dfrac{d^4 u_1}{dx^4}, & \dfrac{d^4 u_1}{dx^3\, dy}, & \dfrac{d^4 u_1}{dx^2\, dy^2} \\ \dfrac{d^4 u_2}{dx^3\, dy}, & \dfrac{d^4 u_2}{dx^2\, dy^2}, & \dfrac{d^4 u_2}{dx\, dy^3} \\ \dfrac{d^4 u_3}{dx^2\, dy^2}, & \dfrac{d^4 u_3}{dx\, dy^3}, & \dfrac{d^4 u_3}{dy^4} \end{vmatrix}$$

is a covariant of u_1, u_2, u_3.

Now, on putting $u_1 = u_2 = u_3 = u$, each of the determinants included in the sum becomes identical with the given determinant, which is therefore a covariant of u.

Notice that this covariant is $\overline{12}^2 \cdot \overline{23}^2 \cdot \overline{34}^2 \ldots$ for a binary quantic.

If the quantic u is of the $2n^{\text{th}}$ order, the covariant reduces to what we may call, by analogy, the catalecticant of the q-ary $2n$-ic, which is therefore an invariant.

4. We have just shown how to express $\Delta_n \times \Delta_n$ as a determinant. We shall now proceed to show how to operate on Δ_n with Δ_{n-1}. For brevity, take a very simple example, viz., to show that

$$\begin{vmatrix} \dfrac{d^3 u}{dx^3}, & \dfrac{d^3 u}{dx^2\, dy}, & \dfrac{d^3 u}{dx\, dy^2} \\ \dfrac{d^3 u}{dx^2\, dy}, & \dfrac{d^3 u}{dx\, dy^2}, & \dfrac{d^3 u}{dy^3} \\ y^2, & -xy, & x^2 \end{vmatrix}$$

is a covariant of u.

We have seen that

$$\begin{vmatrix} \frac{d^2u}{dx^2}, & \frac{d^2u}{dx\,dy}, & \frac{d^2u}{dy^2} \\ \frac{d^2v}{dx^2}, & \frac{d^2v}{dx\,dy}, & \frac{d^2v}{d\,y} \\ y^2, & -xy, & x^2 \end{vmatrix}$$

or

$$\begin{vmatrix} a_1^2, & a_1 b_1, & b_1^2 \\ a_2^2, & a_2 b_2, & b_2^2 \\ y^2, & -xy, & x^2 \end{vmatrix} uv \qquad (= \Delta, \text{ say})$$

is a covariant of u and v.

Consider $$\Delta_1 = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

Take the first element of Δ_1, viz., $a_1 \,.\, b_2$, and multiply the first row of Δ by a_1, and the second row by b_2, permute the first and second row in each determinant, and add the results. Then, as before,

$$\Delta_1 \,.\, \Delta = \Sigma \begin{vmatrix} a_1^3, & a_1^2 b_1, & a_1 b_1^2 \\ a_2^2 b_2, & a_2 b_2^2, & b_2^3 \\ y^2, & -xy, & x^2 \end{vmatrix} uv.$$

Now put $u = v$, and we obtain the theorem stated.

Similarly, we get a concomitant of any binary quantic, viz.,

$$\begin{vmatrix} \frac{d^{2n-1}u}{dx^{2n-1}}, & \frac{d^{2n-1}u}{dx^{2n-2}dy}, & \dots & \\ \frac{d^{2n-1}u}{dx^{2n-2}dy}, & & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \xi^n, & & \xi^n\eta, & \dots \end{vmatrix},$$

where ξ, η are any contragredient quantities. As a particular case, we may replace ξ, η by $\frac{d}{dx}$, $\frac{d}{dy}$. Thus we can replace the last row by

$$\frac{d^n v}{dx^n}, \quad \frac{d^n v}{dx^{n-1}y}, \quad \dots,$$

where again we may replace v by u.

As u is a binary quantic, we may also replace ξ, η by $\dfrac{du}{dx}, \dfrac{du}{dy}$, or by $-y, x$.

The theorem in the case of a q-ary quantic is that the determinant whose successive rows are the terms of

$$\left(\frac{d}{dx_1}\right)^{n-1}\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\left(\frac{d}{dx_1}\right)^{n-2}\left(\frac{d}{dx_2}\right)\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\ldots+\frac{d}{dx_q}\right)^n u,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

where the multipliers $\left(\dfrac{d}{dx_1}\right)^{n-1}, \left(\dfrac{d}{dx_1}\right)^{n-2}\dfrac{d}{dx_2}, \ldots$ consist of the terms of $\left(\dfrac{d}{dx_1}+\dfrac{d}{dx_2}+\ldots\right)^{n-1}$, the rest of the rows being the terms of such expansions as $(\xi+\eta+\ldots)^n$, where $\xi, \eta, \ldots$ are any contragredient quantities, is a concomitant of u. We may, of course, replace any row $(\xi+\eta+\ldots)^n$ by $\left(\dfrac{d}{dx_1}+\dfrac{d}{dx_2}+\ldots\right)^n v$, and so obtain a joint covariant of several quantics.

5. The above result is obtained by operating, say, with Δ_{n-1} on Δ_n. Other theorems may be obtained by operating with Δ_{n-r} on Δ_n. For example, to prove that

$$\begin{vmatrix} \dfrac{d^4u}{dx^4}, & \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3} \\ \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3}, & \dfrac{d^4u}{dy^4} \\ \dfrac{d^3u}{dx^3}, & \dfrac{d^3u}{dx^2dy}, & \dfrac{d^3u}{dx\,dy}, & \dfrac{d^3u}{dy^3} \\ y^3, & -x^2y, & xy^2, & -x^3 \end{vmatrix}$$

is a covariant of any binary quantic.

Start with the invariant operator

$$\begin{vmatrix} a_1^3, & a_1^2b_1, & a_1b_1^2, & b_1^3 \\ a_2^3, & \ldots & \ldots & \ldots \\ a_3^3, & \ldots & \ldots & \ldots \\ a_4^3, & \ldots & \ldots & b_4^3 \end{vmatrix},$$

and operate with $(a_1b_2-a_2b_1)$; afterwards put $u_1=u_2=u_3$, and finally replace the last row by the terms of $(y-x)^3$.

6. We shall next show how to operate on Δ_n successively with operators of the same kind. Take a simple example.

To prove that

$$\begin{vmatrix} \dfrac{d^4u}{dx^4}, & \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3} \\ \dfrac{d^4u}{dx^3dy}, & \dfrac{d^4u}{dx^2dy^2}, & \dfrac{d^4u}{dx\,dy^3}, & \dfrac{d^4u}{dy^4} \\ \dfrac{d^4v}{dx^4}, & \dfrac{d^4v}{dx^3dy}, & \dfrac{d^4v}{dx^2dy^2}, & \dfrac{d^4v}{dx\,dy^3} \\ \dfrac{d^4v}{dx^3dy}, & \dfrac{d^4v}{dx^2dy^2}, & \dfrac{d^4v}{dx\,dy^3}, & \dfrac{d^4v}{dy^4} \end{vmatrix}$$

is a covariant of any two binary quantics.

Start with the invariant operator

$$\begin{vmatrix} a_1^3, & a_1^2b_1, & a_1b_1^2, & b_1^3 \\ a_2^3, & a_2^2b_2, & a_2b_2^2, & b_2^3 \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3^3 \\ a_4^3, & a_4^2b_4, & a_4b_4^2, & b_4^3 \end{vmatrix}.$$

As before, multiply by $(a_1b_2-a_2b_1)$. This gives the invariant operator

$$\Sigma \begin{vmatrix} a_1^4, & a_1^3b_1, & a_1^2b_1^2, & a_1b_1^3 \\ a_2^3b_2, & a_2^2b_2^2, & a_2b_2^3, & b_2^4 \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3^3 \\ a_4^3, & a_4^2b_4, & a_4b_4^2, & b_4^3 \end{vmatrix},$$

where Σ means the sum of the determinants obtained by permuting the subscripts 1 and 2.

In this put $u_1=u_2=u$; then

$$\left|\begin{matrix} a^4, & a^3b, & a^2b^3, & ab^3 \\ a^3b, & a^2b^2, & ab^3, & b^4 \\ a_3^3, & a_3^2b_3, & a_3b_3^2, & b_3^3 \\ a_4^2, & a_4^2b_4, & a_4b_4^2, & b_4^3 \end{matrix}\right| \begin{matrix} u \\ u \\ u_3 \\ u_4 \end{matrix}$$

is a covariant, where

$$| \; a^4, \quad a^3b, \quad a^2b^2, \quad ab^3 \; | \, u$$

is an abbreviation of

$$\left| \frac{d^4u}{dx^4}, \quad \dots, \quad \frac{d^4u}{dx\,dy^3} \right|,$$

and so on.

Now, multiply by $(a_3b_4 - a_4b_3)$, and put $u_3 = u_4 = v$, and we get the theorem stated.

If we start with Δ_n, we get similarly a covariant of two binary quantics which reduces to a joint invariant in certain cases.

7. In a similar way, we may obtain joint covariants and invariants of several binary quantics. For example,

$$\begin{vmatrix} a_1^7, & a_1^6b_1, & a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5 \\ a_1^6b_1, & a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5, & a_1b_1^6 \\ a_1^5b_1^2, & a_1^4b_1^3, & a_1^3b_1^4, & a_1^2b_1^5, & a_1b_1^6, & b_1^7 \\ a_2^6, & a_2^5b_2, & a_2^4b_2^2, & a_2^3b_2^3, & a_2^2b_2^4, & a_2b_2^5 \\ a_2^5b_2, & a_2^4b_2^2, & a_2^3b_2^3, & a_2^2b_2^4, & a_2b_2^5, & b_2^6 \\ a_3^5, & a_3^4b_3, & a_3^3b_3^2, & a_3^2b_3^3, & a_3b_3^4, & b_3^5 \end{vmatrix},$$

or $$\Delta_1 \,.\, \Delta_2 \,.\, \Delta_5\,,$$

gives us a joint covariant of three binary quantics, and also a joint invariant of a binary 7-ic, a binary 6-ic, and a binary 5-ic.

We can also extend the method to quantics in several variables. For example,

$$\begin{vmatrix} a_1^3, & a_1^2b_1, & a_1^2c_1, & a_1b_1^2, & a_1b_1c_1, & a_1c_1^2 \\ a_1^2b_1, & a_1b_1^2, & a_1b_1c_1, & b_1^3, & b_1^2c_1, & b_1c_1^2 \\ a_1^2c_1, & a_1b_1c_1, & a_1c_1^2, & b_1^2c_1, & b_1c_1^2, & c_1^3 \\ a_2^3, & \dots & \dots & \dots & \dots & \dots \\ a_2^2b_2, & \dots & \dots & \dots & \dots & \dots \\ a_2^2c_2, & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

gives a joint covariant of two ternary quantics, and a joint invariant of two ternary cubics, as we see by operating upon the determinant operator derived from $(a+b+c)^2$ successively by $|a_1 b_2 c_3|$ and $|a_4 b_5 c_6|$.

Similarly, by starting with the determinant operator derived from $(a+b+c)^3$, which is of the tenth order, we can obtain a joint covariant of any three ternary quantics, and also a joint invariant of a ternary quintic, a ternary quartic, and a ternary cubic, viz., by operating with the operator derived from $(a+b+c)^2$ on the first six rows, with the operator $|a_7 b_8 c_9|$ on the next three rows, and leaving the last row unaltered.

Note on the Potential of Rings. By A. L. Dixon.

Received and read April 8th, 1897.

In a paper recently read before this Society,* Dr. E. W. Hobson gave the following result, viz., that

$$\iiint \dots \frac{\left\{1-\sum\limits_{r=1}^{r=n} \frac{\xi_r^2}{a_r^2}\right\}^{\lambda-1}}{\left\{\sum\limits_{r=1}^{r=n} \overline{x_r-\xi_r}^2+h^2\right\}^{\frac{1}{2}m}} F\left\{\xi_1, \xi_2, \xi_3 \dots \xi_n\right\} d\xi_1 d\xi_2 \dots d\xi_n$$

(where the integral is taken for all real values of the ξ's which make $1-\Sigma \frac{\xi^2}{a^2}$ positive or zero)

$$= \frac{m\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} U^{\frac{1}{2}(n-m)+\lambda-1}$$

$$\times \left\{1+\frac{U\theta D}{2 \,.\, n-m+2\lambda}+\frac{U^2\theta^2 D^2}{2 \,.\, 4 \,.\, n-m+2\lambda \,.\, n-m+2\lambda+2}\right.$$

$$\left.+\frac{U^3\theta^3 D^3}{2 \,.\, 4 \,.\, 6 \,.\, n-m+2\lambda \,.\, n-m+2\lambda+2 \,.\, n-m+2\lambda+4}+\dots\right\}$$

$$\times F\left\{\frac{a_1^2 x_1}{a_1^2+\theta}, \frac{a_2^2 x_2}{a_2^2+\theta}, \dots \frac{a_n^2 x_n}{a_n^2+\theta}\right\} d\theta,$$

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* *Proceedings*, Vol. XXVII., p. 524.

where $$P = \left\{(a_1^2+\theta)(a_2^2+\theta)(a_3^2+\theta)\ \ldots\ (a_n^2+\theta)\right\}^{\frac{1}{2}},$$

$$U = 1 - \frac{x_1^2}{a_1^2+\theta} - \frac{x_2^2}{a_2^2+\theta} \cdots \frac{x_n^2}{a_n^2+\theta} - \frac{h^2}{\theta},$$

$$D = \frac{a_1^2+\theta}{a_1^2}\,\frac{d^2}{dx_1^2} + \frac{a_2^2+\theta}{a_2^2}\,\frac{d^2}{dx_2^2} + \ldots\ \frac{a_n^2+\theta}{a_n^2}\,\frac{d^2}{dx_n^2},$$

and θ_0 is the positive root of $U = 0$.

By interpreting $x_1 x_2 \ldots \xi_1 \xi_2 \ldots$ geometrically as Cartesian coordinates in space of $n+1$ dimensions, Dr. Hobson deduced many interesting results as to the potentials of "ellipsoids" and "ellipsoidal discs." The object of the present communication is to show how, by a somewhat curious geometrical interpretation, these results may be extended to the determination of the potential at any point on the axis of rotation of an "ellipsoidal" ring, by which I mean a solid formed in space of $n+1$ dimensions, by the rotation of an "ellipsoid" of n dimensions about a line parallel to one of its principal axes.

It will be sufficient to confine the investigation to space of three dimensions; the generalizations that can be made after the manner of Dr. Hobson's paper will be obvious.

The equation of a ring of elliptic section, the circular axis of which is a circle of radius c, is obviously

$$\frac{(\sqrt{x^2+y^2}-c)^2}{a^2} + \frac{z^2}{b^2} = 1,$$

the axis of rotation being the axis of z, and the centre of the circular axis the origin.

Write r for $\sqrt{x^2+y^2}$, and ρ for $r-c$.

An element of volume is

$$2\pi r\,dr\,dz \quad \text{or} \quad 2\pi\,(\rho+c)\,d\rho\,dz,$$

and the potential at any point z' on the axis of rotation is

$$\iint \frac{2\pi r\,dr\,dz}{(r^2+\overline{z-z'}^2)^{\frac{1}{2}m}} \left\{1 - \frac{(r-c)^2}{a^2} - \frac{z^2}{b^2}\right\}^{\lambda-1} f(r, z),$$

where $$\left\{1 - \frac{(r-c)^2}{a^2} - \frac{z^2}{b^2}\right\}^{\lambda-1} f(r, z)$$

is the density at such point of the ring. Now, putting ρ for $r-c$, this becomes

$$2\pi\iint\frac{(\rho+c)\,d\rho\,dz}{(\overline{\rho+c}^2+\overline{z-z'}^2)^{\frac{1}{2}m}}\left(1-\frac{\rho^2}{a^2}-\frac{z^2}{b^2}\right)^{\lambda-1}F(\rho,z),$$

which, by Dr. Hobson's transformation, is equal to

$$\frac{2\pi m\,\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\lambda-\frac{m}{2}\right)}\tfrac{1}{2}\pi ab\int_{\theta_0}^{\infty}\frac{\theta^{-\frac{1}{2}m}\,U^{\lambda-\frac{1}{2}m}}{(a^2+\theta)^{\frac{1}{2}}(b^2+\theta)^{\frac{1}{2}}}$$

$$\times\left(1+\frac{U\theta D}{2.\,2\lambda+2-m}+\frac{U^2\theta^2D^2}{2.\,4.\,2\lambda+2-m.\,2\lambda+4-m}+\ldots\right)\left(\frac{a^2\rho}{a^2+\theta}+c\right)$$

$$\times F\left(\frac{a^2\rho}{a^2+\theta},\ \frac{b^2z}{b^2+\theta}\right),$$

where
$$U=1-\frac{\rho^2}{a^2+\theta}-\frac{z^2}{b^2+\theta},$$

$$D=\frac{a^2+\theta}{a^2}\,\frac{d^2}{d\rho^2}+\frac{b^2+\theta}{b^2}\,\frac{d^2}{dz^2},$$

and ρ is put equal $-c$, and z to z' after the operations D, D^2, &c., have been performed.

For instance, taking $m=1$, $\lambda=1$, $F=1$, we have as the potential at any point on the axis of a ring of elliptic cross-section and of uniform density—the law of attraction being that of the inverse square of the distance—the expression

$$4\pi ab\int_{\theta_0}^{\infty}\frac{\theta^{-\frac{1}{2}}\left(1-\dfrac{c^2}{a^2+\theta}-\dfrac{z^2}{b^2+\theta}\right)^{\frac{1}{2}}}{(a^2+\theta)^{\frac{1}{2}}(b^2+\theta)^{\frac{1}{2}}}\;\frac{c\theta}{a^2+\theta}\,d\theta,$$

or
$$4\pi abc\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}}(\theta-\theta_0)^{\frac{1}{2}}(\theta-\theta_1)^{\frac{1}{2}}\,d\theta}{(a^2+\theta)^2(b^2+\theta)},$$

where θ_0 is the positive root, and θ_1 the other root of the equation

$$(a^2+\theta)(b^2+\theta)-c^2(b^2+\theta)-z^2(a^2+\theta)=0.$$

Putting $a=b$, we get

$$\theta_0=c^2+z^2-a^2,\quad \theta_1=-a^2,$$

and the potential for a ring of circular section is

$$4\pi a^2c\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}}(\theta-\theta_0)^{\frac{1}{2}}}{(a^2+\theta)^{\frac{5}{2}}}\,d\theta.$$

Putting $\theta = (c^2+z^2)\,\mathrm{ds}^2 u,$

where $\kappa^2 = a^2/(c^2+z^2),$

we get

$$\frac{4M}{\pi}\int_0^K \frac{\mathrm{cn}^2 u\,\mathrm{dn}^2 u\,du}{(c^2+z^2)^{\frac{1}{2}}},$$

which is the same as

$$\frac{4M}{3\pi\sqrt{c^2+z^2}}\left\{\left(1+\frac{1}{\kappa^2}\right)E+\left(1-\frac{1}{\kappa^2}\right)K\right\},$$

the expression found in Mr. F. W. Dyson's paper "On the Potential of an Anchor Ring" (*Phil. Trans.*, 1893, A, p. 58).

For a density $\frac{1}{r}$, we get for a ring of elliptic section

$$4\pi ab\int_{\theta_0}^{\infty}\frac{(\theta-\theta_0)^{\frac{1}{2}}(\theta-\theta_1)^{\frac{1}{2}}\,d\theta}{\theta^{\frac{1}{2}}(a^2+\theta)(b^2+\theta)},$$

and for a ring of circular section

$$4\pi a^2\int_0^K \mathrm{cn}^2 u\,du \qquad [\kappa^2 = a^2/(c^2+z^2)].$$

On the Deformation of a Plane Closed Polygon so that a certain Function remains Constant. *By* F. S. MACAULAY, M.A.

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We use the term *deformation* in the general sense of *change of shape.* By a plane *closed polygon* we mean one whose complete boundary can be continuously traced by a point starting from any vertex and ending at the same, no finite part being traced twice, except where it may happen that one part of the boundary coincides with another part.

In thus tracing the boundary the sides are traced *in order.* Each side has a definite magnitude and direction, the direction namely in which it is traced. Changing the order of the sides, without altering their magnitudes and directions, gives rise to a new closed polygon. for which, however, the function considered below remains constant.

We shall use the term *double segment* for a closed rectilineal figure which has only two vertices, and consequently two equal and opposite sides.

We suppose the lengths of all straight lines to be positive, so that the lines BC, CB differ only in direction, not in magnitude or sign. We shall denote points by capital letters $A, B, C, \dots$; and lengths of lines either by small letters $a, b, c, \dots$, or by capitals $BC, CA, \dots$. Also, if a, BC are the lengths of lines with definite directions, we use the notation (a), (BC) for $a.e^{\iota\theta}$, $BC.e^{\iota\phi}$, where θ, ϕ are the inclinations of the lines a, BC respectively to a fixed arbitrary line in the plane. Thus, if A, B, C are any three points, and α, β, γ the angles that BC, CA, AB make with the fixed line or axis, we have

$$(BA) = (AB)\,e^{\iota\pi} = -(AB),$$

and
$$(BC)+(CA)+(AB) = BC.e^{\iota\alpha}+CA.e^{\iota\beta}+AB.e^{\iota\gamma}$$

$$=\overline{BC\cos\alpha+CA\cos\beta+AB\cos\gamma}+\iota\,\overline{BC\sin\alpha+CA\sin\beta+AB\sin\gamma}=0.$$

Hence, if O be any fixed origin in the plane, we have

$$(BO)+(OA)+(AB)=0,$$

or
$$(AB)=(OB)-(OA), \qquad (1)$$

since $(BO) = -(OB)$, from above.

Similarly for any number of points $A, B, \dots H, K$, we have

$$(AB)+(BC)+\dots+(HK)+(KA)=0,$$

and all the geometrical properties of a figure, involving lengths of lines, are expressible in equations of this type.

In saying that two triangles ABC, $A'B'C'$ are similar, we mean not only that the angles A, B, C are equal to A', B', C' respectively, but also that the rotations denoted by ABC, $A'B'C'$ are in the same sense, both positive or both negative. When such similarity exists it is clear that the ratios $(BC):(B'C')$, $(CA):(C'A')$, $(AB):(A'B')$ are all equal. Also the *sufficient* condition for the similarity is that two of these ratios are equal. For, if

$$(BC):(B'C') = (CA):(C'A'),$$

then
$$BC.e^{\iota\alpha} : B'C'.e^{\iota\alpha'} = CA.e^{\iota\beta} : C'A'.e^{\iota\beta'};$$

therefore
$$\frac{BC}{B'C'}\cos(\alpha-\alpha') = \frac{CA}{C'A'}\cos(\beta-\beta'),$$

and
$$\frac{BC}{B'C'}\sin(\alpha-\alpha') = \frac{CA}{C'A'}\sin(\beta-\beta');$$

therefore
$$\frac{BC}{B'C'}=\frac{CA}{C'A'},$$

and
$$\alpha-\alpha'=\beta-\beta'+2n\pi;$$

the solution $\frac{BC}{B'C'}=-\frac{CA}{C'A'}$ being inadmissible, since all lengths are positive. Hence the triangles ABC, $A'B'C'$ are similar in the complete sense.

By means of the formula (1) we can write the condition for the similarity of the triangles ABC, $A'B'C'$ in either of the symmetrical forms

$$\begin{aligned}(OA)(B'C')+(OB)(C'A')+(OC)(A'B')&=0,\\(OA')(BC)+(OB')(CA)+(OC')(AB)&=0.\end{aligned}\tag{2}$$

The same condition may also be given in the forms

$$\begin{vmatrix}1 & (OA) & (O'A')\\ 1 & (OB) & (O'B')\\ 1 & (OC) & (O'C')\end{vmatrix}=0,\qquad \left\|\begin{matrix}(BC), & (CA), & (AB)\\ (B'C'), & (C'A'), & (A'B')\end{matrix}\right\|=0,$$

the two equations involved in the last form being equivalent to only one.

Similarly, in saying that the triangle (a), (b), (c) is similar to PQR, we mean that the angles opposite (a), (b), (c) are equal to the angles P, Q, R respectively, and that the rotations denoted by (a), (b), (c) and PQR are in the same sense. The condition for the similarity is the same as above, viz.:

$$(p)(a)+(q)(b)+(r)(c)=0,$$

where (p), (q), (r) stand for (OP), (OQ), (OR) respectively, O being any fixed point in the plane.

The general property of closed polygons which we give below can, in the cases of the triangle and the quadrilateral, be proved by elementary geometry. The property of the triangle is probably well known, and possibly also that of the quadrilateral. These two cases may be enunciated as follows:—

If ABC, PQR be any two triangles, and on the sides of ABC three other triangles $BP'C$, $CQ'A$, $AR'B$ be described similar to QPR, RQP, PRQ respectively, then the pairs of lines AP', QR, and BQ', RP, and CR', PQ have one and the same product, and are equally inclined to one and the same line. More definitely,

$$(AP')(QR)=(BQ')(RP)=(CR')(PQ).$$

If $ABCD$, $PQRS$ *be any two quadrilaterals, and triangles* $AP'B$, $BQ'C$, $CR'D$, $DS'A$ *be described similar to* QPS, RQP, SRQ, PSR *respectively, then the pairs of lines* $P'R'$, QS, *and* $Q'S'$, PR *have the same product, and are equally inclined to the same line, or, in symbolic notation,*

$$(P'R')(QS)+(Q'S')(PR)=0.$$

We have here only partly enunciated the theorem for the quadrilateral; the complete theorem is obtained by combining it with the preceding one.

The general theorem for any closed polygon is as follows:—

Let (a_1), (a_2), ... (a_n) *be the sides of any closed polygon taken in order to which correspond respectively any* n *given points* S_1, S_2, ... S_n. *On any side* (a_p) *of the polygon describe a triangle similar to* $S_{p+1}\,S_p\,S_{p-1}$, *the side* (a_p) *being homologous to the side* $S_{p+1}\,S_{p-1}$; *and deform the polygon by moving the two vertices on the side* (a_p) *up to the vertex of the triangle, without displacing any other vertices. We thus obtain a new closed polygon whose sides, taken in order, are*

$$(a_1),\ \dots\ (a_{p-2}),\ (a'_{p-1}),\ (a'_{p+1}),\ (a_{p+2})\ \dots\ (a_n),$$

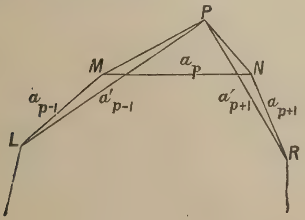
corresponding respectively to the points

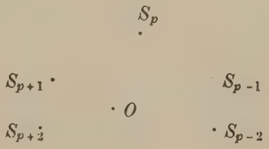
$$S_1,\ S_2,\ \dots\ S_{p-1},\ S_{p+1},\ \dots\ S_n.$$

By applying a similar process to the new polygon, and continuing it, we finally arrive at a double segment whose sides HK, KH *correspond to two of the points* S, *say* S_q, S_r. *Then the lines* HK, S_q, S_r *have a fixed product and are equally inclined to a fixed line.*

Thus the ratios and inclinations of the $\frac{1}{2}n\,(n-1)$ double segments which can be derived from the polygon (a_1), (a_2), ... (a_n) depend only on the relative positions of the points S_1, S_2, ... S_n; *i.e.*, they in no way depend on the polygon from which they are derived. To complete the theorem, one other property remains to be mentioned:—

If the vertices H, K *of one of the double segments coincide, the same will be true of any other double segment, and in this case any triangle derived from the polygon, whose sides correspond to* S_p, S_q, S_r, *will be similar to the triangle* $S_pS_qS_r$.





To prove the theorem, we take any fixed point O, and consider the function $\sum\limits_{m=1}^{m=n}(a_m)(OS_m)$, or $\Sigma\,(s_m)(a_m)$, where (s_m) stands for (OS_m). This function remains the same wherever O is taken. Let LM, MN, NR be the sides (a_{p-1}), (a_p), (a_{p+1}) and MPN the triangle similar to $S_{p+1}\,S_p\,S_{p-1}$ described on the side (a_p). Then from these triangles we have, by (2),

$$(s_{p-1})(MP)+(s_p)(NM)+(s_{p+1})(PN)=0,$$

or $$(s_{p-})(MP)+(s_{p+1})(PN)=(s_p)(MN)=(s_p)(a_p).$$

Adding $(s_{p-1})(a_{p-1})+(s_{p+1})(a_{p+1})$ to each side, we have, by (1),

$$(s_{p-1})(a'_{p-1})+(s_{p+1})(a'_{p+1})=(s_{p-1})(a_{p-1})+(s_p)(a_p)+(s_{p+1})(a_{p+1}).$$

Thus the function $\Sigma\,(s_m)(a_m)$ does not alter when $LMNR$ is replaced by LPR. In fact, if $PM'N'$ is any triangle similar to PMN, the value of the function $\Sigma\,(s_m)(a_m)$ remains constant when $LMNR$ is replaced by $LM'N'R$; and when the triangle $PM'N'$ is made to vanish the term $(s_p)(a'_p)$ also vanishes, and disappears from the function. Hence, if we continue the reduction until we arrive at a double segment whose sides HK, KH correspond to s_q, s_r respectively, we have

$$(s_q)(HK)+(s_r)(KH)=\Sigma\,(s_m)(a_m),$$

or $$\{(s_q)-(s_r)\}(HK)=\Sigma\,(s_m)(a_m),$$

or $$(HK)(S_qS_r)=-\Sigma\,(s_m)(a_m)=\text{constant}.$$

This proves the theorem. It is clear that a double segment cannot vanish unless $\Sigma\,(s_m)(a_m)$ vanishes, assuming that none of the points S are at infinity. If $\Sigma\,(s_m)(a_m)$ vanishes, every double segment vanishes, and in this case, if the reduction of the polygon be carried down to a triangle ABC, whose sides correspond to S_p, S_q, S_r, then

$$(s_p)(BC)+(s_q)(CA)+(s_r)(AB)=0,$$

so that the triangle ABC is similar to $S_pS_qS_r$.

Thursday, May 13th, 1897.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Ten members present.

The President read a letter from Mr. J. H. Enthoven, thanking the Society for the letter of condolence which had been drawn up by the President (*cf.* p. 393).

The following communications were made:—

On Cubic Curves as connected with certain Triangles in Perspective: Mr. S. Roberts.

Determination of certain Primes: Mr. F. W. Lawrence.

An Analogue of Anharmonic Ratio: Mr. J. Brill.

An Essay on the Geometrical Calculus (*continuation*): Herr E. Lasker.

On the Partition of Numbers: Mr. G. B. Mathews.

Notes on Synthetic Geometry: Mr. W. Esson.

The following presents were made to the Library:—

Queen's College, Galway, "Calendar for 1896–7."

Miller, W. J. C.—"Mathematical Questions and Solutions," Vol. LXVI., 8vo; London, 1897.

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. 3, No. 185; April, 1897.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 3; Leipzig, 1897.

"Proceedings of the Royal Society," Vol. LXI., No. 370.

"Monatshefte für Mathematik und Physik," Jahrgang VIII., 1897, Pt. 2; Wien.

"Proceedings of the Physical Society," Vol. XV., Pt. 4; April, 1897.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXX., Liv. 5; Harlem, 1897.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," 1896, V., VI.; Leipzig, 1897.

"Œuvres Mathématiques d'Evariste Galois," avec une Introduction par E. Picard, 8vo; Paris, 1897.

"Bulletin of the American Mathematical Society," Vol. III., No. 7; New York, 1897.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIII., No. 1; Coimbra, 1897.

"Bulletin de la Société Mathématique de France," Tome XXV., Nos. 2, 3; Paris, 1897.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich," 1896, Supplement.

"Bulletin des Sciences Mathématiques," Tome XXI., Avril, 1897; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 3, Vol. III., Fasc. 3, 4; 1897.

"Atti della R. Accademia delle Science Fisiche e Matematiche di Napoli," Vol. VIII.; 1897.

"Rendiconti del Circolo Matematico di Palermo," Tomo XI., Fasc. 3; 1897.

"Journal of the College of Science, Imperial University, Japan," Vol. IX., Pt. 2.

Fischer, O.—"Beiträge zu einer Muskeldynamik," No. 6, 8vo; Leipzig, 1897.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 6, 7, Sem. 1; Roma, 1897.

"Educational Times," May, 1897.

"Annales de la Faculté des Sciences de Toulouse," Tome XI., 1897, Fasc. 1; Paris.

"Journal für die reine und angewandte Mathematik," Bd. CXVII., Heft 4; Berlin, 1897.

"Indian Engineering," Vol. XXI., Nos. 12–16, March 20–April 17, 1897.

See, T. J. J.—"Researches on the Evolution of the Stellar Systems," Vol. I., 4to; Lynn, Mass., 1896.

On Cubic Curves as connected with certain Triangles in Perspective.

By SAMUEL ROBERTS. Received and read May 13th, 1897.

1. Consider a triangle $A'B'C'$ in perspective with the triangle of reference ABC, the connectors AA', BB', CC' meeting in O, the centre of perspective (or homology). The coordinates of the vertices A', B', C' may be supposed proportional to

$$\begin{array}{ccc} h_1x, & y, & z, \\ x, & h_2y, & z, \\ x, & y, & h_3z, \end{array}$$

where x, y, z are the trilinear coordinates of O.

Write x', y', z' for the coordinates of A'; then

$$\frac{x'}{x} = \frac{A'D}{OD}, \quad \frac{y'}{y} = \frac{z'}{z} = \frac{AA'}{AO},$$

D being the point in which AO meets the side of ABC opposite A.

Hence we have

$$x' : y' : z' :: \frac{AO}{OD} \cdot \frac{A'D}{AA'} x : y : z,$$

and h_1 is the anharmonic ratio of the range $AA'OD$; in like manner h_2, h_3 are the anharmonic ratios of the ranges $BB'OE$, $CC'OF$, if E, F are the points in which BO, CO meet the sides of ABC opposite to B, C.

2. Take other three points A'', B'', C'' whose coordinates are proportional to

$$(X_1,\ Y_1,\ Z_1),$$
$$(X_2,\ Y_2,\ Z_2),$$
$$(X_3,\ Y_3,\ Z_3),$$

respectively, and let the triangle $A''B''C''$ be in perspective with the triangle $A'B'C'$, corresponding vertices being denoted by the same letters. If α, β, γ are the coordinates of the centre of perspective, the following conditions hold:—

$$\left.\begin{aligned} xh_1(\gamma Y_1-\beta Z_1)+y\ (\alpha Z_1-\gamma X_1)+z\ (\beta X_1-\alpha Y_1) &= 0, \\ x\ (\gamma Y_2-\beta Z_2)+yh_2(\alpha Z_2-\gamma X_2)+z\ (\beta X_2-\alpha Y_2) &= 0, \\ x\ (\gamma Y_3-\beta Z_3)+y\ (\alpha Z_3-\gamma X_3)+zh_3(\beta X_3-\alpha Y_3) &= 0. \end{aligned}\right\} \qquad (1)$$

If h_1, h_2, h_3 are constant, the eliminant with regard to x, y, z, equated to zero, gives a cubic locus. The equation is

$$\begin{aligned} &\gamma^2\alpha\,\{X_2Y_1Y_3\,(h_1h_2h_3-h_2)+X_3Y_1Y_2\,(1-h_1)+X_1Y_2Y_3\,(1-h_3)\} \\ &-\beta^2\alpha\,\{X_3Z_1Z_2\,(h_1h_2h_3-h_3)+X_2Z_1Z_3\,(1-h_1)+X_1Z_2Z_3\,(1-h_2)\} \\ &+\alpha^2\beta\,\{Y_3Z_1Z_2\,(h_1h_2h_3-h_3)+Y_1Z_2Z_3\,(1-h_2)+Y_2Z_1Z_3\,(1-h_1)\} \\ &-\gamma^2\beta\,\{X_2X_3Y_1\,(h_1h_2h_3-h_1)+X_1X_2Y_3\,(1-h_2)+X_1X_3Y_2\,(1-h_3)\} \\ &+\beta^2\gamma\,\{X_2X_3Z_1\,(h_1h_2h_3-h_1)+X_1X_3Z_2\,(1-h_3)+X_1X_2Z_3\,(1-h_2)\} \\ &-\alpha^2\gamma\,\{Y_1Y_3Z_2\,(h_1h_2h_3-h_2)+Y_2Y_3Z_1\,(1-h_3)+Y_1Y_2Z_3\,(1-h_1)\} \\ &+\alpha\beta\gamma\,\{(X_3Y_1Z_2-X_2Y_3Z_1)(h_1h_2h_3-1)+(X_3Y_2Z_1-X_2Y_1Z_3)\,h_1 \\ &\qquad+(X_2Y_1Z_3-X_1Y_3Z_2)\,h_2+(X_1Y_3Z_2-X_3Y_2Z_1)\,h_3\} = 0. \quad (2) \end{aligned}$$

The curve passes through the vertices of the triangle of reference, and also [as the form of (1) shows] through those of the triangle $A''B''C''$.

There are three constants h_1, h_2, h_3 at our disposal to determine three other points on the curve. Any curve of the third degree can therefore be represented by an equation of the foregoing form, since we can always take six points on the curve, no three of which shall be collinear. In this case there are ten pairs of triangles, and we can take any one of the triangles for reference. Moreover the correspondence of vertices may be varied. If we include collinear triplets of points, the numbers of possible triangles of reference will be so far diminished.

The above result may be stated thus :—If through the vertices of a triangle ABC a variable triplet of right lines concurrent at O be drawn meeting the opposite sides in D, E, F, and the vertices of variable triangles $A'B'C'$ be determined so that the anharmonic ratios of the ranges $AA'OD$, $BB'OE$, $CC'OF$ are constant, the locus of the centres of homology of such of these triangles as are in perspective with a fixed triangle $A''B''C''$ is a cubic curve through the vertices of the two fixed triangles.

We may take the second triangle for reference and represent the same curve by an equation of the same form.

3. Let

$$\left.\begin{aligned} A_1\alpha+B_1\beta+C_1\gamma &= 0, \\ A_2\alpha+B_2\beta+C_2\gamma &= 0, \\ A_3\alpha+B_3\beta+C_3\gamma &= 0, \end{aligned}\right\} \qquad (3)$$

be the equations of the sides of the second triangle opposite respectively to the vertices $A''B''C''$ which correspond to A, B, C.

The coordinates of the intersections of three straight lines through the point (α, β, γ) with the sides (3) respectively are proportional to

$$\begin{array}{ccc} -\dfrac{B_1\beta+C_1\gamma}{A_1}, & \beta, & \gamma, \\ \alpha, & -\dfrac{A_2\alpha+C_2\gamma}{B_2}, & \gamma, \\ \alpha, & \beta, & -\dfrac{A_3\alpha+B_3\beta}{C_3}. \end{array}$$

Now, take straight lines through these points respectively, and through the corresponding opposite vertices of the second fixed triangle $A''B''C''$. The equations are of the form

$$x(Y_1\gamma-Z_1\beta)+y\left(-\frac{B_1\beta+C_1\gamma}{A_1}-X_1\gamma\right)+z\left(\frac{B_1\beta+C_1\gamma}{A_1}+X_1\beta\right)=0.$$

&c. &c.

For brevity, denote $Y_1\gamma - Z_1\beta$ by $(Y_1\gamma)'$, and so in like cases. The condition that the above lines shall meet in a point is

$$\begin{vmatrix} A_1\,(Y_1\gamma)', & B_1\,(Y_1\gamma)'-\Delta\gamma, & C_1\,(Y_1\gamma)'+\Delta\beta \\ A_2\,(Z_2\alpha)'+\Delta\gamma, & B_2\,(Z_2\alpha)', & C_2\,(Z_2\alpha)'-\Delta\alpha \\ A_3\,(X_3\beta)'-\Delta\beta, & B_3\,(X_3\beta)'+\Delta\alpha, & C_3\,(X_3\beta)' \end{vmatrix} = 0, \quad (4)$$

where Δ means the determinant

$$\begin{vmatrix} X_1\,Y_1\,Z_1 \\ X_2\,Y_2\,Z_2 \\ X_3\,Y_3\,Z_3 \end{vmatrix},$$

A_1, B_1, C_1 are $(Y_2Z_3)'$, $(Z_2X_3)'$, $(X_2Y_1)'$, respectively, and the sets A_2, B_2, C_2, and A_3, B_3, C_3 have analogous values.

The determinant (4) is separable into

$$\begin{vmatrix} A_1\,(Y_1\gamma)', & B_1\,(Y_1\gamma)', & C_1\,(Y_1\gamma)' \\ A_2\,(Z_2\alpha)', & B_2\,(Z_2\alpha)', & C_2\,(Z_2\alpha)' \\ A_3\,(X_3\beta)', & B_3\,(X_3\beta)', & C_3\,(X_3\beta)' \end{vmatrix} (a)$$

$$+ \begin{vmatrix} A_1\,(Y_1\gamma)', & B_1\,(Y_1\gamma)', & \Delta\beta \\ A_2\,(Z_2\alpha)', & B_2\,(Z_2\alpha)', & -\Delta\alpha \\ A_3\,(X_3\beta)', & B_3\,(X_3\beta)', & 0 \end{vmatrix} (b)$$

$$+ \begin{vmatrix} 0, & B_1\,(Y_1\gamma)', & C_1\,(Y_1\gamma)' \\ \Delta\gamma, & B_2\,(Z_2\alpha)', & C_2\,(Z_2\alpha)' \\ -\Delta\beta, & B_3\,(X_3\beta)', & C_3\,(X_3\beta)' \end{vmatrix} (c)$$

$$+ \begin{vmatrix} A_1\,(Y_1\gamma)', & -\Delta\gamma, & C_1\,(Y_1\gamma)' \\ A_2\,(Z_2\alpha)', & 0, & C_2\,(Z_2\alpha)' \\ A_3\,(X_3\alpha)', & \Delta\alpha, & C_3\,(X_3\alpha)' \end{vmatrix} (d)$$

$$+ \begin{vmatrix} 0, & B_1\,(Y_1\gamma)', & \Delta\beta \\ \Delta\gamma, & B_2\,(Z_2\alpha)', & -\Delta\alpha \\ -\Delta\beta, & B_3\,(X_3\beta)', & 0 \end{vmatrix} (e) \quad + \begin{vmatrix} 0, & -\Delta\gamma, & C_1\,(Y_1\gamma)' \\ \Delta\gamma, & 0, & C_2\,(Z_2\alpha)' \\ -\Delta\beta, & \Delta\alpha, & C_3\,(X_2\beta)' \end{vmatrix} (f)$$

$$+ \begin{vmatrix} A_1\,(Y_1\gamma)', & -\Delta\gamma, & \Delta\beta \\ A_2\,(Z_2\alpha)', & 0, & -\Delta\alpha \\ A_3\,(X_3\beta)', & \Delta\alpha, & 0 \end{vmatrix} (g) \quad + \begin{vmatrix} 0, & -\Delta\gamma, & \Delta\beta \\ \Delta\gamma, & 0, & -\Delta\alpha \\ -\Delta\beta, & \Delta\alpha, & 0 \end{vmatrix} (h).$$

But $(e)+(f)+(g)$ gives

$$\alpha^2\gamma\,(Y_1Y_2Z_3+Y_2Y_3Z_1)-\alpha^2\beta\,(Y_1Z_2Z_3+Y_2Z_1Z_3)$$
$$+\beta^2\alpha\,(X_2Z_1Z_3+X_1Z_2Z_3)-\&\text{c.}$$
$$+\alpha\beta\gamma(X_3Y_1Z_2-X_2Y_3Z_1)+2\,(Y_1\gamma)'\,(Z_2\alpha)'\,(X_3\beta)'.$$

Moreover $(a)+(b)+(c)+(d)$ gives

$$-2\,(Y_1\gamma)'\,(Z_2\alpha)'\,(X_3\beta)',$$

and (h) is zero. Therefore the net result is

$$\alpha^2\gamma\,(Y_1Y_2Z_3+Y_2Y_3Z_1)-\&\text{c.}+\alpha\beta\gamma\,(X_3Y_1Z_2-X_2Y_3Z_1)=0.$$

This equation is the same, except the reversal of signs, as that obtained from (2) by making

$$h_1=h_2=h_3=0,$$

and the new variable triangle is inscribed in the second triangle, while the locus is the same.

4. The foregoing result depends on the following geometrical theorem, which is probably known, though I have not met with it: Given two triangles ABC, $A'B'C'$, and three right lines through the vertices of ABC concurrent in O_1, and meeting the opposite sides in D, E, F; if DEF, $A'B'C'$ are in perspective, and P is the centre of perspective, right lines through A, B, C concurrent in P will meet corresponding sides of $A'B'C'$ in points D', E', F' such that right lines joining these points to opposite vertices will be concurrent in O_2.

Thus, in Fig. 1, we have

$$\frac{AF}{FB}=\frac{\sin APF}{\sin FPB}\,\frac{AP}{BP},\quad \frac{BD}{DC}=\frac{\sin BPD}{\sin DPC}\,\frac{BP}{CP},\quad \frac{CE}{EA}=\frac{\sin CPE}{\sin EPA}\,\frac{CP}{AP},$$

$$\frac{A'F'}{F'B'}=\frac{\sin DPC}{\sin CPE}\,\frac{A'P}{B'P},\quad \frac{B'D'}{D'C'}=\frac{\sin EPA}{\sin APF}\,\frac{B'P}{C'P},\quad \frac{C'E'}{E'A'}=\frac{\sin FPB}{\sin BPD}\,\frac{C'P}{A'P}$$

and since, by the theorem of Ceva,

$$\frac{AF}{FB}\,\frac{BD}{DC}\,\frac{CE}{EA}=1,$$

we have also
$$\frac{A'F'}{F'B'}\,\frac{B'D'}{D'C'}\,\frac{C'E'}{E'A'}=1.$$

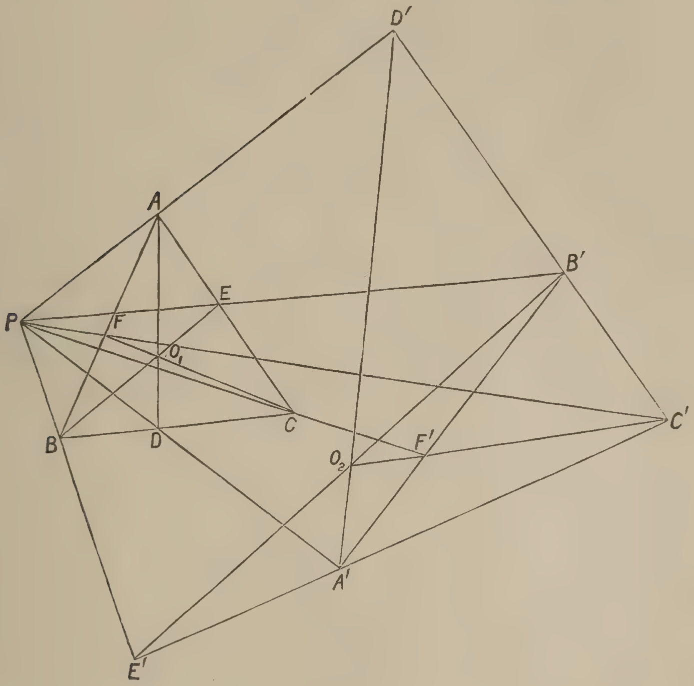


FIG. 1.

5. It appears that (2) gives the same equation when, instead of being zero, h_1, h_2, h_3 have the harmonic value -1. In fact, the variable triangle and the centre of perspective relative to the triangle of reference then constitute a quadrangle whose harmonic triangle is that of reference. Thus in (Fig. 2) a, b, c, d being the quadrangle, the right lines through any point P and through a, b, c concurrent in P meet the corresponding sides of the harmonic triangle ABC in D, E, F, and right lines through these points and corresponding vertices meet in O.

6. When h_1, h_2, h_3 are made equal to h in (2), the equation becomes, after division by $1-h$,

$$(h^2+h)\{(Y_1\gamma)'(Z_2\alpha)'(X_3\beta)'\}$$
$$+\{(Z_1\alpha)'(X_2\beta)'(Y_3\gamma)'+(X_1\beta)'(Y_2\gamma)'(Z_3\alpha)'\}=0.$$

The function multiplied by h^2+h, equated to zero, represents the right lines joining corresponding vertices of the two fixed triangles;

the remaining function, equated to zero, gives the locus for $h = 0$ or -1. Thus we have for different values of the anharmonic ratio h a sheaf of cubics passing through the same nine points on the three right lines joining corresponding vertices of the two triangles.

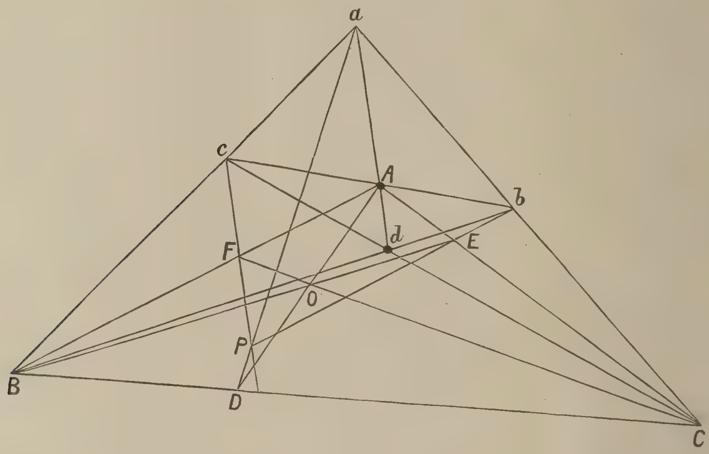


FIG. 2.

7. If, from the conditions (1), α, β, γ are eliminated, instead of x, y, z, we get the equation of another cubic locus.

In fact, the conditions may be written

$$\begin{aligned}
\alpha\,(yZ_1-zY_1)+\beta\,(zX_1-xZ_1h_1)+\gamma\,(xY_1h_1-yX_1) &= 0,\\
\alpha\,(yZ_2h_2-zY_2)+\beta\,(zX_2-xZ_2)+\gamma\,(xY_2-yX_2h_2) &= 0,\\
\alpha\,(yZ_3-zY_3h_3)+\beta\,(zX_3h_3-xZ_2)+\gamma\,(xY_3-yX_3) &= 0.
\end{aligned}$$

Now, if we divide the left-hand member of the first of these equations by h_1, it becomes

$$\alpha\,\frac{1}{h_1}\,(yZ_1-zY_1)+\beta\left(z\frac{X_1}{h_1}-xZ_1\right)+\gamma\left(xY_1-y\,\frac{X_1}{h_1}\right)=0,$$

practically the same form as the first of the set (1), (α, β, γ) and (x, y, z) being interchanged, and $\frac{1}{h_1}$, $\frac{X_1}{h_1}$ being written for h_1 and X_1.

Applying the same remark, *mutatis mutandis*, to the remaining equations, we can derive the locus when α, β, γ are eliminated from that obtained when x, y, z are eliminated.

The equation thus obtained is

$$\begin{aligned} &z^2x\left\{X_2Y_1Y_3(1-h_1h_3)+X_3Y_1Y_2(h_1h_2-h_3)+X_1Y_2Y_3(h_3-1)\right\} \\ -&y^2x\left\{X_3Z_1Z_2(1-h_1h_2)+X_2Z_1Z_3(h_1h_3-h_2)+X_1Z_2Z_3(h_2-1)\right\} \\ +&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ +&xyz\left\{\begin{array}{r}(X_3Y_1Z_2-X_2Y_3Z_1)(1-h_1h_2h_3)+(X_2Y_1Z_3-X_3Y_2Z_1)h_1 \\ +(X_1Y_3Z_2-X_2Y_1X_3)h_2+(X_3Y_2Z_1-X_1Y_3Z_2)h_3\end{array}\right\}=0. \quad (5)\end{aligned}$$

The coefficient *xyz* here is the same as that of $\alpha\beta\gamma$ in (2) with changed sign, showing that this term disappears from the two equations simultaneously, if at all.

The curve passes through the points given by

$$\frac{X_1}{h_1},\quad Y_1,\quad Z_1,$$

$$X_1,\quad \frac{Y_2}{h_2},\quad Z_2,$$

$$X_3,\quad Y_3,\quad \frac{Z_3}{h_3},$$

and through the vertices of the triangle of reference. Of course, it is the locus of the centres of homology of the triangle of reference and the variable triangle.

8. It appears from (2) and (5) that, if $h_1=h_2=h_3$, and the coefficients of $\alpha\beta\gamma$ or xyz vanish, the triangle $A''B''C''$ must be in perspective with the triangle ABC. By way of introducing this condition, write

$$k_1X,\ Y,\ Z,\quad \text{for}\quad X_1,\ Y_1,\ Z_1,$$

$$X,\ k_2Y,\ Z,\quad ,,\quad X_2,\ Y_2,\ Z_2,$$

$$X,\ Y,\ k_3Z,\quad ,,\quad X_3,\ Y_3,\ Z_3.$$

The first equation becomes

$$\begin{aligned} &\gamma^2aXY^2\left\{h_1h_2h_3-k_2h_1-h_2-k_1k_2h_3+k_1k_2+k_2\right\} \\ -&\beta^2aXZ^2\left\{h_1h_2h_3-k_3h_1-k_1k_3h_2-h_3+k_1k_3+k_3\right\} \\ +&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ +&a\beta\gamma XYZ\left\{(k_2-k_3)h_1+(k_3-k_1)h_2+(k_1-k_2)h_3\right\}=0; \qquad (6)\end{aligned}$$

the curve now passes through the point (X, Y, Z), a seventh point determined.

The second equation becomes

$$
\begin{aligned}
&y^2xZ^2X\,(1+h_1h_2k_3+k_1k_3h_2-k_3h_1-h_1h_2-k_1k_3)\\
&-z^2xY^2X\,(1+h_1h_3k_2+k_1k_2h_3-k_2h_3-h_1h_3-k_1k_2)\\
&+\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
&+xyzXYZ\,\{(k_2-k_3)\,h_1+(k_3-k_1)\,h_2+(k_1-k_2)\,h_3\}=0, \qquad (7)
\end{aligned}
$$

and may be derived from the previous form by writing $\frac{1}{h_1}$, $\frac{1}{h_2}$, $\frac{1}{h_3}$ for h_1, h_2, h_3, $\frac{k_1}{h_1}$, $\frac{k_2}{h_2}$, $\frac{k_3}{h_3}$ for k_1, k_2, k_3, and x, y, z, for α, β, γ, respectively, and *vice versa.*

The coefficients of $\alpha\beta\gamma$, xyz vanish for

$$h_1=h_2=h_3 \quad \text{or} \quad k_1=k_2=k_3.$$

If

$$h_1=h_2=h_3=0 \text{ or } -1,$$

or if

$$k_1=k_2=k_3=0 \text{ or } -1,$$

the tangents to either of the curves (6) and (7) at the vertices of the triangle of reference meet in one point, and the equations are reducible directly to the form

$$A\,\frac{\alpha}{m}\left(\frac{\beta^2}{n^2}-\frac{\gamma^2}{p^2}\right)+B\,\frac{\beta}{n}\left(\frac{\gamma^2}{p^2}-\frac{\alpha^2}{m^2}\right)+C\,\frac{\gamma}{p}\left(\frac{\alpha^2}{m^2}-\frac{\beta^2}{n^2}\right)=0,$$

which I formerly treated of (*Quarterly Journal of Mathematics*, Vol. v., pp. 54 *et seq.*). The equation has also been the subject of papers by Michael Roberts (*Q. J.*, Vol. v., p. 363), by Herr Eckardt ("Ueber eine gewisse Classe von Curven dritter Ordnung," *Schlömilch Z.*, XIII., pp. 263–6), and geometrically the same curves have been noticed by Emil Weyr ("Zur Erzeugung der Curven dritter Ordnung," Wien, *Sitzungsberichte der Kaizerl. Akad.*, LVIII., pp. 633–44).

It would not be convenient to restate here the results arrived at by the examination of the equation. Maclaurin gave the theorem that, if four points of contact of tangents to a cubic curve from a point thereon are given, the harmonic or diagonal triangle of the quadrangle has its vertices on the curve. The equation in question shows that the converse is true. The form is possible for a general curve having no double point, but it is not possible when there is one double point. Mr. Michael Roberts gave the discriminant in the form

$$\sqrt{-R} = \frac{3\omega(1-\omega)}{m^6n^6p^6}(A^2-B^2)(A^2-C^2)(B^2-C^2),$$

where ω is a cube root of unity.

The equality of two of the squares A^2, B^2, C^2 indicates the breaking up of the curve into a conic and right line; the other factor $\frac{1}{m^6n^6p^6}$, equated to zero, also breaks up the curve.

Thus, if, in (6), we make

$$k_1 = k_2 = k_3 = -1,$$

and

$$h_1h_2h_3+h_1-h_2-h_3 = h_1h_2h_3-h_1+h_2-h_3,$$

or

$$h_1 = h_2,$$

we get

$$(h_1^2h_3-h_3)\left\{\frac{\alpha}{X}\left(\frac{\beta^2}{Y^2}-\frac{\gamma^2}{Z^2}\right)+\frac{\beta}{Y}\left(\frac{\gamma^2}{Z^2}-\frac{\alpha^2}{X^2}\right)\right\}$$
$$-(h_1^2h_3-2h_1+h_3)\frac{\gamma}{Z}\left(\frac{\beta^2}{Y^2}-\frac{\alpha^2}{X^2}\right)=0,$$

and the curve consists of the line

$$\frac{\beta}{Y}-\frac{\alpha}{X}=0$$

and the conic

$$(h_1^2h_3-h_3)\left(\frac{\gamma^2}{Z^2}+\frac{\alpha\beta}{XY}\right)-(h_1^2h_3-2h_1+h_3)\frac{\gamma}{Z}\left(\frac{\beta}{Y}+\frac{\alpha}{X}\right)=0,$$

the line passing through two points of the quadrangle and one vertex of the harmonic triangle, the conic passing through the remaining two points of the quadrangle and the remaining vertex of the harmonic triangle.

9. A case will now be briefly treated in which the variable triangle degenerates into a triplet of collinear points. We shall suppose a linear transversal to meet the sides of the triangle of reference in three points (8).

The coordinates of these points may be taken to be $(0, \beta_1, \gamma_1)$, $(\alpha_2, 0, \beta_2)$, $(\alpha_3, \beta_3, 0)$, and they will be collinear if

$$\alpha_2\beta_3\gamma_1+\alpha_3\beta_1\gamma_2=0. \qquad (9)$$

As previously, let

$$(X_1,\ Y_1,\ Z_1),$$
$$(X_2,\ Y_2,\ Z_2),$$
$$(X_3,\ Y_3,\ Z_3)$$

be proportional to the coordinates of the vertices of a second fixed triangle. The linear segments determined by the points (8) will be in perspective with this triangle if the following conditions hold :—

$$\alpha(\beta_1 Z_1 - \gamma_1 Y_1) + \beta(\gamma_1 X_1) - \gamma\beta_1 X_1 = 0,$$
$$-\alpha\gamma_2 Z_2 + \beta(\gamma_2 X_2 - \alpha_2 Z_2) + \gamma\alpha_2 Y_2 = 0,$$
$$\alpha\beta_3 Z_3 - \beta\alpha_3 Z_3 + \gamma(\alpha_3 Y_3 - \beta_3 X_3) = 0.$$

By means of the condition (9) we can eliminate the coordinates of the points (8), and the result is

$$(\alpha Z_1)'(\beta X_2)'(\gamma Y_3)' - (\beta X_1)'(\gamma Y_2)'(\alpha Z_3)' = 0.$$

The expanded form is

$$\alpha^2\{(Y_1Z_3)'Y_2\gamma + (Z_1Y_2)'Z_3\beta\} + \beta^2\{(Z_2X_1)'Z_3\alpha + (Z_3X_2)'X_1\gamma\}$$
$$+\gamma^2\{(X_3Y_2)'X\beta + (X_1Y_2)'Y_2X\} + \alpha\beta\gamma\{Z_1X_2Y_3 + X_3Z_2Y_1 - 2X_1Y_2Z_3\}$$
$$= 0. \qquad (10)$$

If we place the two fixed triangles in perspective, by writing for $X_1 Y_1 Z_1$, &c.,

$$k_1 X,\quad Y,\quad Z,$$
$$X,\quad k_2 Y,\quad Z,$$
$$X,\quad Y,\quad k_3 Z,$$

the equation takes the form

$$\alpha^2\{Y^2Z(k_3-1)k_2\gamma + Z^2Y(k_2-1)k_3\beta\}$$
$$+\beta^2\{Z^2X(k_1-1)k_3\alpha + X^2Z(k_3-1)k_1\gamma\}$$
$$+\gamma^2\{X^2Y(k_2-1)k_1\beta + Y^2X(k_1-1)k_2\alpha\}$$
$$+2\alpha\beta\gamma XYZ\{1-k_1k_2k_3\} = 0;$$

when $k_1 = k_2 = k_3 = k$, this is

$$(\alpha Y + \beta X)(\beta Z + \gamma Y)(\gamma X + \alpha Z) - 2\alpha\beta\gamma XYZ\frac{(k+1)^2}{k} = 0,$$

and the curve is transformed into itself by writing

$$\frac{X}{\alpha},\ \frac{Y}{\beta},\ \frac{Z}{\beta}\quad\text{for}\quad\frac{\alpha}{X},\ \frac{\beta}{Y},\ \frac{\gamma}{Z}.$$

10. We have seen that right lines through the point α, β, γ, and the respective vertices of the triangle of reference meet the corresponding sides of the second fixed triangle in the points whose coordinates are proportional to

$$\left(-\frac{B_1\beta+C_1}{A_1},\quad \beta,\quad \gamma\right),$$

$$\left(\alpha,\quad -\frac{A_2\alpha+C_2\gamma}{B_2},\quad \gamma\right),$$

$$\left(\alpha,\quad \beta,\quad -\frac{A_3\alpha+B_3\beta}{C_3}\right),$$

and, if these are collinear, we get the locus

$$\alpha^2\left\{(A_2B_1)'A_3\beta+(A_3C_1)'A_2\gamma\right\}+\beta^2\left\{(B_1A_2)'B_3\alpha+(B_3C_2)'B_1\gamma\right\}$$
$$+\gamma^2\left\{(C_1A_3)'C_2\alpha+(B_3C_2)'C_1\beta\right\}+\alpha\beta\gamma\left\{B_1A_3C_2+C_1A_2B_3-2A_1B_2C_3\right\}=0.$$

The coefficient of $\alpha\beta\gamma$ is

$$A_3(B_1C_2)'+B_2(C_1A_3)'+A_2(C_1B_3)'+C_3(B_1A_2)'.$$

Now, if we replace A_1 by $(Y_2Z_3)'$, B_1 by $(Z_2X_3)'$, C_1 by $(X_2Y_3)'$, A_2 by $(Y_3Z_1)'$, B_2 by $(Z_3X_1)'$, C_2 by $(X_3Y_1)'$, A_3 by $(Y_1Z_2)'$, B_3 by $(Z_1X_2)'$, C_3 by $(X_1Y_2)'$, the above equation reduces to the same form as (10).

This result depends on the following theorem :—If three concurrent lines pass severally through a different vertex of a fixed triangle, and also through the points where a transversal meets the sides of a second fixed triangle, a triplet of right lines through the same point of concurrence passing severally through a different vertex of the second triangle will determine three collinear points on the corresponding sides of the first triangle. The geometrical proof of this is easy. For, following the notation of Fig. 3, we have

$$\frac{AY}{YC}=\frac{\sin APY}{\sin YPC}\,\frac{PA}{PC},\quad \frac{CX}{XB}=\frac{\sin CPX}{\sin XPB}\,\frac{PC}{PB},$$

$$\frac{BZ}{ZA}=\frac{\sin BPZ}{\sin ZPA}\,\frac{PB}{PA},$$

and, according to the theorem of Menelaus, we assume

$$\frac{AY\,.\,CX\,.\,BZ}{YC\,.\,XB\,.\,ZA}=-1.$$

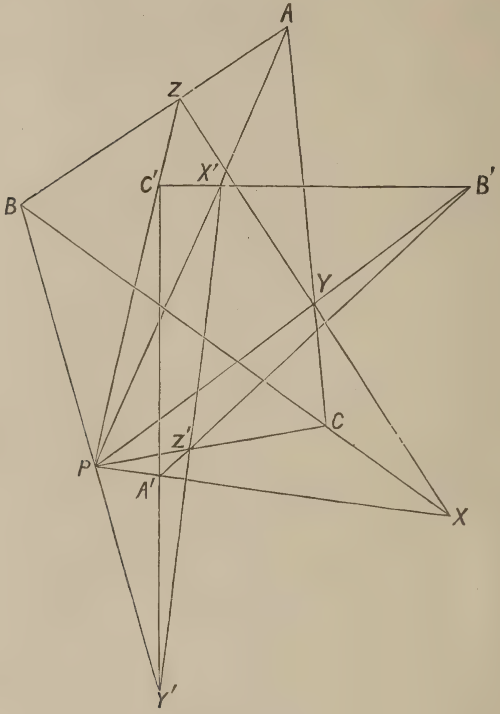


FIG. 3.

Again, $\dfrac{A'Y'}{Y'C'} = \dfrac{\sin XPB}{\sin BPZ} \dfrac{PA'}{PC'}, \quad \dfrac{C'X'}{X'B'} = \dfrac{\sin ZPA}{\sin APY} \dfrac{PC'}{PB'},$

$$\frac{B'Z'}{Z'A'} = \frac{\sin YPC}{\sin CPX} \frac{PB'}{PA'};$$

and therefore $$\frac{A'Y'\,.\,C'X'\,.\,B'Z'}{Y'C'\,.\,X'B'\,.\,Z'A'} = -1.$$

11. Another case is noteworthy, in which a variable triplet of collinear points occurs.

The equations

$$\beta y - \gamma z = \gamma x - \alpha z = \alpha y - \beta x = 0, \tag{11}$$

represent three concurrent lines passing respectively through the vertices of the triangle of reference *ABC*.

Let $$A\alpha+B\beta+C\gamma=0$$

be the equation of any straight line. Its intersections with the lines (11) have coordinates proportional to

$$-\frac{By+Cz}{A},\quad y,\quad z, \tag{1}$$

$$x,\quad -\frac{Ax+Cz}{B},\quad z, \tag{2}$$

$$x,\quad y,\quad -\frac{Ax+By}{C}; \tag{3}$$

take, as before, three points whose coordinates are

$$(X_1,\ Y_1,\ Z_1), \tag{4}$$

$$(X_2,\ Y_2,\ Z_2), \tag{5}$$

$$(X_3,\ Y_3,\ Z_3). \tag{6}$$

Right lines through (1) and (4), (2) and (5), (3) and (6), respectively, are given by

$$\left.\begin{aligned} \alpha\,(Y_1z-Z_1y)-\beta\left(Z_1\frac{By+Cz}{A}+X_1z\right)+\gamma\left(Y_1\frac{By+Cz}{A}+X_1y\right)&=0,\\ \alpha\left(Z_2\frac{Ax+Cz}{B}+Y_2z\right)+\beta\,(Z_2x-X_2z)-\gamma\left(X_2\frac{Ax+Cz}{B}+Y_2x\right)&=0,\\ -\alpha\left(Y_3\frac{Ax+By}{C}+Z_3y\right)+\beta\left(X_3\frac{Ax+By}{C}+Z_3x\right)+\gamma(X_3y-Y_3x)&=0. \end{aligned}\right\} \tag{12}$$

Write P, Q, R for $AX_1+BY_1+CZ_1$, $AX_2+BY_2+CZ_2$, $AX_3+BY_3+CZ_3$, and the above system becomes

$$\alpha A+\beta\left(B-\frac{Px}{(Y_1z)'}\right)+\gamma\left(C+\frac{Py}{(Y_1z)'}\right)=0,$$

$$\alpha\left(A+\frac{Qz}{(Z_2\alpha)'}\right)+\beta B+\gamma\left(C-\frac{Qx}{(Z_2\alpha)'}\right)=0,$$

$$\alpha\left(A-\frac{Ry}{(X_3y)'}\right)+\beta\left(B-\frac{Rx}{(X_3y)'}\right)+\gamma C=0.$$

Eliminating α, β, γ, and equating to zero, we get

$$\begin{vmatrix} A, & B-\dfrac{Pz}{(Y_1z)'}, & C+\dfrac{Py}{(Y_1z)'} \\ \dfrac{Qz}{(Z_2x)'}, & \dfrac{Pz}{(Y_1z)'}, & -\dfrac{Qx}{(Z_2x)'}-\dfrac{Py}{(Y_1z)'} \\ \dfrac{Qz}{(Z_2x)'}+\dfrac{Ry}{(X_3y)'}, & -\dfrac{Rx}{(X_3y)'}, & -\dfrac{Qx}{(Z_2x)'} \end{vmatrix} = 0.$$

Expanding and reducing this result, we find without difficulty

$$(Ax+By+Cz)\{PQz\,(X_3y)'+QRx\,(Y_1z)'+PRy\,(Z_3x)'\} = 0,$$

so that the locus consists of the given line and a conic.

12. On the other hand, we may eliminate x, y, z instead of α, β, γ. Write L for $A\alpha+B\beta+C\gamma$, P, Q, R retaining their meaning. The first equation of the system (12) is

$$z\left(Y_1\alpha-\beta Z_1\frac{C}{A}-\beta X_1+\gamma Y_1\frac{C}{A}\right)$$
$$+y\left(\gamma X_1+\gamma Y_1\frac{B}{A}-\beta Z_1\frac{B}{A}-Z_1x\right) = 0,$$

or $$z\,(Y_1L-P\beta)+y\,(P\gamma-Z_1L) = 0,$$

and in like manner the other two equations are

$$x\,(Z_2L-Q\gamma)+z\,(Q\alpha-X_2L) = 0,$$
$$y\,(X_3L-R\alpha)+x\,(R\beta-Y_3L) = 0.$$

Hence, finally,

$$(Y_1L-\beta P)(Z_2L-\gamma Q)(X_3L+R\beta)$$
$$-(Z_1L-P\gamma)(X_2L-Q\alpha)(Y_3L-R\beta) = 0,$$

or $$L^3\,(X_3Z_2Y_1-X_2Y_3Z_1)$$
$$+L^2\,\{R\,(Z_1X_2\beta-Y_1Z_2\alpha)+Q\,(Z_1Y_3\alpha-Y_1X_3\gamma)\}+P\,(X_2Y_3\gamma-Z_2X_3\beta)$$
$$+L\,\{QR\alpha\,(Y_1\gamma)'-PR\,(Z_2\alpha)'\,\beta+PQ\,(X_3\beta)'\,\gamma\} = L\,.\,\psi = 0,$$

when $\psi = 0$ gives a conic having the same chord as the former one cut off from the line $L = 0$.

13. The foregoing results give the following construction. Take two conics (I.) and (II.) having common intersections P and Q. From A, B, C, three points on (I.), draw three straight lines AO, BO,

CO, meeting in O, a point on the same conic, and cutting the chord PQ (produced if necessary) in a, b, c. Through a, b, c draw three straight lines $A'O'$, $B'O'$, $C'O'$ meeting the conic (II.) in O', and again in A', B', C' respectively. Then, if through A, B, C we draw straight lines AO'', BO'', CO'' meeting the conic (I.) in O'', and the line PQ in a', b', c', the straight lines $a'A'$, $b'B'$, $c'C'$ will intersect in O''', a point on (II.).

But, since two lines determine their own intersection, it is not necessary to pay attention to more than two lines of the variable triplets through A, B, C. Therefore, if we take any conic (I.), and two points P, Q upon it, and are given besides three points A', B', C', which determine with P, Q a second conic (II.), this conic can be constructed by the process indicated.

For suppose we draw straight lines $A'C'$, $B'C'$, meeting the straight line through P and Q in a, b, and through a point O on (I.) we draw aO, bO, meeting the same conic again in A, B; we can then proceed to determine points on (II.) by drawing AO', BO' through O', a point on (I.), and cutting PQ in $a'b'$. For straight lines $a'A'$, $b'B'$ will meet in a point O'' on (II.).

Now it will be remarked that we do not make explicit use of the points P, Q, and this is important, for we have a method of describing a conic through three given real points and two imaginary points given as the intersections of a circle with a given real external straight line.

We may take a circle for conic (I.) and for the line PQ a real line external to the circle. The construction is easily verified by taking two circles and their radical axis. From a geometrical point of view, the straight lines through A, B, and a variable point O of the conic (I.) determine two corresponding homographic ranges on the line PQ. The double points of these are points on the curve. Having therefore determined the points A, B by means of A', B', C' on the conic (II.), this can be constructed through the points A', B', C', P, Q.

Also a pair of real straight lines may be taken for the conic (I.). The five points through which the constructed conic passes are then manifestly real.

14. The form of the equation

$$(Z_1\alpha)'(X_2\beta)'(Y_3\gamma)'-(Y_1\alpha-X_1\beta)(X_3\gamma-Z_3\alpha)(Z_3\beta-Y_2\gamma)=0$$

shows that the product of three perpendiculars from a point P on the curve to the lines BA', CB', AC' is in a constant ratio to the product of three perpendiculars to the lines CA', BC, AB', and this is easily proved geometrically.

Write Pa_2 for the perpendicular from P on BC', and so forth. Then (Fig. 4)

$$\frac{Pa_2}{Pb_1} = \frac{\sin BC'Z}{\sin ZC'A}, \quad \frac{Pc_1}{Pa_3} = \frac{\sin AB'Y}{\sin YB'C}, \quad \frac{Pb_3}{Pc_2} = \frac{\sin CA'X}{\sin XA'B},$$

$$\frac{AY}{YC} = \frac{\sin AB'Y}{\sin YB'C}\,\frac{AB'}{CB'}, \quad \frac{CX}{XB} = \frac{\sin CA'X}{\sin XA'C}\,\frac{CA'}{BA'}, \quad \frac{BZ}{ZA} = \frac{\sin BC'Z}{\sin ZC'A}\,\frac{BC'}{AC'}.$$

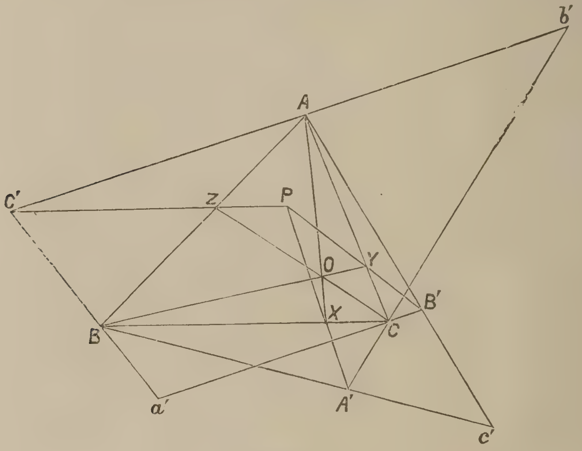


FIG. 4.

and, since AX, BY, CZ are concurrent,

$$\frac{Pa_2 \,.\, Pc_1 \,.\, Pb_3}{Pb_1 \,.\, Pa_3 \,.\, Pc_2} = \frac{AC' \,.\, BA' \,.\, CB'}{AB' \,.\, CA' \,.\, BC'}.$$

In like manner, when the triangle of reference is cut by a linear transversal, the equation

$$(Z_1\alpha)'(X_2\beta)'(Y_3\gamma)' + (\gamma_1\alpha)'(X_3\gamma)'(Z_2\beta)' = 0$$

leads us, following an analogous notation, to the form

$$\frac{Pa_2 \,.\, Pc_1 \,.\, Pb_3}{Pb_1 \,.\, Pa_3 \,.\, Pc_2} = -\frac{AC' \,.\, BA' \,.\, CB'}{AB' \,.\, CA' \,.\, BC'}.$$

The results may be included in one form by postulating

$$\frac{BX \,.\, CY \,.\, AZ}{XC \,.\, YA \,.\, ZB} = \mu,$$

and the condition connecting the coordinates of X, Y, Z becomes

$$\gamma_1\alpha_2\beta_3 - \mu\beta_1\gamma_2\alpha_3 = 0.$$

Determination of certain Primes. *By* F. W. LAWRENCE.

Received and Read May 13th, 1897.

The method of "factorisation" is that of my paper in the *Quarterly Journal of Mathematics* (No. CXI., 1896), except that, "strips" being not required, all the work is shown directly.

The primes found are:

77,843,839,397,	a factor of	$10^{29}-1$,
182,521,213,001,	,, ,,	$10^{25}-1$,
1,001,523,179,	,, ,,	$3^{23}-1$,
12,014,633,		
13,477,969.		

In order to avoid continual reference, I quote here the notation used in my paper, and certain propositions proved there, which will be used in the investigation.

Notation.—N is the number under consideration,

$n = \lambda N$ is the multiple of N we are attacking,

a is the $\frac{1}{2}$ sum of two complementary factors of n,

b ,, $\frac{1}{2}$ difference.

PROP. I.—If every factor of n is congruent to 1 (mod P), where P is any odd integer, then

$$a \equiv 1 \pmod{P}$$

and
$$a^2 \equiv n \pmod{P^2}. \qquad \text{(by § 9)}$$

Thus a is limited to 1 case in P^2 (not only to 1 case in P, as stated by misprint in my paper).

PROP. II.—If p be any odd prime, then $a \pmod{p}$ is limited to $\frac{1}{2}(p+1)$ or $\frac{1}{2}(p-1)$ cases, according as n is or is not a quadratic residue of p. (§ 3)

(The table on page 467 of this notice enables these cases to be written out for small values of p.)

Prop. III.—

$$\left.\begin{array}{ll} \text{i. If } n \equiv 5 \pmod 8\text{, then } a^2 \equiv n+4 \pmod{32}. & \\ \text{ii. If } n \equiv 1 \pmod 8\text{, then either } a^2 \equiv n+16 \pmod{128} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{or } a^2 \equiv n \pmod{64}. & \end{array}\right\} \quad (\S\,6)$$

Prop. IV.—If 2 is a quadratic residue of n, so that all factors of n are either ± 1 (mod 8), then, if $n \equiv 9 \pmod{16}$, we have

$$a^2 \equiv n+16 \pmod{128}. \qquad (\S\,13)$$

Prop. V.—i. If $n \equiv 2 \pmod 3$, then $a \equiv 0 \pmod 3$. (§ 2)

ii. If $n \equiv 1 \pmod 3$, then $a^2 \equiv n \pmod 9$. (§ 7)

Prop. VI.—

$$\left.\begin{array}{ll} \text{i. If } n \equiv 2 \pmod 5\text{, then} & a \equiv \pm 1 \pmod 5. \\ \text{ii. If } n \equiv -2 \pmod 5\text{, then} & a \equiv \pm 2 \pmod 5. \\ \text{iii. If } n \equiv \pm 1 \pmod 5\text{, then either} & a \equiv 0 \pmod 5 \\ & \text{or } a^2 \equiv n \pmod{25}. \end{array}\right\} \quad (\S\,8)$$

Prop. VII.—If no integer from $\sqrt{n}$ up to $\sqrt{n}+t$ is possible as a value of a, there can be no factor of n between

$$\sqrt{n}+t \pm \sqrt{2t\sqrt{n}+t^2}.$$

(This is proved in a paper by me in *Messenger of Mathematics*, Nov., 1894.)

Prop. VIII.—Attacking N directly (*i.e.*, $\lambda = 1$, $n = N$), in order to prove that N has no factor from $\sqrt{N}$ down to $\dfrac{\sqrt{N}}{k}$, it is only necessary to show that there is no possible value for a from $\sqrt{N}$ up to $\sqrt{N}\,(1+\Theta)$, where

$$\Theta = \tfrac{1}{2}k\left(1-\frac{1}{k}\right)^2. \qquad (\S\,16)$$

Prop. IX.—Attacking λN, in order to show that N has no factor from $\sqrt{N}$ down to $\dfrac{\sqrt{N}}{\lambda}$, it is only necessary to show that there is no possible value for a from $\sqrt{\lambda N}$ up to $\sqrt{\lambda N}+\Theta\sqrt{N}$, where

$$\Theta = \tfrac{1}{2}\lambda\left(1-\frac{1}{\sqrt{\lambda}}\right)^2. \qquad (\S\,17)$$

The following table enables the possible values of a (mod p), where $p = 5, 7, 11, 13, 17$, or 19, to be written down at once for any value of n:—

Table to find a (mod p).

p	Possible values of a (mod p) satisfying $a^2 - b^2 \equiv 1$ (mod p).	δ	Possible values of a (mod p) satisfying $a^2 - b^2 \equiv \delta$ (mod p).
5	0 or ± 1	2	± 1
7	± 1 or ± 3	-1	0 or ± 1
11	± 1, ± 2, or ± 4	-1	0, ± 2, or ± 5
13	0, ± 1, ± 2, or ± 6	2	± 1, ± 4, or ± 5
17	0, ± 1, ± 3, ± 4, or ± 6	3	± 1, ± 2, ± 4, or ± 6
19	± 1, ± 5, ± 6, ± 8, or ± 9	-1	0, ± 2, ± 4, ± 5, or ± 9

δ is the numerically least non-residue of p.

Now we must be able to write either

$$n \equiv x^2 \text{ (mod } p) \quad \text{or} \quad n \equiv y^2 \delta \text{ (mod } p),$$

according as n is or is not a quadratic residue of p.

And the required values will therefore be found in the first case by multiplying by x those tabulated in the second column, and in the second case by multiplying by y those tabulated in the fourth column.

For, if a_1 satisfies $\quad a_1^2 - b_1^2 \equiv 1 \text{ (mod } p),$

then $a_1 x$ satisfies $\quad (a_1 x)^2 - (b_1 x)^2 \equiv x^2 \text{ (mod } p);$

and, if a_2 satisfies $\quad a_2^2 - b_2^2 \equiv \delta \text{ (mod } p),$

then $a_2 y$ satisfies $\quad (a_2 y)^2 - (b_2 y)^2 \equiv \delta y^2 \text{ (mod } p).$

1. $N = 77{,}843{,}839{,}397$.

This number is a residuary factor of $10^{29} - 1$.

Accordingly, it is known that every factor of N is of the form $29m + 1$.[*] And the tables for residue, index 10, show that N has no factor below 100,000.

---

[*] The algebraic factor of $10^{29} - 1$, viz. $(10 - 1)$, and others having been already removed, we also have 10 a quadratic residue; but no advantage can be taken of this here.

Now $$280{,}000 > \sqrt{N} > 270{,}000;$$

therefore we need only search for factors from $\sqrt{N}$ down to $\frac{\sqrt{N}}{2{\cdot}8}$. And we can do this by attacking N direct and testing for values of "a" from $\sqrt{N}$ up to $\sqrt{N}(1+\Theta)$,

where $$\Theta = \frac{2{\cdot}8}{2}\left(1-\frac{1}{2{\cdot}8}\right)^2 < {\cdot}58, \qquad \text{(Prop. VIII.)}$$

so that we need only test for a from 270,000 to 443,000.

Now $N \equiv 1+19\times 29 \pmod{29^2} \equiv 5 \pmod{32} \equiv 2 \pmod 3$;

therefore $a^2 \equiv 1+19\times 29 \pmod{29^2} \equiv 9 \pmod{32}$. (Props. I. and III. i.)

And also $a \equiv 1 \pmod{29}$ and $\equiv 0 \pmod 3$; (Props. I. and V. i.)

therefore $a \equiv 1+24\times 29 \pmod{29^2} \equiv \pm 3 \pmod{16} \equiv 0 \pmod 3$.

Or, combining, $a \equiv 697+20\times 29^2$ or $697+26\times 29^2 \pmod{48\times 29^2}$,

i.e., $$a \equiv 17517 \text{ or } 22563 \pmod{40368}.$$

And we may write $a = 40368x_1+17517$,

or $ = 40368x_2+22563$,

where each value of x must be integral, and the limits assigned above to a give $6 < x < 11$.

	mod 5	mod 7	mod 11
Now we have $N \equiv$	2	2	5
therefore we must have $a \equiv$	± 1	± 2 or ± 3	± 3, ± 4, or ± 5
by the table on page 467.			
Moreover $40368 \equiv$	3	6	9
$17517 \equiv$	2	3	5
$22563 \equiv$	3	2	2

Therefore the only values left between the limits	for x_1	and for x_2
after testing for 5 are	8 or 9	7
" " 7 "	8	7
" " 11 "	none	none.

Therefore there is no possible value for a,

and $N = 77{,}843{,}839{,}397$ *is prime.*

2. $N = 182{,}521{,}213{,}001$.

This number is a residuary factor of $10^{25}-1$.

Accordingly, every factor of N is of the form $25m+1$; also 10 is a quadratic residue of every factor; hence here 2 is a quadratic residue, and all factors must be of the form $8m\pm 1$. Moreover, N is known from the tables for residue, index 10, to have no factor $< 100{,}000$.

Now $$430{,}000 > \sqrt{N} > 420{,}000;$$

therefore we need only search for factors of N from $\sqrt{N}$ down to $\dfrac{\sqrt{N}}{4{\cdot}3}$.

As before, we attack N direct, searching for values of "a" from $\sqrt{N}$ up to $\sqrt{N}(1+\Theta)$, where

$$\Theta = \frac{4{\cdot}3}{2}\left(1-\frac{1}{4{\cdot}3}\right)^2 < 1{\cdot}3; \qquad \text{(Prop. VIII.)}$$

therefore we need only test for "a" from 420,000 up to 990,000.

Now $$N \equiv 1+20\times 25 \pmod{25^2};$$

therefore $$a \equiv 1+10\times 25 \pmod{25^2}. \qquad \text{(Prop. I.)}$$

Moreover $$N \equiv 9 \pmod{16};$$

therefore $$a^2 \equiv N+16 \pmod{128}; \qquad \text{(Prop. IV.)}$$

therefore, since $$N \equiv 73 \pmod{128},$$

$$a^2 \equiv 89 \pmod{128} \quad \text{and} \quad a \equiv \pm 27 \pmod{64}.$$

And $$N \equiv 2 \pmod 3;$$

therefore $$a \equiv 0 \pmod 3). \qquad \text{(Prop. v. i.)}$$

Combining,

$$a \equiv 251+10\times 25^2 \text{ or } 251-32\times 25^2 \pmod{3\times 64\times 25^2},$$

i.e., $$a \equiv 6501, \text{ or } -19749 \pmod{120{,}000}.$$

And we may write $$a = 120{,}000x_1+6501,$$

or $$= 120{,}000x_2-19749,$$

where x_1 and x_2 are integers, and for either we have

$$3 < x < 9$$

(from a consideration of the limits of "a" given above).

	mod 7	mod 11	mod 13	mod 17
Now $N \equiv$	1	6	10	4
therefore we must have $a \equiv$	± 1, or ± 3	$0, \pm 2$ or ± 3	$0, \pm 1, \pm 3$ or ± 6	$0, \pm 2, \pm 5, \pm 6, \pm 8$
from tables, p. 467.				
Further $120{,}000 \equiv$	-1	1	10	-3
$6501 \equiv$	5	0	1	7
$-19749 \equiv$	-2	-4	-2	5

Therefore the only possible values left	for x_1	and	for x_2
after testing for 7 are	4, 6, or 8		4, 6, or 8
" " 11 "	8		4 or 6
" " 13 "	8		4 or 6
" " 17 "	8		none.

On trial $(8 \times 120{,}000 + 6501)^2 - N = 751{,}602{,}970{,}000$,

and this evidently cannot be a perfect square; therefore no value of a satisfies, and

$$N = 182{,}521{,}213{,}001 \text{ \textit{is prime.}}$$

3. $N = 1{,}001{,}523{,}179$.

N is a residuary factor of $3^{23}-1$. In fact,

$$3^{23} - 1 = 94N.$$

Every possible factor of N is congruent to 1 (mod 23), and also, since 3 is a quadratic residue, is of the form ± 1 (mod 12).

Moreover, since we have to search all the way from $\sqrt{N}$ downwards for factors of N, it will be well to attack a multiple of N.

Selecting $47N \left(i.e., \dfrac{3^{23}-1}{2}\right)$ as our value of "n," we shall be able to search for all factors of N from $\sqrt{N}$ to $\dfrac{\sqrt{N}}{47}$ by searching for values of a from $\sqrt{47N}$ up to $\sqrt{47N} + \Theta\sqrt{N}$, where

$$\Theta = \frac{47}{2}\left(1 - \frac{1}{\sqrt{47}}\right)^2 < 17{\cdot}2; \qquad \text{(Prop. IX.)}$$

therefore we have to search for a from 210,000 up to 770,000.

Moreover, since $47 \equiv 1 \pmod{23} \equiv -1 \pmod{12}$, all factors of n are of the forms $23m+1$, $12m' \pm 1$.

And a pair of complementary factors of n [which $\equiv 13 \pmod{24}$] must evidently either be $24m''+1$ and $24m'''+13$, or else $24m''-1$ and $24m'''-13$.

So that we have $$a = \pm 5 \pmod{12}. \qquad (\alpha)$$

Now

$$\left.\begin{aligned} n &\equiv 1+14\times 23 \pmod{23^2} \equiv 5 \pmod{32} \equiv 4 \pmod{9}; \\ &\text{therefore} \\ a^2 &\equiv 1+14\times 23 \pmod{23^2} \equiv 9 \pmod{32} \equiv 4 \pmod{9}; \\ &\text{therefore} \\ a &\equiv 1+\ \ 7\times 23 \pmod{23^2} \equiv \pm 3 \pmod{16} \equiv \pm 2 \pmod{9}. \end{aligned}\right\}$$

(Props. I., III., V.)

Combining the last two results, we obtain

$$a \equiv 29,\ 61,\ 83, \text{ or } 115 \pmod{144}.$$

But, since, by (α), $a \equiv \pm 5 \pmod{12}$,

61 and 83 are rendered impossible. Thus, finally,

$$a \equiv 1+7\times 23+23^2 \quad \text{or} \quad 1+7\times 23+107\times 23^2 \pmod{144\times 23^2},$$

i.e., $a \equiv 691$ or $56765 \pmod{76176}$.

And we may write

$$a = 76176x_1+691 \quad \text{or} \quad = 76176x_2+56765,$$

where, owing to the limits for a, $2<x_1<11$, $2<x_2<10$.

	mod 5	mod 7	mod 11	mod 13	mod 17
Now $n \equiv$	3	2	2	4	5
therefore by tables, p. 467, we must have $a \equiv$	± 2	± 2, or ± 3	$0, \pm 4$ or ± 5	$0, \pm 1, \pm 2$ or ± 4	$\pm 1, \pm 2, \pm 3$ or ± 8
Further $76176 \equiv$	1	2	1	9	-1
$691 \equiv$	1	5	9	2	11
$56765 \equiv$	0	2	5	7	2

therefore the only possible values left	for x_1	for x_2
after testing for 5 are	6 or 7	3, 7, or 8
,, ,, 7 ,,	6 or 7	7 or 8
,, ,, 11 ,,	6 or 7	none
,, ,, 13 ,,	6 or 7	
,, ,, 17 ,,	none.	

Thus there are no factors of N lying between $\sqrt{N}$ and $\frac{\sqrt{N}}{47}$, and it only remains for us to try factors below $\frac{\sqrt{N}}{47}$, *i.e.*, below 700. Only three primes below 700 satisfy the required conditions; these are 47, 277, 599. And none of these divide N.

Therefore $N = 1{,}001{,}523{,}179$ *is prime.*

4. $N = 12{,}014{,}633$. (See bottom of p. 474.)

Here it is known that any factor must be of the form $7m+1$.

Now $N \equiv 1 \pmod{8} \equiv 2 \pmod{3}$.

Therefore, in order to obtain n of the form 5 (mod 8) and 1 (mod 3), we attack $29N$, and, in order to search for factors from $\sqrt{N}$ down to $\frac{\sqrt{N}}{29}$, we search for values of a from $\sqrt{29N}$ up to

$$\sqrt{29N} + \frac{29}{2}\left(1 - \frac{1}{\sqrt{29}}\right)^2 \sqrt{N}.$$

This will be done if we search from 18,500 up to 54,000.

$$\left.\begin{aligned} \text{Now } 29N &\equiv 5 \pmod{32} \equiv 4 \pmod{9} \equiv 1+7 \pmod{49}; \\ \text{therefore } a^2 &\equiv 9 \pmod{32} \equiv 4 \pmod{9} \equiv 1+7 \pmod{49}; \\ \text{therefore } a &\equiv \pm 3 \pmod{16} \equiv \pm 2 \pmod{9} \equiv 1 + 4 \times 7 \pmod{49}. \end{aligned}\right\}$$

(Props. I., III., V.)

Combining, $a \equiv 29, 61, 83,$ or $115 \pmod{144}$, and $\equiv 29 \pmod{49}$;

therefore $a \equiv 29,\ 29+54\times 49,\ 29+80\times 49,$ or $29+134\times 49$ $\pmod{49\times 144}$,

i.e., $a \equiv 29,\ 2675,\ 3949,$ or $6595 \pmod{7056}$.

And we may put

$$a = 7056x_1+29 \quad \text{or} \quad 7056x_2+2675,$$

or

$$7056x_3+3949 \quad \text{or} \quad 7056x_4+6595.$$

And, from the limits of a, $2<x<8$ for x_1, x_2, x_3,

$$1<x<7 \quad ,, \quad x_4.$$

		mod 5	mod 11	mod 13
Now	$29N \equiv$	2	6	6
therefore we must have	$a \equiv$	± 1	$0, \pm 2$, or ± 3	$\pm 3, \pm 4$, or ± 6
Also	$7056 \equiv$	1	5	10
	$29 \equiv$	4	7	3
	$2675 \equiv$	0	2	10
	$3949 \equiv$	4	0	10
	$6595 \equiv$	0	6	4

Thus there are left possible values of	x_1	x_2	x_3	x_4
after testing for 5	5, 7	4, 6	5, 7	4, 6
,, ,, 11	7	4	5, 7	6
,, ,, 13	none	none	none	none.

Thus there is no factor between $\sqrt{N}$ and $\dfrac{\sqrt{N}}{29}$, and the only primes below this limit satisfying the required conditions are 29, 43, 71, 113.

And none of these divide N; therefore

$$N = 12{,}014{,}633 \textit{ is prime.}$$

5. $N = 13{,}477{,}969$. (See bottom of p. 474.)

All factors are of the form $7m+1$.

We have $N \equiv 1 \pmod{8} \equiv 1 \pmod{3}$,

and we might multiply by 29 or 43, and proceed somewhat as before, or proceed directly with N as shown below.

Thus $N \equiv 1+80 \pmod{128} \equiv 1 \pmod{9} \equiv 1+4\times 7 \pmod{7^2}$.

And either $a^2 \equiv 81+16 \pmod{128}$ or $\equiv 81 \pmod{64}$;

(Prop. III.)

also $a^2 \equiv 1 \pmod{9} \equiv 1+4\times 7 \pmod{7^2}$.

Thus either $a \equiv \pm 49 \pmod{64}$ or $\equiv \pm 9 \pmod{32}$,

and $a \equiv \pm 1 \pmod{9} \equiv 1 \pm 2\times 7 \pmod{7^2}$.

Combining, either

$$a \equiv 64 \pm 49 \pmod{64\times 49} \quad \text{or} \quad a \equiv 407 \text{ or } 1289 \pmod{32\times 49},$$

and $a \equiv \pm 1 \pmod{9}$.

Finally, either

$$a \equiv 3151,\ 9521,\ 15695,\ \text{or } 25201 \pmod{28224},$$

or $a \equiv 5111,\ 5993,\ 6679,\ \text{or } 7561 \pmod{14112}$.

And, in order to search for factors from $\sqrt{N}$ down to $\frac{\sqrt{N}}{16}$, we must have "a" running from $\sqrt{N}$ up to $\sqrt{N}\left[1+\frac{16}{2}(1-\frac{1}{16})^2\right]$, (Prop. VIII.) and to do this we search from 3,600 up to 30,000.

Now $N \equiv 19 \pmod{25}$;

therefore either $a \equiv 0 \pmod{5}$ or $a^2 \equiv 19 \pmod{25}$, (Prop. VI.)

i.e., " $a \equiv 0 \pmod{5}$ or $a \equiv \pm 12 \pmod{25}$,

and the only values between the limits which are possible are

$$15695 \quad \text{or} \quad 20105 \qquad (\textit{i.e.}, 5993+14112).$$

Now, mod 11 :—$N \equiv 10$; therefore a must be 0, ± 2, or ± 5;

$$15695 \equiv -2 \text{ and satisfies;}$$
$$20105 \equiv -3 \text{ and fails.}$$

And mod 13 :—$N \equiv -2$; therefore a must be ± 1, ± 5, or ± 6;

$$15695 \equiv 4 \text{ and fails.}$$

Therefore there is no factor of N between $\sqrt{N}$ and $\frac{\sqrt{N}}{16}$.

Below this limit the only possible primes are 29, 43, 71, 113, 127, 197, 211.

None of these divide N. Therefore

$N = 13{,}477{,}969$ *is prime.*

[*Note.*—12,014,633 and 13,477,969 are norms of 7^{ic} trinomial integers

$$N = X^2+7Y^2 = N_7 \quad (a+2b\rho+b\rho^5),$$

$N_7 f(\rho)$ denoting $\qquad f(\rho)\,.\,f(\rho^2)\ldots f(\rho^6)$,

ρ a true 7th root of 1.

Thus $\quad N_7(1+16\rho+8\rho^5) = 3221^2+7\,.\,484^2 = 12{,}014{,}633$,

$\qquad\quad N_7(3+16\rho+8\rho^5) = 3471^2+7\,.\,452^2 = 13{,}477{,}969$.

They were sent by Col. Cunningham to Mr. Bickmore, who sent them to me.]

An Extension of the Theorem $\dfrac{\Pi(\gamma-\alpha-\beta-1)\,\Pi(\gamma-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} = F_1(\alpha, \beta, \gamma)$.

By the Rev. F. H. Jackson, M.A. Received March 27th, 1897. Read April 8th, 1897.

1.

The chief object of this paper is to investigate the following theorem :—

$$\underset{\kappa=\infty}{\mathrm{L}}\ \frac{(p^{\gamma-\beta}-1)\ldots(p^{\gamma-\beta+\kappa-1}-1)}{(p^{\gamma-\beta-\alpha}-1)\ldots(p^{\gamma-\beta-\alpha+\kappa-1}-1)}\ \frac{(p^{\gamma-\alpha}-1)\ldots(p^{\gamma-\alpha+\kappa-1}-1)}{(p^{\gamma}-1)\ldots(p^{\gamma+\kappa-1}-1)}\,p^{\alpha\beta}$$

$$= 1+p^{\gamma-\alpha-\beta}\,\frac{(p^{\alpha}-1)(p^{\beta}-1)}{(p^{\gamma}-1)(p-1)}$$

$$+p^{2(\gamma-\alpha-\beta)}\,\frac{(p^{\alpha}-1)(p^{\alpha+1}-1)(p^{\beta}-1)(p^{\beta+1}-1)}{(p^{\gamma}-1)(p^{\gamma+1}-1)(p-1)(p^2-1)}+\ldots,$$

which may be written

$$\frac{\Pi(p^{\gamma-\alpha-\beta-1})\,\Pi(p^{\gamma-1})}{\Pi(p^{\gamma-\alpha-1})\,\Pi(p^{\gamma-\beta-1})} = F_1(p^{\alpha}.p^{\beta}.p^{\gamma}). \tag{A}$$

When $p=1$, the series (A) reduces to the hypergeometric series in which the element x equals unity. The series is convergent for all values of $p>1$, and, if $p=1$, subject to the condition $\gamma-\alpha-\beta>0$.

2.

Let $\begin{Bmatrix} a \\ n \end{Bmatrix}$ denote the function

$$\underset{\kappa=\infty}{\mathrm{L}}\ \frac{(p^{a-n+1}-1)(p^{a-n+2}-1)\ldots(p^{a-n+\kappa}-1)}{(p^{a+1}-1)(p^{a+2}-1)\ldots(p^{a+\kappa}-1)}$$

$$\times\frac{(p^{n+1}-1)(p^{n+2}-1)\ldots(p^{n+\kappa}-1)}{(p-1)(p^2-1)\ldots(p^{\kappa}-1)}\,p^{n(a-n)}. \tag{1}$$

When n is a positive integer, the function $\left\{\begin{matrix} a \\ n \end{matrix}\right\}$ reduces to the product

$$\frac{(p^a-1)(p^{a-1}-1)\dots(p^{a-n+1}-1)}{(p^n-1)(p^{n-1}-1)\dots(p-1)}.$$

For, in general, $\left\{\begin{matrix} a \\ n \end{matrix}\right\} = \underset{\kappa=\infty}{\mathrm{L}} \dfrac{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^{a-n+\kappa}-1)}{(p^{a+1}-1)(p^{a+2}-1)\dots(p^{a+\kappa}-1)}$

$$\times \frac{(p^{n+1}-1)(p^{n+2}-1)\dots(p^{n+\kappa}-1)}{(p-1)(p^2-1)\dots(p^\kappa-1)} p^{n(a-n)},$$

which may be written, if n is a positive integer, in the form

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^{a-1}-1)(p^a-1)(p^{a+1}-1)(p^{a+2}-1)\dots(p^{a-n+\kappa}-1)}{(p^{a+1}-1)(p^{a+2}-1)\dots(p^{a-n+\kappa}-1)(p^{a-n+\kappa+1}-1)(p^{a-n+\kappa+2}-1)\dots(p^{a+\kappa}-1)}$$

$$\times \frac{(p^{n+1}-1)(p^{n+2}-1)\dots(p^{n+\kappa}-1)}{(p-1)(p^2-1)\dots(p^\kappa-1)} p^{n(a-n)},$$

which, by cancelling common factors of the numerator and denominator and rearranging, reduces to

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^a-1)}{(p-1)(p^2-1)\dots(p^n-1)} \times \frac{(p^{n+1}-1)(p^{n+2}-1)\dots(p^{n+\kappa}-1)}{(p^{n+1}-1)(p^{n+2}-1)\dots(p^\kappa-1)}$$

$$\times \frac{p^{n(a-n)}}{(p^{a-n+\kappa+1}-1)(p^{a-n+\kappa+2}-1)\dots(p^{a+\kappa}-1)},$$

which, by cancelling of common factors, again reduces to

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^a-1)}{(p-1)(p^2-1)\dots(p^n-1)}$$

$$\times \frac{(p^{\kappa+1}-1)(p^{\kappa+2}-1)\dots(p^{\kappa+n}-1)}{(p^{a-n+\kappa+1}-1)(p^{a-n+\kappa+2}-1)\dots(p^{a+\kappa}-1)} \times p^{n(a-n)}. \qquad \text{(B)}$$

Now the part of the product which involves κ may be written

$$\frac{\left(p-\dfrac{1}{p^\kappa}\right)\left(p^2-\dfrac{1}{p^\kappa}\right)\dots\left(p^n-\dfrac{1}{p^\kappa}\right)}{\left(p^{a-n+1}-\dfrac{1}{p^\kappa}\right)\dots\left(p^a-\dfrac{1}{p^\kappa}\right)},$$

which, when $p > 1$ and κ is infinite, reduces to

$$\frac{p^{1+2+3+\dots+n}}{p^{a+(a+1)+(a+2)+\dots(a-n+1)}} = \frac{p^{\frac{1}{2}[n(n+1)]}}{p^{\frac{1}{2}n(2a-n+1)}} = p^{n^2-an}.$$

So the function (B) becomes

$$\frac{(p^{a-n+1}-1)\dots(p^a-1)}{(p^n-1)(p^{n-1}-1)\dots(p-1)} p^{n^2-an} p^{n(a-n)} = \frac{(p^{a-n+1}-1)\dots(p^a-1)}{(p^n-1)\dots(p-1)},$$

which, if $p = 1$, reduces to

$$\frac{a \,.\, a-1 \,.\, a-2 \,\ldots\, a-n+1}{n\,!} = \frac{(a)_n}{(n_n)}.$$

3.

We shall first consider the theorem

$$\begin{Bmatrix} a+b \\ n \end{Bmatrix} = \begin{Bmatrix} a \\ n \end{Bmatrix} + \sum_{r=1}^{r=\infty} p^{r(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix} \begin{Bmatrix} b \\ r \end{Bmatrix}, \qquad \text{(C)}$$

which is a generalized form of the Vandermonde Theorem in Prof. Lloyd Tanner's Paper on "Enumeration of Groups of Totitives," § 25, page 347, *Lond. Math. Soc. Proc.*, Vol. XXVII.

It will first be shown that, if $\begin{Bmatrix} a+b \\ n \end{Bmatrix}$ can be expanded in a series of the form

$$C_0 + C_1 \begin{Bmatrix} b \\ 1 \end{Bmatrix} + C_2 \begin{Bmatrix} b \\ 2 \end{Bmatrix} + \ldots + C_r \begin{Bmatrix} b \\ r \end{Bmatrix} + \ldots,$$

then the coefficients C_0, C_1, ... C_r, ... will be independent of b, and will follow the law

$$C_r = p^{r(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix}.$$

A rigorous proof of the Theorem (C) would involve a comparison of the two sides of (C), which is not attempted in this paper.*

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* [I am greatly indebted to one of the referees for the following remarks, as indicating that there is a stronger *prima facie* case for the theorem than appears in the paper:—

(*a*) The infinite products in $\begin{Bmatrix} a \\ n \end{Bmatrix}$ may each be written in the form

$$p^x (1-q^{y+1})(1-q^{y+2}) \ldots (1-q^{y+k}), \text{ where } q = \frac{1}{p},$$

and in $\begin{Bmatrix} a \\ n \end{Bmatrix}$ the factors p^x cancel. The four products $\Pi_x (1-q^x)$ are each of them convergent if $0 < q < 1$, that is, if $p > 1$.

It is assumed that p^x when x is not integral means its absolute value. Thus $\begin{Bmatrix} a \\ n \end{Bmatrix}$ is finite, single valued, &c., unless a is a negative integer. In the exceptional cases $\begin{Bmatrix} a \\ n \end{Bmatrix}$ is infinite or indeterminate.

(*b*) On the assumption that the form of the series is suitable, it is shown that

$$\begin{Bmatrix} a+b \\ n \end{Bmatrix} = \begin{Bmatrix} a \\ n \end{Bmatrix} + \sum_{r=1}^{r=\infty} p^{(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix} \begin{Bmatrix} b \\ r \end{Bmatrix}.$$

This series is convergent (in virtue of the conditions already assigned) for all real values of n, and for all real values of a, b, except negative integers, and this gives a strong presumption for the suitability of the series, especially as the particular assumptions for determing C_r affect b and not n.]

4.

The following expansions will also be considered:—

$$\frac{a^x-1}{a-1} = \left\{\begin{matrix} x \\ 1 \end{matrix}\right\} + (a-p)\left\{\begin{matrix} x \\ 2 \end{matrix}\right\} + \dots$$
$$\dots + (a-p)(a-p^2)\dots(a-p^r)\left\{\begin{matrix} x \\ r+1 \end{matrix}\right\} + \dots, \qquad (2)$$

$$x = \left\{\begin{matrix} x \\ 1 \end{matrix}\right\} + (1-p)\left\{\begin{matrix} x \\ 2 \end{matrix}\right\} + \dots$$
$$\dots + (1-p)(1-p^2)\dots(1-p^r)\left\{\begin{matrix} x \\ r+1 \end{matrix}\right\} + \dots. \qquad (3)$$

5.

The following lemmata will be useful in the subsequent work:—

Lemma (1).— $$\frac{\left\{\begin{matrix} a \\ n \end{matrix}\right\}}{\left\{\begin{matrix} a \\ n-r \end{matrix}\right\}} = \frac{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^{a-n+r}-1)}{(p^{n-r+1}-1)(p^{n-r+2}-1)\dots(p^n-1)},$$

$p > 1$, n unrestricted, r an integer, which may be proved by substituting for $\left\{\begin{matrix} a \\ n \end{matrix}\right\}$ the infinite product, and by substituting a similar product for $\left\{\begin{matrix} a \\ n-r \end{matrix}\right\}$.

Lemma (2).— $$\left\{\begin{matrix} a+r-s \\ r-s \end{matrix}\right\}$$
$$= \frac{(p^{a+r-s}-1)\dots(p^{a+1}-1)}{(p^{r-s}-1)\dots(p-1)}$$
$$\equiv (-1)^{r-s}p^{a+1}\cdot p^{a+2}\dots p^{a+r-s}\frac{(p^{-a-1}-1)(p^{-a-2}-1)\dots(p^{-a-r+s}-1)}{(p^{r-s}-1)(p^{r-s-1}-1)\dots(p-1)}$$
$$\equiv (-1)^{r-s}p^{\frac{1}{2}[(r-s)(2a+r-s+1)]}\left\{\begin{matrix} -a-1 \\ r-s \end{matrix}\right\},$$

r and s being positive integers, and $r > s$.

Lemma (3).— $$\left\{\begin{matrix} a+r-s \\ n \end{matrix}\right\} \equiv \frac{(p^s)!\,(p^{r-s})!\,(p^{n-r})!}{p^n!}$$
$$\times \left\{\begin{matrix} a \\ n-r \end{matrix}\right\}\left\{\begin{matrix} a-n+r \\ s \end{matrix}\right\}\left\{\begin{matrix} a+r-s \\ r-s \end{matrix}\right\},$$

in which $(p^s)!$ denotes $(p^s-1)(p^{s-1}-1)\dots(p-1)$.

Lemma (4).—By combining Lemmata (2) and (3), we obtain

$$\left\{\begin{matrix} a+r-s \\ n \end{matrix}\right\} \equiv (-1)^{r-s}p^{\frac{1}{2}[(r-s)(2a+r-s+1)]}\frac{(p^s)!\,(p^{r-s})!\,(p^{n-r})!}{(p^n)!}$$
$$\times \left\{\begin{matrix} a \\ n-r \end{matrix}\right\}\left\{\begin{matrix} a-n+r \\ s \end{matrix}\right\}\left\{\begin{matrix} -a-1 \\ r-s \end{matrix}\right\}.$$

6.

Consider any function of $a+b$, viz., $f(a+b)$, supposed capable of expansion in the form

$$C_0+C_1\left\{ {b \atop 1} \right\}+C_2\left\{ {b \atop 2} \right\}+\ldots+C_{(r)}\left\{ {b \atop r} \right\}+\ldots, \tag{4}$$

where $C_0, C_1, C_2, \ldots C_{(r)}$ are functions of a only. Write

$$(a+b)=C_0+C_1\left\{ {b \atop 1} \right\}+C_2\left\{ {b \atop 2} \right\}+\ldots,$$

substituting successively

$$\begin{aligned} b&=0,\\ b&=1,\\ b&=2,\\ b&=3,\\ \ldots&\;\ldots. \end{aligned}$$

We obtain $f(a)=C_0$,

$$\begin{aligned} f(a+1)&=C_0+C_1,\\ f(a+2)&=C_0+\frac{p^2-1}{p-1}C_1+\frac{p^2-1\,.\,p-1}{p^2-1\,.\,p-1}C_2,\\ \ldots&\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots,\\ f(a+r)&=C_0+\frac{p^r-1}{p-1}C_1+\frac{p^r-1\,.\,p^{r-1}-1}{p^2-1\,.\,p-1}C_2+\ldots+C_r,\\ \ldots&\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots, \end{aligned} \tag{5}$$

from which we have

$$\begin{aligned} C_0&=f(a),\\ C_1&=f(a+1)-f(a),\\ C_2&=f(a+2)-\frac{p^2-1}{p-1}f(a+1)+pf(a),\\ C_3&=f(a+3)-\frac{p^3-1}{p-1}f(a+2)+p\frac{p^3-1}{p-1}f(a+1)-p^3f(a); \end{aligned}$$

and, in general,

$$\begin{aligned} C_r=f(a+r)&-\left\{ {r \atop 1} \right\}f(a+r-1)+p\left\{ {r \atop 2} \right\}f(a+r-2)-\ldots\\ &\ldots+(-1)^s p^{\frac{1}{2}(s\,.\,s-1)}\left\{ {r \atop s} \right\}f(a+r-s)+\ldots+(-1)^r p^{\frac{1}{2}(r\,.\,r-1)}f(a), \end{aligned} \tag{6}$$

$$C_r=\sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}(s\,.\,s-1)}\left\{ {r \atop s} \right\}f(a+r-s),$$

which may be established by induction as follows:—

We have, by (5),

$$f(a+r) = C_0 + \begin{Bmatrix} r \\ 1 \end{Bmatrix} C_1 + \begin{Bmatrix} r \\ 2 \end{Bmatrix} C_2 + \ldots + \begin{Bmatrix} r \\ r-1 \end{Bmatrix} C_{r-1} + C_r. \qquad \text{(D)}$$

Assume that

$$\begin{aligned} C_{r-1} = f(a+r-1) - \begin{Bmatrix} r-1 \\ 1 \end{Bmatrix} f(a+r-2) + \ldots \\ \ldots + (-1)^s p^{\frac{1}{2}(s.s-1)} \begin{Bmatrix} r-1 \\ s \end{Bmatrix} f(a+r-1-s) + \ldots \\ \ldots + (-1)^{r-1} p^{\frac{1}{2}(r-1.r-2)} f(a), \end{aligned}$$

$$\begin{aligned} C_{r-2} = f(a+r-2) - \ldots + (-1)^{s-1} p^{\frac{1}{2}(s-1.s-2)} \begin{Bmatrix} r-2 \\ s-1 \end{Bmatrix} f(a+r-1-s) + \ldots \\ \ldots + (-1)^{r-2} p^{\frac{1}{2}(r-2.r-3)} f(a), \end{aligned}$$

$$\begin{aligned} C_{r-3} = f(a+r-3) - \ldots + (-1)^{s-2} p^{\frac{1}{2}(s-2.s-3)} \begin{Bmatrix} r-3 \\ s-2 \end{Bmatrix} f(a+r-1-s) + \ldots \\ \ldots + (-1)^{r-3} p^{\frac{1}{2}(r-3.r-4)} f(a), \end{aligned}$$

&c. &c.

Then, from (D),

$$\begin{aligned} C_r = f(a+r) - \begin{Bmatrix} r \\ r-1 \end{Bmatrix} \Big[f(a+r-1) - \begin{Bmatrix} r-1 \\ 1 \end{Bmatrix} f(a+r-2) + \ldots \\ \ldots + (-1)^s p^{\frac{1}{2}(s.s-1)} \begin{Bmatrix} r-1 \\ s \end{Bmatrix} f(a+r-1-s) + \ldots \\ \ldots + (-1)^{r-1} p^{\frac{1}{2}(r-1.r-2)} f(a) \Big] \\ - \begin{Bmatrix} r \\ r-2 \end{Bmatrix} \Big[f(a+r-2) - \begin{Bmatrix} r-2 \\ 1 \end{Bmatrix} f(a+r-3) + \ldots \\ \ldots + (-1)^{s-1} p^{\frac{1}{2}(s-1.s-2)} \begin{Bmatrix} r-2 \\ s-1 \end{Bmatrix} f(a+r-1-s) + \ldots \\ \ldots + (-1)^{r-2} p^{\frac{1}{2}(r-2.r-3)} f(a) \Big] \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ - \begin{Bmatrix} r \\ r-s-1 \end{Bmatrix} \Big[f(a+r-s-1) - \begin{Bmatrix} r-s-1 \\ 1 \end{Bmatrix} f(a+r-s-2) + \ldots \\ \ldots + (-1)^{r-s-1} p^{\frac{1}{2}(r-s-1.r-s-2)} f(a) \Big]. \end{aligned}$$

Collecting the terms and rearranging them so that the coefficients of $f(a+r), f(a+r-1), \dots f(a+r-s)$, respectively, may be together, we have

$$C_r = f(a+r) - \genfrac\{\}{0pt}{}{r}{r-1} f(a+r-1) + \left[\genfrac\{\}{0pt}{}{r}{r-1} \genfrac\{\}{0pt}{}{r-1}{1} - \genfrac\{\}{0pt}{}{r}{r-2} \right] \times f(a+r-2) - \dots$$

$$= f(a+r) - \genfrac\{\}{0pt}{}{r}{1} f(a+r-1) + p \genfrac\{\}{0pt}{}{r}{2} f(a+r-2) - \dots,$$

the coefficient of $f(a+r-s)$ being

$$(-1)^{s+1} \left[p^{\frac{1}{2}(s.s-1)} \genfrac\{\}{0pt}{}{r-1}{s} \genfrac\{\}{0pt}{}{r}{r-1} - p^{\frac{1}{2}(s-1.s-2)} \genfrac\{\}{0pt}{}{r-2}{s-1} \genfrac\{\}{0pt}{}{r}{r-2} + p^{\frac{1}{2}(s-2.s-3)} \genfrac\{\}{0pt}{}{r-3}{s-2} \genfrac\{\}{0pt}{}{r}{r-3} - \dots + (-1)^s \genfrac\{\}{0pt}{}{r}{r-s-1} \right]. \quad \text{(E)}$$

Now it may easily be shown, by substitution, that

$$\genfrac\{\}{0pt}{}{r-1}{s} \genfrac\{\}{0pt}{}{r}{r-1} = \genfrac\{\}{0pt}{}{r}{s+1} \genfrac\{\}{0pt}{}{s+1}{1},$$

$$\genfrac\{\}{0pt}{}{r-2}{s-1} \genfrac\{\}{0pt}{}{r}{r-2} = \genfrac\{\}{0pt}{}{r}{s+1} \genfrac\{\}{0pt}{}{s+1}{2},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots,$$

and generally that

$$\genfrac\{\}{0pt}{}{r-t}{s-t+1} \genfrac\{\}{0pt}{}{r}{r-t} = \genfrac\{\}{0pt}{}{r}{s+1} \genfrac\{\}{0pt}{}{s+1}{t},$$

whence the expression (E) becomes

$$(-1)^{s+1} \genfrac\{\}{0pt}{}{r}{s+1} \left[p^{\frac{1}{2}(s.s-1)} \genfrac\{\}{0pt}{}{s+1}{1} - p^{\frac{1}{2}(s-1.s-2)} \genfrac\{\}{0pt}{}{s+1}{2} + p^{\frac{1}{2}(s-2.s-3)} \genfrac\{\}{0pt}{}{s+1}{3} - \dots + (-1)^s \genfrac\{\}{0pt}{}{s+1}{s+1} \right]. \quad \text{(F)}$$

Now we know that for m a positive integer

$$p^{\frac{1}{2}(m.m-1)} \genfrac\{\}{0pt}{}{m}{0} - p^{\frac{1}{2}(m-1.m-2)} \genfrac\{\}{0pt}{}{m}{1} + p^{\frac{1}{2}(m-2.m-3)} \genfrac\{\}{0pt}{}{m}{2} - \dots + (-1)^m \genfrac\{\}{0pt}{}{m}{m} \equiv 0,$$

whence we have the identity

$$p^{\frac{1}{2}(s+1.s)} \equiv p^{\frac{1}{2}(s.s-1)} \genfrac\{\}{0pt}{}{s+1}{1} - p^{\frac{1}{2}(s-1.s-2)} \genfrac\{\}{0pt}{}{s+1}{2} + \dots + (-1)^s \genfrac\{\}{0pt}{}{s+1}{s+1},$$

and the expression (F) becomes

$$(-1)^{s+1} \genfrac\{\}{0pt}{}{r}{s+1} p^{\frac{1}{2}(s+1,s)},$$

and $$C_r = f(a+r) - \genfrac\{\}{0pt}{}{r}{1} f(a+r-1) + \dots$$
$$\dots + (-1)^{s+1} \genfrac\{\}{0pt}{}{r}{s+1} p^{\frac{1}{2}(s+1 . s)} f(a+r-s-1) + \dots$$
$$\dots + (-1)^r p^{\frac{1}{2}(r . r-1)} f(a).$$

That is, C_r is of the form given above, if $C_{r-1}, C_{r-2}, \dots C_0$ are of the same form; now we have shown that $C_1, C_2, C_3 \dots$ are of the form; therefore the form of the coefficients has been established generally.

7.

If the expansion of $\genfrac\{\}{0pt}{}{a+b}{n}$ is required, we have for this particular function

$$C_r = \sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}(s . s-1)} \genfrac\{\}{0pt}{}{r}{s} \genfrac\{\}{0pt}{}{a+r-s}{n}.$$

Using Lemma (4), this expression reduces to

$$\sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}(s . s-1)} \genfrac\{\}{0pt}{}{r}{s} (-1)^{r-s} p^{\frac{1}{2}[(r-s)(2a+r-s+1)} \frac{(p^s)!\,(p^{r-s})!\,(p^{n-r})!}{(p^n)!}$$
$$\times \genfrac\{\}{0pt}{}{a}{n-r} \genfrac\{\}{0pt}{}{a-n+r}{s} \genfrac\{\}{0pt}{}{-a-1}{r-s},$$

and, since $$\genfrac\{\}{0pt}{}{r}{s} = \frac{(p^r)!}{(p^s)!\,(p^{r-s})!},$$

the above expression becomes

$$\sum_{s=0}^{s=r} (-1)^r p^{\frac{1}{2}(r^2+2ar+r+2s^2-2s-2rs-2as)} \frac{(p^r)!\,(p^{n-r})!}{(p^n)!}$$
$$\times \genfrac\{\}{0pt}{}{a}{n-r} \genfrac\{\}{0pt}{}{a-n+r}{s} \genfrac\{\}{0pt}{}{-a-1}{r-s}$$

$$\equiv (-1)^r p^{\frac{1}{2}(r^2+2ar+r)} \frac{(p^r)!\,(p^{n-r})!}{(p^n)!} \genfrac\{\}{0pt}{}{a}{n-r}$$
$$\times \Big[\genfrac\{\}{0pt}{}{-a-1}{r} + p^{-r-a} \genfrac\{\}{0pt}{}{-a-1}{r-1} \genfrac\{\}{0pt}{}{a-n+r}{1} + \dots$$
$$\dots + p^{s(s-1-r-a)} \genfrac\{\}{0pt}{}{-a-1}{r-s} \genfrac\{\}{0pt}{}{a-n+r}{s} + \dots \Big].$$

The bracket contains $r+1$ terms, and has been shown in Prof. Lloyd Tanner's paper "On Enumeration of Groups of Totitives," equal to $\begin{Bmatrix} -n+r-1 \\ r \end{Bmatrix}$; therefore

$$C_r = (-1)^r p^{\frac{1}{2}(r^2+2ar+r)} \frac{(p^r)!\,(p^{n-r})!}{(p^n)!} \begin{Bmatrix} a \\ n-r \end{Bmatrix} \begin{Bmatrix} -n+r-1 \\ r \end{Bmatrix}$$

$$= (-1)^r p^{\frac{1}{2}(r^2+2ar+r)} \frac{(p^r)!\,(p^{n-r})!}{(p^r)!} \begin{Bmatrix} a \\ n-r \end{Bmatrix} \frac{(p^{-n+r-1}-1)\dots(p^{-n}-1)}{(p^r-1)\dots(p-1)}.$$

Now $(p^{-n+r-1}-1)(p^{-n+r-2}-1)\dots(p^{-n}-1) = (-1)^r p^{\frac{1}{2}[-r(2n-r+1)]} \dfrac{(p^n)!}{(p^{n-r})!}$.

Therefore $C_r = (-1)^{2r} p^{r(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix} = p^{r(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix}$.

Hence $$\begin{Bmatrix} a+b \\ n \end{Bmatrix} = \begin{Bmatrix} a \\ n \end{Bmatrix} + \dots + p^{r(a-n+r)} \begin{Bmatrix} a \\ n-r \end{Bmatrix} \begin{Bmatrix} b \\ r \end{Bmatrix} + \dots, \qquad (7)$$

when n is not restricted to being a positive integer, subject to conditions for convergence.

8.

To obtain a generalized form of the series $F(\alpha, \beta, \gamma)$ from the above, we proceed as follows:—

Divide both sides of (7) by $\begin{Bmatrix} a \\ n \end{Bmatrix}$; then we obtain

$$\frac{\begin{Bmatrix} a+b \\ n \end{Bmatrix}}{\begin{Bmatrix} a \\ n \end{Bmatrix}} = 1 + \dots + p^{r(a-n+r)} \frac{\begin{Bmatrix} a \\ n-r \end{Bmatrix}}{\begin{Bmatrix} a \\ n \end{Bmatrix}} \begin{Bmatrix} b \\ r \end{Bmatrix} + \dots. \qquad (8)$$

Using Lemma (1), the general term of the series becomes

$$p^{r(a-n+r)} \frac{(p^{n-r+1}-1)(p^{n-r+2}-1)\dots(p^n-1)(p^b-1)\dots(p^{b-r+1}-1)}{(p^{a-n+1}-1)(p^{a-n+2}-1)\dots(p^{a-n+r}-1)(p^r-1)\dots(p-1)}.$$

On making the substitutions $n = -\alpha$, $b = -\beta$, $a-n+1 = \gamma$, the general term becomes

$$p^{r(\gamma-\alpha-\beta)} \frac{(p^\alpha-1)(p^{\alpha+1}-1)\dots(p^{\alpha+r-1}-1)(p^\beta-1)(p^{\beta+1}-1)\dots(p^{\beta+r-1}-1)}{(p^\gamma-1)(p^{\gamma+1}-1)\dots(p^{\gamma+r-1}-1)(p^r-1)(p^{r-1}-1)\dots(p-1)}$$

Similarly, by Lemma (1), the expression on the left side of (8) becomes

$$\underset{\kappa=}{\mathrm{L}} \frac{(p^{a+b-n+1}-1)\dots(p^{a+b-n+\kappa}-1)}{(p^{a+b+1}-1)\dots(p^{a+b+\kappa}-1)} \frac{(p^{a+1}-1)\dots(p^{a+\kappa}-1)}{(p^{a-n+1}-1)\dots(p^{a-n+\kappa}-1)} p^{bn}.$$

Making the substitutions $n = -\alpha$, $b = -\beta$, $a-n+1 = \gamma$, this becomes

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(p^{\gamma-\beta}-1)\ldots(p^{\gamma-\beta+\kappa-1}-1)}{(p^{\gamma-\beta-\alpha}-1)\ldots(p^{\gamma-\beta-\alpha+\kappa-1}-1)} \, \frac{(p^{\gamma-\alpha}-1)\ldots(p^{\gamma-\alpha+\kappa-1}-1)}{(p^{\gamma}-1)\ldots(p^{\gamma+\kappa-1}-1)} p^{\alpha\beta},$$

which is equal to

$$1+p^{\gamma-\alpha-\beta}\frac{(p^{\alpha}-1)(p^{\beta}-1)}{(p^{\gamma}-1)(p-1)}$$

$$+p^{2(\gamma-\alpha-\beta)}\frac{(p^{\alpha}-1)(p^{\alpha+1}-1)(p^{\beta}-1)(p^{\beta+1}-1)}{(p^{\gamma}-1)(p^{\gamma+1}-1)(p-1)(p^{2}-1)}+\ldots, \tag{9}$$

which is a generalized form of

$$\frac{\Pi(\gamma-\alpha-\beta-1)\,\Pi(\gamma-1)}{\Pi(\gamma-\alpha-1)\,\Pi(\gamma-\beta-1)} = 1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}+\frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,.\,2\,.\,\gamma\,.\,\gamma+1}+\ldots. \tag{10}$$

The general form may be written

$$\frac{\Pi\,(p^{\gamma-\alpha-\beta-1})\,\Pi\,(p^{\gamma-1})}{\Pi\,(p^{\gamma-\alpha-1})\,\Pi\,(p^{\gamma-\beta-1})} = F_1\,(p^{\alpha}\,.\,p^{\beta}\,.\,p^{\gamma}), \tag{11}$$

reducing to (10) when $p = 1$.

9.

To obtain the expansion of a^x, we have

$$a^x = C_0+C_1\left\{\begin{matrix}x\\1\end{matrix}\right\}+C_2\left\{\begin{matrix}x\\2\end{matrix}\right\}+\ldots+C_r\left\{\begin{matrix}x\\r\end{matrix}\right\}+\ldots,$$

in which

$$C_r = \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}(s\,.\,s-1)}\left\{\begin{matrix}r\\s\end{matrix}\right\}a^{r-s}$$

$$= a^r - a^{r+1}\frac{p^r-1}{p-1}+a^{r-2}p\,\frac{p^r-1\,.\,p^{r-1}-1}{p^2-1\,.\,p-1}+\ldots+(-1)^r p^{\frac{1}{2}(r\,.\,r-1)},$$

r being a positive integer.

The above $= (a-1)(a-p)\ldots(a-p^{r-1})$;

therefore $a^x = 1+(a-1)\left\{\begin{matrix}x\\1\end{matrix}\right\}+(a-1)(a-p)\left\{\begin{matrix}x\\2\end{matrix}\right\}+\ldots,$

whence

$$\frac{a^x-1}{a-1} = \left\{\begin{matrix}x\\1\end{matrix}\right\}+(a-p)\left\{\begin{matrix}x\\2\end{matrix}\right\}+(a-p)(a-p^2)\left\{\begin{matrix}x\\3\end{matrix}\right\}+\ldots, \tag{12}$$

subject to conditions for convergence.

To expand x in a series of the form

$$C_0+C_1\left\{\begin{matrix}x\\1\end{matrix}\right\}+C_2\left\{\begin{matrix}x\\2\end{matrix}\right\}+\dots+C_r\left\{\begin{matrix}x\\r\end{matrix}\right\}+\dots.$$

In this case

$$C_r=r-\frac{p^r-1}{p-1}(r-1)+p\frac{p^r-1\,.\,p^{r-1}-1}{p^2-1\,.\,p-1}(r-2)-\dots+0$$
$$=(1-p)(1-p^2)\dots(1-p^{r-1});$$

therefore

$$x=\left\{\begin{matrix}x\\1\end{matrix}\right\}+(1-p)\left\{\begin{matrix}x\\2\end{matrix}\right\}+(1-p)(1-p^2)\left\{\begin{matrix}x\\3\end{matrix}\right\}+\dots$$
$$\dots+(1-p)\dots(1-p^{r-1})\left\{\begin{matrix}x\\r\end{matrix}\right\}+\dots, \qquad (13)$$

subject to conditions for convergence.

This expansion might have been derived from (12) by putting $a=1$, when the left side $\dfrac{a^x-1}{a-1}=x$, and the right side of (12) becomes the same as series (13).

10.

We must now consider the convergence of the expansion of $\left\{\begin{matrix}a+b\\n\end{matrix}\right\}$.

The ratio
$$\frac{u_{r+1}}{u_r}=\frac{p^{(r+1)(a-n+r+1)}}{p^{r(a-n+r)}}\frac{\left\{\begin{matrix}a\\n-r-1\end{matrix}\right\}}{\left\{\begin{matrix}a\\n-r\end{matrix}\right\}}\frac{\left\{\begin{matrix}b\\r+1\end{matrix}\right\}}{\left\{\begin{matrix}b\\r\end{matrix}\right\}}$$
$$=p^{a-n+2r+1}\frac{(p^{n-r}-1)(p^{b-r}-1)}{(p^{a-n+r+1}-1)(p^{r+1}-1)};$$

making $r=\infty$, the ratio $\dfrac{u_{r+1}}{u_r}$ in the limit becomes equal to $\dfrac{1}{p}$.

Therefore the series is convergent for all values of p greater than unity.

In the case when $p=1$, the series becomes the well known $F_1(\alpha, \beta, \gamma)$, which may be shown to be convergent if $\gamma-\alpha-\beta$ is positive.

Again, in the expansion of $\dfrac{a^x-1}{a-1}$ the ratio of

$$\frac{u_{r+1}}{u_r}=\frac{(a-p^r)\left\{\begin{matrix}x\\r+1\end{matrix}\right\}}{\left\{\begin{matrix}x\\r\end{matrix}\right\}}=\frac{(a-p^r)(p^{x-r}-1)}{(p^{r+1}-1)}=\frac{\left(\frac{a}{p^r}-1\right)\left(\frac{p^x}{p^r}-1\right)}{\left(p-\frac{1}{p^r}\right)},$$

and, when $r = \infty$, the limit of the ratio $= \dfrac{1}{p}$. Therefore the series will be convergent if $p > 1$.

11.

In considering the series $F_1(p^\alpha, p^\beta, p^\gamma)$, it naturally occurs that one should wish to consider the series which will be the generalization of $F(\alpha, \beta, \gamma, x)$ and the differential equation connected with it.

The series $$1+p^{\gamma-\alpha-\beta}\frac{p^\alpha-1\,.\,p^\beta-1}{p^\gamma-1\,.\,p-1}x+\dots$$
is not the series sought, for, on differentiating the powers of x, we do not obtain coefficients which cancel with the factors $p-1$, p^2-1, p^3-1, &c., of the denominators.

It seems that the ordinary notion of differentiation is not sufficient, and that differentiation must be generalized into some operation of the following type:—

$$u = \left(\frac{p^x-1}{p-1}\right)^{(p^n-1)/(p-1)},$$

$$\frac{du}{d\,(x)} = \frac{p^n-1}{p-1}\left(\frac{p^x-1}{p-1}\right)^{(p^{n-1}-1)/(p-1)},$$

reducing when $p=1$ to ordinary differentiation.

On the Partition of Numbers. By G. B. MATHEWS, M.A. Read May 13th, 1897. Received, in revised form, June 25th, 1897.

The problem of the partition of a given multipartite number $(m, m', m'', \dots)$ into assigned parts $(a, a', a'', \dots)$, $(b, b', b'', \dots)$, &c., is identical with that of finding integers $x, y, z, \dots t$, none of which are negative, although some of them may be zero, so that

$$\left.\begin{aligned} u &= ax + by + cz + \dots + lt = m, \\ u' &= a'x + b'y + c'z + \dots + l't = m', \\ u'' &= a''x + b''y + c''z + \dots + l''t = m'', \\ &\&\text{c.}, \end{aligned}\right\} \qquad (1)$$

the number of equations being equal to that of the elements m, m', m'', &c.

I propose to show, by very elementary considerations, that this general problem is reducible, in an indefinite number of ways, to one of simple partition. For convenience, the term "solution" will be used for a solution in which $x, y, z, \ldots t$ satisfy the condition above stated.

In order to avoid the discussion of exceptional cases, it will be supposed that none of the coefficients $a, b, \ldots l$; $a', b', \ldots l'$; &c., is zero. No real loss of generality arises from this; because if, for instance, we have the two equations $u = m$, $u' = m'$, and zero coefficients occur in them, we may replace them by the equivalent pair

$$u + u' = m + m', \quad u + 2u' = m + 2m',$$

in which the coefficients are all positive, and a similar process may be applied in the general case.

Suppose, in the first place, that we have the two equations

$$\left.\begin{aligned} u &= ax + by + cz + dw = m, \\ u' &= a'x + b'y + c'z + d'w = m'; \end{aligned}\right\} \tag{2}$$

then, if λ, μ are any two positive integers, every solution of (2) is also a solution of

$$(\lambda a + \mu a')\, x + (\lambda b + \mu b')\, y + (\lambda c + \mu c')\, z + (\lambda d + \mu d')\, w = \lambda m + \mu m'. \tag{3}$$

But, conversely, if λ, μ are suitably chosen, then every solution of (3) is also a solution of (2). For suppose λ, μ are prime to each other; then every solution of (3) must give

$$u = ax + by + cz + dw = m \pm \mu\theta,$$

$$u' = a'x + b'y + c'z + d'w = m' \mp \lambda\theta,$$

where θ is some integer or zero. If we take $\lambda \overset{=}{>} m'$ and $\mu \overset{=}{>} m$, one or other of the quantities $m \mp \mu\theta$, $m' \mp \lambda\theta$ will be zero or negative, except when $\theta = 0$. But every solution reckoned makes u and u' positive;* therefore $\theta = 0$, and every solution of (3) is also a solution of (2). In other words, the problem of bipartition represented by the pair of equations (2) has been reduced to that of a simple partition, corresponding to the equation (3).

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* The value zero for u or u' is excluded by the supposition that all the coefficients of u and u' are positive.

As a particular case, if m, m' are prime to each other, we may put $\lambda = m'$, $\mu = m$, so that (3) becomes

$$(m'a+ma')\,x+(m'b+mb')\,y+(m'c+mc')\,z+(m'd+md')\,w = 2mm'.$$

For example, the solutions of

$$x+y+z+w = 9, \quad x+2y+3z+4w = 20$$

coincide with those of

$$29x+38y+47z+56w = 360.$$

The argument has not been affected by the number of unknown quantities x, y, z, &c., so that the general problem of the partition of a bipartite number has been reduced to that of a simple partition. But the process may evidently be extended so as to cover the most general case of multiple partition; thus suppose that there are three equations

$$\left.\begin{aligned} u &= ax+by+cz+dw = m, \\ u' &= a'x+b'y+c'z+d'w = m', \\ u'' &= a''x+b''y+c''z+d''w = m''; \end{aligned}\right\} \tag{4}$$

then, by suitably choosing λ, μ, the solutions of the first two may be made to coincide with those of

$$\lambda u+\mu u' = \lambda m+\mu m',$$

and now we may again choose integers λ', μ' so that the solutions common to this and the third of the given equations coincide with those of the single equation

$$\lambda'(\lambda u+\mu u')+\mu' u'' = \lambda'(\lambda m+\mu m')+\mu' m'';$$

or, changing the notation, we may say that it is possible, in an infinite number of ways, to choose positive integers λ, μ, ν so that the solutions of (4) coincide with those of

$$(\lambda a+\mu a'+\nu a'')\,x+\ldots = \lambda m+\mu m'+\nu m''.$$

It is clear that, by a process of induction, a similar result may be established for any number of equations.

In the case of two equations with three undetermined quantities

$$ax+by+cz-m = 0, \quad a'x+b'y+c'z-m' = 0,$$

we have a geometrical interpretation; namely, that, if there are integral points on the line of intersection of the planes represented by these equations, then an indefinite number of planes

$$\lambda\,(ax+by+cz-m)+\mu\,(a'x+b'y+c'z-m')=0$$

can be found (λ, μ being positive integers), so as to pass through no integral points in the first octant except those on their common line of intersection. A quasi-geometrical interpretation of the same kind may, of course, be given to the result in the general case.

It may be observed that the above method of finding λ, μ for the reduction of a bipartition is not the only one; nor does it necessarily give the most convenient values. Another way of proceeding is as follows:—Let the two given equations be

$$\left.\begin{aligned} ax+by+cz+dw&=m, \\ a'x+b'y+c'z+d'w&=m'; \end{aligned}\right\} \qquad (5)$$

then (i.), if there is any solution at all, one at least of the quantities

$$\frac{m'a}{a'},\quad \frac{m'b}{b'},\quad \frac{m'c}{c'},\quad \frac{m'd}{d'}$$

must be equal to or exceed m; and (ii.), if i is any positive integer which exceeds the greatest of the four quantities, the solutions of (5) coincide with those of

$$(a+ia')\,x+(b+ib')\,y+(c+ic')\,z+(d+id')\,w=m+im'. \qquad (6)$$

To prove the first statement, suppose

$$m'a<ma',\quad m'b<mb',\quad \&c.;$$

then $$m'(ax+by+cz+dw)<m\,(a'x+b'y+c'z+d'w),$$

if none of the quantities x, y, z, w is negative, and they are not all zero. But every solution of (5) makes

$$m'(ax+by+cz+dw)=m\,(a'x+b'y+c'z+d'w);$$

therefore the assumed inequalities cannot all hold if the given equations are soluble.

Every solution of (6) leads to

$$\left.\begin{aligned} ax+by+cz+dw &= m+iq, \\ a'x+b'y+c'z+d'w &= m'-q, \end{aligned}\right\} \qquad (7)$$

where q is zero or some positive integer. But, since $ia' > m'a$, &c., it follows that

$$i\,(a'x+b'y+c'z+d'w) > m'\,(ax+by+cz+dw)\,;$$

and therefore, if $$ax+by+cz+dw > i,$$

it follows, *a fortiori*, that

$$a'x+b'y+c'z+d'w > m'\,;$$

so that the equations (7) are inconsistent, except for $q = 0$. This proves that every solution of (6) is also a solution of (5).

Thus, if, in the numerical example above given, we take $m = 9$, $m' = 20$, the process just explained gives $i \overset{=}{>} 21$, and the derived equation for $i = 21$ is

$$22x+43y+64z+85w = 429,$$

while, if we take $m = 20$, $m' = 9$, we have $i \overset{=}{>} 37$, and the derived equation for $i = 37$ is

$$38x+39y+40z+41w = 353.$$

The second process, like the first, may be used to replace any number of linear indeterminate equations by a single equivalent one.

It may be of interest to state that the argument upon which this paper is based occurred to me after reading the memoirs of Dedekind and Kronecker on the theory of ideal numbers. Kronecker makes extensive use of "undetermined quantities," which are, in fact, what Sylvester called *umbræ;* in Dedekind's ideals we have undetermined integers, or compound moduli, which serve the same purpose as Kronecker's umbræ.

Notes on Synthetic Geometry. *By* W. Esson, F.R.S.

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In a course of lectures on Synthetic Geometry, delivered at Oxford in 1894, I endeavoured to place the fundamental conceptions of the subject on a purely synthetic basis. In almost all the treatises results are assumed which have been proved by analytic processes. This is especially the case in the determination of Plücker's characteristics, and in the theory of united elements in correspondences. I hope to show in the following notes that these subjects admit of a purely synthetic treatment.

Definitions.

1. The degree of a plane curve is the number of points determined by the curve upon a straight line. This number remains the same whatever be the position of the straight line in the plane of the curve.

It is understood that the curve is described by a purely graphic method, so that the number of points in which the curve meets a straight line can be ascertained by the mode of description.

2. The class of a plane curve is the number of tangents determined by the curve at a point. This number remains the same whatever be the position of the point in the plane of the curve.

Plücker's Characteristics.

3. The class of a curve described in a given manner is m. By an alteration in the relative positions of the elements of description, a double point, not previously existing, is introduced. The class of the altered curve is $m-2$.

The tangent to each branch of the curve at the double point meets the branch to which it is a tangent in two consecutive points, and the other branch in one point ultimately coincident, but not consecutive, with the former points. This tangent counts as two at the consecutive points and one in addition as drawn from the point on the other branch. The number of remaining tangents determined by the curve at the double point is $m-6$. But in the altered curve the tangents at the double point are one for each branch, each counting as two. The class of the altered curve is therefore

$$m-6+4=m-2.$$

It is seen that the original curve of class m degenerates into a curve of class $m-2$ and two points coinciding with the double point.

Cor.—If δ double points are introduced, the class of the altered curve is $m-2\delta$.

4. If the altered curve has a cusp, not previously existing, the class is $m-3$.

A cusp arises when by a further alteration in the elements of description the tangents at the double point become coincident and the loop at the point degenerates into a point.

A tangent to the loop from a point on one branch of the curve adjacent to the double point is lost in the altered curve, and the class of the curve is further diminished by 1.

Cor.—If κ cusps are introduced, the class of the altered curve is $m-3\kappa$.

4. If the original curve is of the n^{th} degree and has no double point or cusp, the class of the curve is $n(n-1)$.

Let u_n be the class of the original curve. Suppose that by an alteration in the elements of description the curve degenerates into a similar curve of the $(n-1)^{\text{th}}$ degree and a straight line. The altered curve has $n-1$ double points at the intersections of the curves of the $(n-1)^{\text{th}}$ degree and straight line, so that the class of the altered curve is, by § 3, diminished by $2(n-1)$, but the class is u_{n-1}, the number of tangents determined at a point by the similar curve of the $(n-1)^{\text{th}}$ degree. Hence

$$u_n-u_{n-1}=2(n-1),$$

and
$$u_1=0;$$

therefore
$$u_n=n(n-1).$$

Cor.—The class of a curve of the n^{th} degree which has δ double points and κ cusps is $n(n-1)-2\delta-3\kappa$.

6. The degree of a curve described in a given manner is n. By an alteration in the relative positions of the elements of description a double tangent, not previously existing, is introduced. The degree of the altered curve is $n-2$.

The point of contact of the double tangent on each branch of the curve determines two consecutive tangents to that branch and one to the other branch ultimately coincident, but not consecutive, with the former tangents. This point of contact counts as two on the consecutive tangents and one in addition as determined by the tangent to the other branch. The number of remaining points determined by the curve on the double tangent is $n-6$. But in the altered curve the points of contact of the double tangent are one for each branch, each counting as two. The degree of the altered curve is therefore

$$n-6+4 = n-2.$$

It is seen that the original curve of degree n degenerates into a curve of degree $n-2$ and two straight lines coinciding with the double tangent.

Cor.—If τ double tangents are introduced, the degree of the altered curve is $n-2\tau$.

7. If the altered curve has an inflexion, not previously existing, the degree is $n-3$.

An inflexion arises when the points of contact of a double tangent become coincident. A tangent to one of the originally distinct portions of the curve at a point adjacent to the point of contact of the double tangent with this portion loses an intersection with the other portion of the curve which the double tangent touches, and the degree of the curve is further diminished by 1.

Cor.—If ι inflexions are introduced, the degree of the altered curve is $n-3\iota$.

8. If the original curve is of the m^{th} class and has no double tangent or inflexion, the degree of the curve is $m(m-1)$.

Let u_m be the degree of the original curve. Suppose that by an alteration of the elements of description the curve degenerates into a similar curve of the $(m-1)^{\text{th}}$ class and a point. The altered curve has $m-1$ double tangents determined at the point by the curve of the $(m-1)^{\text{th}}$ class, so that the degree of the altered curve is diminished by $2(m-1)$, but the degree is u_{m-1}, the number of points determined on a straight line by the similar curve of the $(m-1)^{\text{th}}$ class.

Hence $\quad u_m - u_{m-1} = 2(m-1), \quad \text{and} \quad u_1 = 0$

therefore $$u_m = m(m-1).$$

Cor.—The degree of a curve of the m^{th} class which has τ double tangents and ι inflexions is

$$m(m-1)-2\tau-3\iota.$$

9. The number of conditions which determine a curve described in a given manner is p. By an alteration of the relative positions of the elements of description δ double points and κ cusps, not previously existing, are introduced. The number of conditions which determine the altered curve is $p-\delta-2\kappa$.

As two branches of the altered curve pass through the same point, one condition is lost for each double point, and when the loop of the double point degenerates into a point an additional condition is lost for each cusp.

10. The number of conditions which determine a curve described in a given manner is q. By an alteration of the relative positions of the elements of description τ double tangents and ι inflexions are introduced. The number of conditions which determine the altered curve is $q-\tau-2\iota$.

As the same straight line is touched twice by the curve, one condition is lost for each double tangent, and when the points of contact of the double tangent become coincident an additional condition is lost for an inflexion.

11. The number of conditions which determine a curve of the n^{th} degree which has no double point and no cusp is $\frac{1}{2}n(n+3)$.

Let u_n be this number. Suppose the curve to degenerate into a similar curve of the $(n-1)^{\text{th}}$ degree and a straight line. The degenerate curve has $n-1$ double points, and the number of conditions which determine the altered curve is $u_n-(n-1)$; but the curve of the $(n-1)^{\text{th}}$ degree is determined by u_{n-1} conditions, and the straight line by 2, so that

$$u_n-(n-1)=u_{n-1}+2,$$

or $$u_n-u_{n-1}=n+1, \quad \text{and} \quad u_1=2;$$

therefore $$u_n=\tfrac{1}{2}n(n+3).$$

Cor.—The number of conditions which determine a curve of the n^{th} degree which has δ double points and κ cusps is

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa.$$

12. The number of conditions which determine a curve of the m^{th} class which has no double tangent and no inflexion is $\frac{1}{2}m\,(m+3)$.

Let u_m be this number. Suppose the curve to degenerate into a similar curve of the $(m-1)^{\text{th}}$ class and a point. The degenerate curve has $m-1$ double tangents, and the number of conditions which determine the altered curve is $u_m-(m-1)$; but the curve of the $(m-1)^{\text{th}}$ class is determined by u_{m-1} conditions, and the point by 2, so that

$$u_m-(m-1)=u_{m-1}+2,$$

or

$$u_m-u_{m-1}=m+1, \quad \text{and} \quad u_1=2;$$

therefore

$$u_m=\tfrac{1}{2}m\,(m+3).$$

Cor. 1.—The number of conditions which determine a curve of the m^{th} class which has τ double tangents and ι inflexions is

$$\tfrac{1}{2}m\,(m+3)-\tau-2\iota.$$

13. The number of conditions which determine a curve is the same in whatever manner it is expressed. Hence for the same curve

$$\tfrac{1}{2}n\,(n+3)-\delta-2\kappa=\tfrac{1}{2}m\,(m+3)-\tau-2\iota.$$

14. The number of points common to two curves one of the n_1^{th} and the other of n_2^{th} degree is n_1n_2.

Consider the curves taken together as a curve of the $(n_1+n_2)^{\text{th}}$ degree which has, in addition to the double points and cusps belonging to each curve, a number of double points equal to the number of points common to the two curves; this number is thus half the difference between

$$(n_1+n_2)(n_1+n_2-1)-2\,(\delta_1+\delta_2)-3\,(\kappa_1+\kappa_2)$$

and

$$m_1\,(n_1-1)-2\delta_1-3\kappa_1+n_2\,(n_2-1)-2\delta_2-3\kappa_2,$$

the class of the compound curve; *i.e.*, the number is n_1n_2.

15. The number of tangents common to two curves one of the m_1^{th} class and the other of the m_2^{th} class is m_1m_2.

Consider the curves taken together as a curve of the $(m_1+m_2)^{\text{th}}$ class, which has, in addition to the double tangents and inflexions common to each, a number of double tangents equal to the number of tangents common to the two curves; this number is thus half the difference between

$$(m_1+m_2)(m_1+m_2-1)-2(\tau_1+\tau_2)-3(\iota_1+\iota_2)$$

and $$m_1(m_1-1)-2\tau_1-3\iota_1+m_2(m_2-1)-2\tau_2-3\iota_2,$$

the degree of the compound curve; *i.e.*, the number is $m_1 m_2$.

16. The maximum number of double points which a proper curve of the n^{th} degree can have is $\frac{1}{2}(n-1)(n-2)$.

If a curve has one more than the maximum number, it degenerates into two curves of lower degree. Let these two curves be a curve of the $(n-1)^{\text{th}}$ degree with the maximum number of double points and a straight line. Let u_n be the maximum number for a curve of the n^{th} degree. The straight line and curve of the $(n-1)^{\text{th}}$ degree have together $u_{n-1}+n-1$ double points, so that

$$u_n+1 = u_{n-1}+n-1,$$

or $$u_n-u_{n-1} = n-2 \quad\text{and}\quad u_2 = 0;$$

therefore $$u_n = \tfrac{1}{2}(n-1)(n-2).$$

17. The maximum number of double tangents which a proper curve of the m^{th} class can have is $\frac{1}{2}(m-1)(m-2)$.

If a curve has one more than the maximum number, it degenerates into two curves of lower class. Let the two curves be a curve of the $(m-1)^{\text{th}}$ class with the maximum number of double tangents and a point. Let u_m be the maximum number for a curve of the m^{th} class. The point and curve of the $(m-1)^{\text{th}}$ class have together $u_{m-1}+m-1$ double tangents, so that

$$u_m+1 = u_{m-1}+m-1,$$

or $$u_m-u_{m-1} = m-2 \quad\text{and}\quad u_2 = 0;$$

therefore $$u_m = \tfrac{1}{2}(m-1)(m-2).$$

18. It is convenient to express Plücker's characteristics in terms of the degree n, class m, and deficiency D.

We have

$$n(n-1)-m-2\delta-3\kappa = m(m-1)-n-2\tau-3\iota = 0,$$

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa = \tfrac{1}{2}m(m+3)-\tau-2\iota,$$

whence, by subtraction,

$$\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa = \tfrac{1}{2}(m-1)(m-2)-\tau-\iota = D,$$

and
$$\kappa = 2(n-1) - m + 2D, \qquad \text{(i.)}$$
$$\iota = 2(m-1) - n + 2D, \qquad \text{(ii.)}$$
$$\delta = \tfrac{1}{2}(n-1)(n-6) + m - 3D, \qquad \text{(iii.)}$$
$$\tau = \tfrac{1}{2}(m-1)(m-6) + n - 3D. \qquad \text{(iv.)}$$

Correspondences.

19. *Definition.*—If two sets of elements of the same kind are such that to one element of one set correspond n elements of the other set, the groups of n elements of the second set are said to be in involution.

20. There are $2(n-1)$ double elements of an involution of the n^{th} order.

Let the involution be represented by rays of a linear pencil each group of n rays of which corresponds to a single ray of another linear pencil.

To the join of the centres of the pencil correspond, (1) n rays of the first pencil which are tangents at the centre of this pencil to the locus of intersection of corresponding rays of the two pencils, (2) one ray of the second pencil which is tangent at the centre of this pencil to the same locus. There is no double ray of the second pencil, and therefore no tangent to the locus from the centre of the first pencil except those at the centre, each of which counts as two. The locus is therefore of the $2n^{\text{th}}$ class. Of the $2n$ tangents determined by the locus at the centre of the second pencil, the tangent at the centre counts as two; the remaining $2(n-1)$ tangents are those rays of the second pencil which correspond to the double rays of the first pencil, of which the number is therefore $2(n-1)$.

21. Two involutions whose corresponding elements admit of coincidence, and which have a (m, n) correspondence, have $m+n$ united corresponding elements.

Let the involutions be projective with two linear pencils. The locus of intersection of their corresponding rays is of the $(m+n)^{\text{th}}$ degree, for the join of their centres meets the locus in $m+n$ points, viz., m at one centre and n at the other, determined by the rays which correspond to the join of the centre.

Any other straight line meets the locus in $m+n$ points, viz., the united corresponding points of the involutions determined on the straight line by the pencils. The number of united corresponding elements of the original involutions is therefore $m+n$.

The locus employed in the preceding proof has interesting characteristics.

22. The class of the locus when it has no cusps and no multiple points except at the centres of the pencils is $2mn$.

At the centre of one pencil the locus determines $2mn$ tangents, viz., $2m$ at the centre and $2m(n-1)$ determined as the rays corresponding to the $2(n-1)$ double rays of the other pencil. The class of the locus is therefore $2mn$.

23. The deficiency of such a locus is $(m-1)(n-1)$.

The multiple points at the centres are equivalent to

$$\tfrac{1}{2}m(m-1)+\tfrac{1}{2}n(n-1)$$

double points, and the deficiency is therefore the excess of

$$\tfrac{1}{2}(m+n-1)(m+n-2)$$

over this number of double points, viz., $(m-1)(n-1)$.

24. The Plücker's characteristics of such a locus are, by § 18,

$$\begin{aligned}\kappa &= 0,\\ \iota &= 3(2mn-m-n),\\ \delta &= \tfrac{1}{2}m(m-1)+\tfrac{1}{2}n(n-1),\\ \tau &= 2(mn-1)(mn-2)-4(m-1)(n-1).\end{aligned}$$

25. When $m = 1$, the locus has no deficiency, and in this case (the locus in § 20, of degree $n+1$ and class $2n$),

$$\begin{aligned}\kappa &= 0,\\ \iota &= 3(n-1),\\ \delta &= \tfrac{1}{2}n(n-1),\\ \tau &= 2(n-1)(n-2).\end{aligned}$$

26. The number of conditions which determine the locus of the $(m+n)^{\text{th}}$ degree and class $2mn$ which has one given multiple point of the m^{th} and one of the n^{th} order is $(m+1)(n+1)+1$.

Each multiple point of the m^{th} order is determined by $\tfrac{1}{2}m(m+1)$ conditions, so that the number of conditions including the points which are assigned to be of multiplicity m and n respectively, the number of conditions is

$$\tfrac{1}{2}(m+n)(m+n+3)-\tfrac{1}{2}m(m+1)-\tfrac{1}{2}n(n+1)+2=(m+1)(n+1)+1.$$

27. A pencil of such curves which has two given multiple centres, one of the m^{th} and another of the n^{th} order, and $(m+1)(n+1)-2$ other given centres, has $(m-1)(n-1)$ other centres determined by the given centres.

By § 26 the pencil is determined by $(m+1)(n+1)$ conditions, *i.e.*, by $(m+1)(n+1)-2$ other points in addition to the two multiple centres; but two curves of the pencil intersect in

$$m^2+n^2+(m+1)(n+1)-2=(m+n)^2-(m-1)(n-1)$$

points at the given centres, and therefore in $(m-1)(n-1)$ other points which are also centres of the pencil of curves.

28. When $m=1$, the number of conditions which determine the locus of the $(n+1)^{\text{th}}$ degree and $2n^{\text{th}}$ class which has one given multiple point of the n^{th} order is $2n+3$.

29. When $m=1$, a pencil of such curves is completely determined by one given multiple centre of the n^{th} order and $2n+1$ other centres.

30. *Example.*—A pencil of cubics of the fourth class is determined by five single centres and one double centre.

Construct the conic a determined by the five single centres, and the conic b determined by the double centre and any four of the single centres. Take any fixed point O on a. A ray through O determines a point P on a and two points Q on b; P and Q determine at the remaining single centre and at the double centre respectively rays which have a (1, 2) correspondence and whose intersections therefore lie on a cubic of the fourth class having a double point at the given double centre. Each position of O determines one such cubic passing through all the given centres.

An Essay on the Geometrical Calculus — (continuation).

By EMANUEL LASKER. Received May 3rd, 1897. Communicated May 13th, 1897, by Mr. TUCKER.

In the former part of this essay* the calculus of plane spaces and plane forms has been established. It was shown that the symbols of plane spaces and plane forms may be composed with each other as if the operation of composition denoted multiplication only; that $\xi\eta$ is not $=\eta\xi$ always, but $=\pm\eta\xi$, according to the rule of signs. The conceptions of the normal form of a space, of the plane space I at infinity and the spherical manifoldness $\mathfrak{J}$ at I, have been introduced, and a multitude of metrical relations shown to take their origin therefrom. In this part, homogeneous algebraical forms of the plane space symbols are considered and shown to be algebraically equivalent to the algebraical formations of geometry. A sign $\times$ (and reciprocally $\ast$) is introduced as an extension of the conception of composition, and its principal laws are discussed. Some properties of the intersections of surfaces are explained, especially of the group of points common to k surfaces in the space S_k. Finally, a few applications are given to show that the symbolism used will yield good results without much effort.

Let the space in which we operate be a straight line l. Let $A, B, \dots L$ denote any point on that line, $\mathfrak{G}$ a group of such points, for instance, A, B, and C. Then $\mathfrak{G}$ will be denoted by $A.B.C$, where the symbol . is expressive of the fact that the various points thus connected are to be considered as a group, collectively. The number of such points is called the *order* of the group. In the natural extension of the symbolism used for plane spaces, $[\mathfrak{G}D]$, or sometimes simply $\mathfrak{G}D$, where D may be any point on l, will denote the magnitude which is the product of the various magnitudes formed by the points of the group $\mathfrak{G}$ with D;

$$\mathfrak{G}D = AD\,.\,BD\,.\,CD.$$

Any equation between groups such as

$$\mathfrak{G}+\mathfrak{G}'=\mathfrak{G}'',$$

or

$$\mathfrak{G}\,.\,\mathfrak{G}'=\mathfrak{G}'',$$

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* Pp. 217–260 *supra*.

signifies that both the right- and left-hand sides composed with an arbitrary point symbol on l are equal. If $\mathfrak{G}$ contains the point A,

then $\mathfrak{G}A = 0,$

and inversely, if $\mathfrak{G}A = 0,$

then $\mathfrak{G}$ must contain the point A, since a product can only vanish when any one of its factors vanishes. If P, Q are any two points on the line l, and λ, μ two parameters,

$$\mathfrak{G}\,(\lambda P+\mu Q)=0$$

will be an equation for λ, μ whose roots determine the position of the points $\lambda P+\mu Q$ of the group $\mathfrak{G}$. The definitions given allow us therefore to treat such groups $\mathfrak{G}$ as algebraical forms of two homogeneous variables and to reduce any equation between groups to algebraical identities.

Let u be any group, or, as we shall sometimes say, *point-form*, of the n^{th} order. Let $A_1, \dots A_n$, be any n points on l, and $\alpha_1, \dots \alpha_n$, any constants. Then $u\,(\alpha_1A_1+\dots+\alpha_nA_n)$ is a magnitude determined by u, the position of the A, and expressible as a homogeneous rational integral function of the $\alpha_1, \dots \alpha_n$, of the n^{th} order. It will therefore contain a term $C\alpha_1, \dots \alpha_n$, where C is a magnitude determined by u and the points $A_1, \dots A_n$, alone. We denote the $n!^{\text{th}}$ part of C by $u\times A_1.A_2.\dots A_n$. The sign × is a symbol of operations whose properties we propose now to study. First of all, it is clear from its definition that

$$(u+u')\times A_1.\dots A_n = u\times A_1.\dots A_n+u'\times A_1.\dots A_n.$$

Secondly, if we develop

$$u\,(\alpha_1A_1+\dots+\alpha_nA_n+\alpha_{n+1}A_{n+1})$$

as a homogeneous form of the α_i, we shall obtain altogether $n+1$ terms which linearly contain the $n+1$ parameters, the form being only of the n^{th} order; and, from the supposition

$$\alpha_{n+1}=0,$$

it is clear that the factor of $n!\,\alpha_1 \dots \alpha_n$ is again $u\times A_1 \dots A_n$; that of $n!\,\alpha_1 \dots \alpha_{n-1}\alpha_{n+1}$ is therefore similarly $u\times A_1 \dots A_{n-1}A_{n+1}$, &c. If now we identify α_{n+1} with α_n, we obtain

$$u\times A_1.A_2.\dots A_{n-1}(A_n+A_{n+1}) = u\times A_1.\dots A_{n-1}A_n+u\times A_1.\dots A_{n-1}.A_{n+1}.$$

It follows then that the symbol × in regard to linear changes of the forms operated upon has all the properties of an ordinary multiplication symbol, and that therefore no error will be produced by treating the $A_1, \dots A_n$ collectively, that is, as a group v.

The fact that $u \times v$ is a magnitude which changes linearly in a corresponding manner to the u and v will be expressed by saying that the operation × is distributive; or, in symbols, the operation × is distributive because

$$(\lambda u + \mu u') \times v = \lambda (u \times v) + \mu (u' \times v),$$

and
$$u \times (\lambda v + \mu v') = \lambda (u \times v) + \mu (u \times v'),$$

λ, μ denoting constants.

To give an instance, let

$$u = A.B, \quad v = C.D,$$

$$u (\alpha C + \beta D) = A (\alpha C + \beta D) . B (\alpha C + \beta D)$$
$$= \alpha^2 AC.BC + \alpha\beta (AC.BD + AD.BC) + \beta^2 AD.BD;$$

therefore
$$u \times v = \tfrac{1}{2} (AC.BD + AD.BC).$$

Or let
$$u = A.B.C, \quad v = D.E.F;$$

then
$$u \times v = \tfrac{1}{6} (AD.BE.CF + AD.BF.CE + AE.BF.CD + AE.BD.CF + AF.BD.CE + AF.BE.CD).$$

If we transpose u and v, the magnitude $u \times v$ changes sign when n is odd, but remains unchanged when n is even. Therefore $u \times u$ is always 0 when n is odd, but may be distinct from 0 when n is even.

It is also immediately seen that uP, where P is any point on l, is the same as $u \times P^n$. Therefore $u \times P^n = 0$ only if P is one of the points of the group u.

$u \times v = 0$ is a single condition for the coefficients of u and v; if u therefore is fixed, $u = A_1 . \dots A_n$, then v restricted by the linear condition $u \times v = 0$ will only contain n independent parameters; and it therefore follows that

$$v = c_1 A_1^n + \dots + c_n A_n^n,$$

where the c are arbitrary constants.

When v is of lower degree than u, say of the m^{th}, $n - m$ being $= \lambda$, then $v \times u$ will be represented by the form w,

$$v \times u = w,$$

so that identically $v.P^{\lambda}\times u = wP,$

where P remains arbitrary.

Let, for instance, $u = A.B,$

and $v = C;$

then $v\times u = \frac{1}{2}(CA.B+CB.A).$

$v\times u$ may also be defined as that form which is represented by the group of the points P for which

$$v.P^{\lambda}\times u = 0.$$

If we replace, in the equation

$$v.P^{\lambda}\times u = wP,$$

P by $\alpha_1.P_1+\ldots+\alpha_{\lambda}.P_{\lambda}$, and, in the development of $v.P^{\lambda}\times u$ and wP as rational integral functions of the α, equate the respective coefficients of $\alpha_1\ldots\alpha_{\lambda}$ to each other, then we obtain, denoting further $P_1.\ldots P_{\lambda}$ by t,

$$v.t\times u = w\times t.$$

If v and u are both of the same order, v and u will be called conjugate or harmonic to each other whenever

$$v\times u = 0.$$

If v is of inferior order to u, then $v\times u$ will be spoken of as the *polar* of v with respect to u, and, if this polar vanishes identically (equivalent to $\lambda+1$ conditions), then v will be called *apolar* with respect to u.

If u has a double point A, n being the order of u, then

$$A^{n-1}\times u = 0,$$

as is immediately clear from its manner of formation. This equation is equivalent to two conditions; therefore it is also the general condition for A to be a double point of u. In the same manner

$$A^{n+1-\lambda}\times u = 0,$$

whenever A is a λ-fold point of u.

If $u = A^{\lambda_1}B^{\lambda_2}\ldots L^{\lambda_k},$

$\lambda_1+\lambda_2+\ldots+\lambda_k$ being of course $=n$, and, if v be a form of the n^{th} order, such that

$$u\times v = 0,$$

then v must be of the form

$$A^{n+1-\lambda_1} v' + B^{n+1-\lambda_2} v'' + \dots + L^{n+1-\lambda_k} v^{(k)},$$

where $v', v'', \dots v^{(k)}$ are arbitrary forms of orders $\lambda_1-1, \lambda_2-1, \dots \lambda_k-1$, respectively. For this form satisfies the equation

$$u \times v = 0,$$

and contains

$$\lambda_1+\lambda_2+\dots+\lambda_k = n;$$

that is the requisite number of arbitrary constants.

If

$$v \times u = 0,$$

where the order of v (m) is smaller than that of u (n), and

$$v = A^{\lambda_1} \dots L^{\lambda_k},$$

then

$$v \times u = 0$$

restricts u $(n-m+1)$-fold; u contains therefore only m arbitrary constants. Hence

$$u = A^{n+1-\lambda_1} u' + B^{n+1-\lambda_2} u'' + \dots,$$

where again $u', u'', \dots$ are arbitrary forms of orders $\lambda_1-1, \lambda_2-1, \dots$, respectively.

If u is given of the nth order, v any form of the mth order, then

$$v \times u = 0,$$

being equivalent to $n-m+1$ conditions, restricts the $m+1$ constants of v $(n-m+1)$-fold, so that $2m-n$ arbitrary constants remain. One factor being necessarily arbitrary, $2m-n$ must be at least $=1$; so that a general form u of the nth order where n is odd $= 2m+1$ determines exactly one form v of order $\frac{n-1}{2}$, apolar to u; v can be found by the solution of $\frac{n-1}{2}$ linear equations. v being found, we may express u in a specially simple manner; for instance, if v is the product of $\frac{n-1}{2}$ distinct points, it may be expressed as the sum of $\frac{n-1}{2}$ nth powers of multiples of these points. This expression for u is sometimes called its *canonical* form.

If $u_1, \dots u_k$ are forms of the nth order, then any form expressible by $c_1u_1 + \dots + c_ku_k$ is said to belong to the *involution* $u_1 \dots u_k$; and

$k-1$ is called its degree of manifoldness. We suppose, of course, that no such identity as

$$c_1u_1+\ldots+c_ku_k=0$$

exists (that the u_i are *linearly independent*). The k equations

$$u_1\times v=0,\quad u_2\times v=0,\quad \ldots\quad u_k\times v=0$$

restrict the form v, supposed to be of the n^{th} order, k-fold; v will therefore be a member of an involution whose degree of manifoldness will be $n-k$, and which may well be called the *reciprocal* of the involution $u_1\ldots u_k$, or the involution *conjugate* to the involution of the u. By means of this conception many truths concerning involutions can immediately be derived from truths known to hold for reciprocal involutions. For example, let

$$k=n;$$

the involution of the v will be of manifoldness 0, and v will be uniquely determined. It is therefore immediately seen that, in general, an involution of manifoldness $n-1$ will contain n n^{th} powers (of points belonging to v); and that the group of these will be harmonic to all members of the involution $u_1\ldots u_n$, which may also be defined by this property. Or let the degree of manifoldness of the involution $u_1\ldots u_k$ be $n-2$, and let v_1, v_2 form the conjugate involution. Then the condition that the involution $u_1\ldots u_k$ should contain an n^{th} power is equivalent to the condition that v_1 and v_2 should have a point in common.

The equation

$$v.t\times u=(v\times u)\times t$$

may be geometrically illustrated as follows. Let v be a point-group of order m, u of order n. Then all groups of order n conjugate to u and comprising the m points v form an involution; and so do the system of the $n-m$ points which combined with v are conjugate to u. This latter involution is reciprocal to a certain group of order $n-m$, which is exactly $v\times u$, the polar of v to u.

To obtain these results it is not in the least essential that the symbols used should be points situated on a certain line. They may be any point or plane space symbols situated anywhere. For the definitions given for the operation × will again apply. If u is a form of order n of S_a symbols, ξ any plane space, $u\xi$ will denote the result of the composition of all terms contained in u with ξ. Let, for instance, u be a point-form in a plane S_2, and let l be any line

$$l=a_1l_1+\ldots+a_nl_n;$$

then $u\,(\alpha_1 l_1 + \ldots + \alpha_n l_n)$ will be a magnitude developable as a rational integral function of the α, and will contain a term $\alpha_1 \ldots \alpha_n$, whose coefficient will depend only upon the mutual situation of u and the l_i, and whose $n!^{\text{th}}$ part may be denoted by $u \times l_1 l_2 \ldots l_n$. It follows in the same manner as before that any linear change of the l may be treated as if the sign × denoted multiplication, and that the l may be treated as if they were multiplied together. It is true that the line-forms v in a plane cannot generally be represented as the product of lines, but nothing prevents us from extending our definitions also to general forms v, since v can be represented as the sum of such products. Hence $u \times v$ is a magnitude uniquely determined by the point-form u and the line-form v; and the operation × is distributive. To give an instance, let

$$u = A.B + C.D,$$

and

$$v = a.b + c.d,$$

the A, B, C, D denoting points; the a, b, c, d denoting lines. Then $u \times v$ will be

$$u \times v = \tfrac{1}{2}\,(Aa.Bb + Ab.Ba + Ca.Db + Cb.Da + Ac.Bd + Ad.Bc + Cc.Dd + Cd.Dc).$$

Or let u be a point-form of the order n, and P any point in the plane of operation. Putting

$$P = \alpha_1 P_1 + \ldots + \alpha_n P_n,$$

uP will be a line-form developable as a power-series of the α; whose coefficients are again line-forms. The coefficients of $\alpha_1 \ldots \alpha_n$ will be a line-form which depends solely upon the situation of u and $P_1 \ldots P_n$. Since the calculus is just the same whatever the symbols may signify, whether lines or points, the results obtained will also be the same. $u \times v$, where u and v are any two point-forms of order n, is therefore a line-form uniquely determined by u and v, and the operation × is also in this instance distributive.

After these explanations, we may announce the general result. The operation × is applicable to forms of symbols of any manifoldness. If u is of order n, $u\xi$ is the same as $u \times \xi^n$. The operation ×, which in the reciprocal geometry will be written $*$, is an extension of that of composition, and is always distributive; $u \times v$ differs from $v \times u$ either not at all or only in sign.

The result of $u \times v$ polarized with any new form w, written $(u \times v) \times w$, can only differ in sign, if at all, from $u \times (v \times w)$ or $(u \times w) \times v$, &c. It suffices to show this when u, v, w are products of plane forms.

Let, for instance,
$$u = \xi_1 . \xi_2,$$
$$v = \eta_1 . \eta_2,$$
$$w = \zeta_1 . \zeta_2,$$
where the ξ, η, ζ are any plane space symbols. Then
$$u \times v = \tfrac{1}{2} (\xi_1\eta_1 . \xi_2\eta_2 + \xi_1\eta_2 . \xi_2\eta_1)$$
and
$$(u \times v) \times w = \tfrac{1}{4} (\xi_1\eta_1\zeta_1 . \xi_2\eta_2\zeta_2 + \xi_1\eta_1\zeta_2 . \xi_2\eta_2\zeta_1 + \xi_1\eta_2\zeta_1 . \xi_2\eta_1\zeta_2 + \xi_1\eta_2\zeta_2 . \xi_2\eta_1\zeta_1).$$
The law of formation makes the statement immediately obvious, so that the wording of the proof seems unnecessary.

We come now to the problem of finding geometrical equivalents to the algebraical forms introduced and identities established. A certain liberty of choice will always exist, but the conception of the old geometers cannot in any way be improved upon. Accordingly, if $h+1$ is the manifoldness of the space containing all symbols of a form u, whose symbols may be S_h forms, then u will be represented by the manifoldness of points P for which uP vanishes, and u will be called a surface. And reciprocally, if u is a point-form, it will find its geometrical equivalent in the manifoldness of all spaces Σ of h manifoldness for which $u\Sigma$ will vanish. But, if u can be represented as the product of points, this group of points—each point counted in its proper multiplicity—is a more direct representation of u. In any case, a S_h form or point-form u and its geometrical equivalent determine each other uniquely, if an arbitrary factor is left out of consideration.

If u is a form of S_a symbols, where a differs from 0 or h, then we might in the same manner represent it by the manifoldness of plane *forms* Σ of manifoldness $h-a$, for which
$$u\Sigma = 0,$$
and to obtain a visible representation we should have to define Σ again by a number of plane *spaces* Σ' of manifoldness a belonging to the involution reciprocal to Σ. So then u might ultimately find its geometrical equivalence in the manifoldness of a group of plane spaces Σ', which stand in a certain relation to each other, in virtue of which the reciprocal Σ to the involution determined by them makes $u\Sigma$ vanish. This representation would certainly have the advantage of defining u uniquely, a factor being left out of consideration. But it is wholly unsuited to aid the geometrical imagination, especially when u is itself defined as the intersection of surfaces or as a curve or geometrical formation of some kind in general. For this reason we must

represent a form u by the manifoldness of plane *spaces* Σ, for which

$$u\Sigma = 0.$$

Remembering that certain relations are identically satisfied by the coordinates of a plane space other than a point or a S_h—in space, for instance, if A, B, C, D represent a pyramid,

$$AB\,.\,CD + AC\,.\,DB + AD\,.\,BC$$

applied to *any* line would vanish—we must bear in mind, that certain forms $\mathfrak{S}_1 \ldots \mathfrak{S}_k$ will in the light of the above definition vanish identically. The definition given above will therefore create a correspondence between u and its geometrical equivalent only modulo $\mathfrak{S}_1 \ldots \mathfrak{S}_k$. This fact, however, does not touch the validity of our equations, if it is understood that they are always to be read modulo, the fixed system of moduli $\mathfrak{S}_1 \ldots \mathfrak{S}_k$.

If, then, u is a geometrical formation, the manifoldness of Σ belonging to it will be that of those spaces Σ which have with u a point in common; for this is true if u reduces to a product of plane spaces, and, since u can always be represented as the sum of such products, in many ways, the general truth of the proposition is easily made evident. For that reason a geometrical formation $\mathfrak{C}$ will define the form u of which it is the representative (modulo $\mathfrak{S} \ldots \mathfrak{S}_k$), and we might consequently apply our equations directly to the formations whose symbols we use.[*]

The geometrical representation which we now have agreed to use will immediately lead to some notable consequences. Let, for instance, the space in which we operate be the plane. A point-group of the second order, such as

$$u = aA^2 + bB^2 + cC^2,$$

will, in general, not be represented by two points, unless its discriminant abc vanishes.

If we compose u with any point P,

$$uP = a\,.\,AP^2 + b\,.\,BP^2 + c\,.\,CP^2,$$

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[*] [A geometrical "formation," as the word is used in this essay, denotes always a manifoldness of points, forming a curve, a surface, &c., in space of any degree of manifoldness. According to the principle established above, an algebraical form (of the symbols used in the geometrical calculus) will correspond to it. It is, however, easily seen that the converse is not true. A form like, for instance, $aAB^2 + bCD^2 + cEF\,.\,GH$, where A, B, C, D, E, F, G, H may be situated in space S_3, will not generally define a conic in space, but a line-manifoldness of the second order. Hence it will be understood that we may speak of algebraical forms which exist as geometrical formations and of such as do not.]

this will represent a line-group which will obviously contain P. If this line-group contains also some other point Q, then

$$uPQ = 0,$$

from which it follows that PQ belongs to the manifoldness of straight lines represented by u. In other words, uP represents the two lines through P which belong to the manifoldness u. Let in a similar manner u be a line-form, a curve, in the plane, and l any line; then in the reciprocal geometry u/l will denote the point-group which u and l have in common.

If $\mathfrak{C}$ is a curve in space S_3, $\mathfrak{C}P$ must denote, according to the definitions given, the manifoldness of points Q for which

$$\mathfrak{C}PQ = 0,$$

which is evidently the cone standing on $\mathfrak{C}$ whose vertex is P. If S is any plane in space, $\mathfrak{C}/S$ will similarly denote the point-group common to $\mathfrak{C}$ and S. Generally, if $\mathfrak{C}$ is any geometrical formation, ξ any space, P any point on $\mathfrak{C}$, then $\mathfrak{C}\xi$ will contain the space $P\xi$; for, if Σ has the point Q in common with $P\xi$, then

$$\xi\Sigma = 0;$$

therefore $$\mathfrak{C}\xi\Sigma = 0.$$

In the same manner $\mathfrak{C}/\xi$ denotes the intersection of $\mathfrak{C}$ by ξ.

To give an instance of the working of this, let u be a conic, and a, b be any two lines in its plane intersecting in P. Through P draw any third line $\lambda a + \mu b$. Its two points of intersection with u are $u/\lambda a + \mu b$, or, what is equivalent to this, $u * (\lambda a + \mu b)^2$. If L and M are these two points of intersection, then it follows that

$$L\,.\,M = \lambda^2\,.\,\mathfrak{A} + \lambda\mu\,.\,\mathfrak{B} + \mu^2\,.\,\mathfrak{C},$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are fixed point-forms of the second order. So, then, if $L_1\,.\,M_1$, $L_2\,.\,M_2$, $L_3\,.\,M_3$, $L_4\,.\,M_4$ are any four such point-groups, a linear relation will connect them; and, if we compose this with L_4, it appears that

$$L_1L_4\,.\,M_1L_4, \quad L_2L_4\,.\,M_2L_4, \quad L_3L_4\,.\,M_3L_4$$

are linearly dependent. Or, in other words, the three point-pairs which are common to a conic u and three concurrent lines projected from any point of u are in involution.

Or, let u be a surface of the second order in space, and l be any line upon it. Any plane through l will be of the form $\lambda a + \mu b$, a and b

representing any two distinct planes through l. This plane will cut u in another line l'. Now

$$u/\lambda a+\mu b \equiv l\,.\,l',$$

or, what is immediately seen to be equivalent to this, four such lines l' are linearly dependent. The lines of the quadric, as easily follows, form two groups, belonging to two conjugate involutions of plane line-forms of manifoldness 2.

This proceeding, which is also applicable to the intersection of curved spaces, gives many interesting results, and can be generalized without difficulty.

If u and v are given forms, what does $u\times v$ signify? To answer this question for the plane will be sufficient to indicate the general idea. Let u and v both be point-forms of the nth order in the plane. $u\times v=w$ is then a line-form of the nth order. Let P be any point upon the curve w; therefore

$$w\times P^n=0;$$

then we have, since

$$w=u\times v,$$

$$u\times v\times P^n=0.$$

In the geometry of lines through P, AP composed with BP is, according to our previous definitions, identical with ABP in point geometry. It therefore follows that $u\times v\times P^n$ in point geometry is the same as $uP\times vP$ in the geometry of lines through P. Hence $u\times v$ is the locus of points P which have the property that the two line-groups uP and vP are conjugate to each other.

As a corollary, if u is a conic, $u\times u$ is its reciprocal. If u is a point-group of the third order on a line, $u^2\times u^2$ is its discriminant. Hence, u representing a plane curve of the third order, $u^2 \ast u^2$ is the point-form for the tangent lines of the cubic u. Similar laws exist for surfaces of any order.

We shall, of course, speak again of involutions of forms, and, in connexion with the equation $u\times v=0$, of conjugate forms and conjugate and reciprocal involutions respectively.

Let u now be any *surface*, in the space S_k, and the order of u be n. Let A, B be any two points in S. The line AB will cut u in n points, to be found by evaluating u/AB, or else by the following method. Any point of the line AB may be represented by $\alpha A+\beta B$, α, β being parameters. For the coordinates α, β of the points $\alpha A+\beta B$ situated upon u, we obtain an equation of the nth degree,

$$u\times(\alpha A+\beta B)^n=\alpha^n\,.\,u\times A^n+n\,.\,\alpha^{n-1}\,.\,\beta\,.\,u\times A^{n-1}\,.\,B+\ldots=0.$$

If A is a point upon u, $\beta = 0$ will be one root of the equation, the equation of the other $n-1$ roots reducing to

$$n\,.\,\alpha^{n-1}\,.\,u \times A^{n-1}\,.\,B + (n)_2\,\alpha^{n-2}\beta\,.\,u \times A^{n-2}\,.\,B^2 + \ldots = 0\,;$$

$$u \times A^{n-1}\,.\,B = 0$$

is therefore the necessary condition for the line AB to have two consecutive points in A in common with u, to touch u. This equation is a single condition restricting the position of the point B. All the points B of this kind are therefore situated in a certain plane space of $k-1$ manifoldness, which is said to touch u at the point A. We may evolve its form by calculating $A^{n-1}\,.\,P \times u$, P being left undetermined, *i.e.*, by calculating the polar of A^{n-1} to u.

If, however, any line through A intersects u in two coincident points, then the equation for B must be an identity; *i.e.*, $A^{n-1} \times u$ must vanish identically, and, *vice versâ*, $A^{n-1} \times u = 0$ is the necessary and sufficient condition for A to be a double point upon u.

In the space of $k-1$ manifoldness $A^{n-1} \times u$, we may subject B to the further condition that $A^{n-2}\,.\,B^2 \times u$ should also vanish. In that case the line AB will have three consecutive points in common with u. The equation for B being then of the second order, the points B will form in the space $A^{n-1} \times u$ a surface of the second order. If C is any point upon AB, C^2 will be linearly dependent upon A^2, $A\,.\,B$, B^2. Hence, if

$$u \times A^n = 0, \quad u \times A^{n-1}\,.\,B = 0, \quad u \times A^{n-2}\,.\,B^2 = 0,$$

then generally $$u \times A^{n-2}\,.\,C^2 = 0.$$

The surface of the second order in question is therefore a cone whose vertex is A. If, further, we select upon that cone only such points B for which also

$$A^{n-3} \times B^3 \dot{\times} u = 0,$$

then AB will have four consecutive points in common with u. These lines AB are therefore found by the intersection of the cone with a certain surface of the third order; or else by their property that they are wholly contained by the surface $A^{n-3} \times u$. Generally AB will have h consecutive points in common with u whenever it is wholly contained by the polar of A^{n+1-h} to u.

If A is a double point upon u, $A^{n-1} \times u$ will vanish identically, as we have seen. The points B for which $A^{n-2}\,.\,B^2 \times u = 0$ will then be such that AB has three coincident points in common with u. The equation for B being of the second order, such points form a surface of the second order, which is, as immediately follows from the co-existence of

$$A^n \times u = 0, \quad A^{n-1}\,.\,B \times u = 0, \quad A^{n-2}\,.\,B^2 \times u = 0,$$

a cone whose vertex is A—the cone of contact at A. If also $A^{n-2} \times u$ identically vanishes, then A will be a triple point upon u, and generally the condition for A to be a λ-fold point upon u is that $A^{n+1-\lambda} \times u$ should vanish identically, and the cone of contact is evidently $A^{n-\lambda} \times u$.

This is indeed only a restatement of Joachimsthal's method.

We have made use of the conception of the polar of a point-form v of order m to a surface u of order n. It may be defined as that surface w of order $n-m$ whose points P satisfy the condition

$$v\,.\,P^{n-m} \times u = 0.$$

As before, it follows, if $v \times u = w$,

that $v\,.\,t \times u = w \times t$.

If $v \times u$ identically $= 0$, v will be called apolar to u. This equation expressing the vanishing of a surface of the $(n-m)^{\text{th}}$ order is equivalent to a $(n+k-m)_k$-fold condition.

If $\mathfrak{C}$ is a curve in space, and l any line intersecting it, we shall have

$$\mathfrak{C} \times l^n = 0.$$

If, therefore, P is any point on l, Q any point in space,

$$\mathfrak{C} \times P^n \times Q^n = 0.$$

From this it follows that $\mathfrak{C} \times P^n$ (or $\mathfrak{C}P$) must identically vanish.

If l is any line intersecting $\mathfrak{C}$ twice, then $l^{n-1}\,.\,\mathfrak{C} = 0$ identically, and, if l has a points in common with $\mathfrak{C}$, then

$$l^{n+1-a} \times \mathfrak{C} = 0.$$

For let P be any point on l not on $\mathfrak{C}$, S any plane through l, and a, b any two lines through P lying upon S. Obviously there are n (different or coincident) lines through P in S intersecting the curve $\mathfrak{C}$. These lines are $\lambda a + \mu b$, where λ and μ are to be determined by the equation of the n^{th} order

$$\mathfrak{C} \times (\lambda a + \mu b)^n = 0.$$

Everything else follows as before.

If P is a double point upon $\mathfrak{C}$, then P^{n-1} will vanish identically, since $(PQ)^{n-1} \times \mathfrak{C}$ vanishes identically wherever Q may be situated. For a λ-fold point upon $\mathfrak{C}$, we have, in a similar manner,

$$P^{n+1-\lambda} \times \mathfrak{C} = 0.$$

Whenever $P^{n-1} . Q \times \mathfrak{C}$ vanishes identically, the line PQ will touch $\mathfrak{C}$ in P, and *vice versâ.* The curve $\mathfrak{C}$ is a singly infinite series of points. It is therefore possible to represent its points Π by a power-series in a variable parameter λ,

$$\Pi \equiv P + \lambda P' + \lambda^2 P'' + \dots,$$

where P is one of the curve-points corresponding to the value $\lambda = 0$. Since

$$\Pi^n \times \mathfrak{C} = 0,$$

we may develop $\Pi^n \times \mathfrak{C}$ according to powers of λ, and equate each coefficient of the various powers of λ to zero. It therefore follows that

$$P^{n-1} . P' \times \mathfrak{C} = 0.$$

But for infinitesimal values of λ the point Π consecutive on $\mathfrak{C}$ to P may be considered as being situated upon the line joining P and P'. So then the statement is verified.

If Q is not a point on that tangent-line, then $P^{n-1} . Q \times \mathfrak{C}$ will represent a plane form of the order n, that is (in space S_3), a *surface.* This surface is represented by the cone of order $n-1$ whose vertex is P, and which contains the curve $\mathfrak{C}$, in conjunction with the plane composed of the tangent-line PP' and Q. For let R be any other curve point, and $\alpha P + \beta R$ any point collinear with P and R; then

$$P^{n-1} . Q \times (\alpha P + \beta R)^n$$

being identically $= n . \alpha\beta^{n-1} (PR)^{n-1} . QP + \beta^n . P^{n-1} . Q \times R^n$,

and both $\mathfrak{C} \times R^n$ and $\mathfrak{C} \times (PR)^{n-1}$ vanishing identically,

$$\mathfrak{C} \times P^{n-1} . Q \times (\alpha P + \beta R)^n = 0.$$

In other words, $\mathfrak{C} \times P^{n-1} . Q$ contains all the points $\alpha P + \beta R$, *i.e.*, the cone in question. $\mathfrak{C} \times P^{n-1} . Q$ contains also any point

$$S \equiv \alpha P + \beta P' + \gamma Q,$$

since $\mathfrak{C} \times P^{n-1} . Q$ identically $= \mathfrak{C} \times P^{n-1} . (\lambda P + \mu P' + Q)$,

and $P^{n-1} . S \times S^n$ vanishes identically.

The more complex formations of geometry may be treated in the same manner. We shall, however, for the present, abstain from rigidly formulating the general laws whose existence is indicated above.

II. A geometrical formation $\mathfrak{C}$ is either irreducible or it is complex. $\mathfrak{C}$ will be called irreducible if algebraically the form representing it is irreducible. If $\mathfrak{C}$ is algebraically reducible, it is geometrically complex. If $\mathfrak{C}$ is the product of several forms

$$\mathfrak{C} \equiv \mathfrak{C}' . \mathfrak{C}'' \dots \mathfrak{C}^{(k)},$$

then, since $$\mathfrak{C}P = 0,$$

either $$\mathfrak{C}'P = 0 \quad \text{or} \quad \mathfrak{C}''P = 0, \;\&\text{c.}$$

$\mathfrak{C}' . \mathfrak{C}'' \dots \mathfrak{C}^{(k)}$ containing an infinity of points, at least one of them, say $\mathfrak{C}'$, will contain an infinity of points. $\mathfrak{C}'$ will therefore geometrically exist. This line of thought being followed further makes it evident that, if a form $\mathfrak{C}$ which has existence as a formation is reducible, its various factors $\mathfrak{C}' . \mathfrak{C}'' \dots \mathfrak{C}^{(k)}$ will also represent forms having geometrical existence; or that, in other words, a reducible formation $\mathfrak{C}$ is always a complex of several irreducible formations (some of which may be identical).

If $\mathfrak{C}$ is irreducible, a surface U contains $\mathfrak{C}$ when containing all of its points. U contains $\mathfrak{C}' . \mathfrak{C}''$ when it contains $\mathfrak{C}'$ as well as $\mathfrak{C}''$. U contains $\mathfrak{C}^{\lambda}$ when it contains $\mathfrak{C}$, and besides $\lambda - 1$ formations *consecutive* to $\mathfrak{C}$ (*not* necessarily coincident with $\mathfrak{C}$). But, if U contains every point of $\mathfrak{C}$ as a λ-fold point, then it may be said to contain $\mathfrak{C}$ λ-fold, *i.e.*, to contain λ formations *coincident* with $\mathfrak{C}$. This definition of "containing" will only be of importance when the work of Brill and Nöther is consulted.

Let now $\mathfrak{C}$ be an irreducible curve in any space S of order n. A surface U in S of order N will have $n . N$ points in common with it, since this is the number of points U would have in common with $\mathfrak{C}$ if U were the product of planes, and since this number must be independent of the exact values of the coefficients of U. But, if N is chosen large enough, $n . N$ will fall short of the degree of manifoldness of surfaces U of order N, and other surfaces U', U'', ... of order N will therefore exist which also contain these $n . N$ points of intersection.

If U and U' are any two of these surfaces, a surface $aU + bU'$ may be constructed where the constants a, b are so adjusted that

$$aU + bU' = V$$

will contain, besides the $n . N$ points, yet another of the points of $\mathfrak{C}$. $\mathfrak{C}$ will then have $n . N + 1$ points in common with V, and, $\mathfrak{C}$ being irreducible, it is in keeping with one of the fundamental principles of Algebra to conclude that $\mathfrak{C}$ must be wholly contained by V.

It may in a similar manner be shown that generally an involution of surfaces of order N exists of which each member contains any given geometrical formation $\mathfrak{C}$, and which is defined by this property; on the supposition only that N is chosen large enough.

Referring to Dr. Salmon's classical treatises upon the theory of curves in space, and on the order of restricted systems of equations, we shall make use of the following fundamental proposition: That, if $\mathfrak{C}$ be the *complete* intersection of surfaces $u, v, \ldots w$, any surface S containing $\mathfrak{C}$ must be of the form

$$S = a\,.\,u + b\,.\,v + \ldots + c\,.\,w,$$

the $a, b, \ldots c$ denoting forms.

A proof of this proposition may be given as follows:—The proposition is true if the complete intersection of $u, v, \ldots w$ is a group of points, as is implicitly verified by the discussion which follows, upon the supposition of the truth of Bézout's theorem only. It is algebraically evident that in this instance the $a, b, \ldots c$ contain the coefficients of $u, v, \ldots w$ and S rationally. If, then, some of the variables in $u, v, \ldots w$ and S are treated as parameters, the truth of the proposition follows quite generally.

We may express the substance of this proposition by the statement that the geometrical substrate of the system of moduli $u, v, \ldots w$ whose resultant does not identically vanish is their complete intersection.*

The form of the complete intersection $\mathfrak{C}$ of the surfaces $u, v, w, \ldots$ can be found as follows:—Any $k+1$ surfaces in space S_k will have a point in common if a magnitude, the resultant of the system, vanishes. If, then, $u, v, \ldots w$ are h surfaces, join any $l = k+1-h$ arbitrary plane spaces $S_1, S_2, \ldots S_l$ to them. The resultant R of this system does not contain the coefficients of $S_1, S_2, \ldots S_l$ *per se*, but only in such combinations as are determinants of the matrix $S_1, S_2, \ldots S_l$. In other words, R depends only on the coefficients of $u, v, \ldots w$, and the coordinates of the space $S_1 \mid S_2 \mid \ldots \mid S_l$ in regard to some pyramid of reference $a, b, \ldots c$. Let these coordinates be called

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* It might be seen and verified in a similar manner, that generally to any geometrical formation $\mathfrak{C}$ belongs a system of moduli $u_1 \ldots u_k$, and *vice versâ*. The work of Brill and Nöther has to a certain extent modified the fundamental proposition, but not so as to embarrass us in its use.

$p_1, p_2, \dots p_N$, and let

$$R = F(p_1, p_2, \dots p_N).$$

Let the border-spaces of h manifoldness of the pyramid $a, b, \dots c$ be $\xi_1, \xi_2, \dots \xi_N$. Let

$$X \equiv S_1 \mid S_2 \mid \dots \mid S_l = p_1\xi_1 + \dots + p_N\xi_N.$$

Let $\bar{\xi}_i$ be the space residual to ξ_i; then

$$p_1 = X\bar{\xi}_1 \,.\, [a, b, \dots c],$$

$$p_2 = X\bar{\xi}_2 \,.\, [a, b, \dots c],$$

&c.,

and $\mathfrak{E}$ is therefore in our notation $F(\bar{\xi}_1, \bar{\xi}_2, \dots \bar{\xi}_N)$.

This proceeding is not of much practical value, since it involves the necessity of the introduction of a pyramid of reference. Theoretically it is sufficient to show some of the most important properties of the form belonging to $\mathfrak{E}$, which again are sufficient to make the direct evaluation unnecessary. These are: If

$u_1 \dots u_k$ are the k surfaces,

$\lambda_1 \dots \lambda_k$ their orders,

$\mathfrak{E}$ will be a form whose coefficients are rational integral functions of the coefficients of the u_i, containing those of u_1 in the order $\lambda_2\lambda_3 \dots \lambda_k$, &c., and, considered as functions of the coefficients of all the u_i, are of the order $\lambda_1\lambda_2 \dots \lambda_k$.

If $\mathfrak{E}$ vanishes identically, $u_1 \dots u_k$ must have in common a formation of higher manifoldness than in general, and *vice versâ.*

Another method for the formation of $\mathfrak{E}$ is this. According to Clebsch's work, the invariants and covariants of a system of surfaces may be symbolically expressed—in the notation used here by means of the symbol $*$ and of the symbols of Algebra, applied in some specified manner to the set of surfaces under consideration. If u, for instance, is a quadric in space S_h, $u * u * u \dots * u$ ($h+1$ times) is its discriminant, $u * u$ therefore the manifoldness of its tangent-lines, $u * u * u$ the manifoldness of its tangent-planes, &c. If u and v are two point-pairs upon a straight line, $4(u * v)^2 - 3 \,.\, u^2 * v^2$ is their resultant; and this form is therefore also the product of the four points common to u and v, if u, v denote conics; the curve of intersection if they denote quadrics in space, &c. If, then, the resultant of k forms of orders $\lambda_1, \dots \lambda_k$ in space of manifoldness $k-1$, in its symbolical expression is known, this same expression gives also the intersection L of surfaces $u_1, \ldots u_k$ of orders $\lambda_1, \ldots \lambda_k$ in any space.

If U is any surface of order α in the space S_k, the manifoldness of surfaces of order n containing it is $p \, . \, u$, where p is an arbitrary surface of order $n-\alpha$. The number of (homogeneous) constants contained by p is $(n-\alpha+k)_k$; or, if we denote $(n+k)_k$ by $\varphi(n)$, it is $\varphi(n-\alpha)$. The order of the condition that a surface of order n should contain all points of u is therefore $\phi(n)-\phi(n-\alpha)$; or $\Delta_\alpha \phi(n)$, by the introduction of a symbol of operation Δ_α, whose definition is obvious.

The reciprocal of the involution of surfaces v containing u is formed by the n^{th} powers of the points upon u, which follows from the definition of the involution v. Hence the n^{th} powers of any $\Delta_\alpha \phi(n)+1$ points upon u will be linearly dependent.

If u and v are two surfaces of orders α, β, any surface of order n containing their intersection is of the form

$$p \, . \, u + q \, . \, v.$$

This form would contain $\phi(n-\alpha)+\phi(n-\beta)$ constants, were it not that this number is diminished by the existence of identical relations, such as

$$pu+qv=0.$$

In fact this relation will be satisfied whenever

$$p=r \, . \, v, \quad q=-r \, . \, u,$$

where r denotes any surface of order $n-\alpha-\beta$. So then the number of independent constants in the identical relations reduces to

$$\phi(n-\alpha)+\phi(n-\beta)-\phi(n-\alpha-\beta),$$

and it follows that the n^{th} powers of any $\Delta_\alpha \Delta_\beta \phi(n)+1$ points upon the intersection of u and v must be linearly connected.

If u, v, w are any three surfaces of orders α, β, γ, the form

$$pu+qv+rw$$

would contain $\phi(n-\alpha)+\phi(n-\beta)+\phi(n-\gamma)$ constants,

but for the existence of identities

$$pu+qv+rw=0.$$

Supposing the intersection of u, v, and w not to vanish identically, the intersection of u and v will not be contained by w. If, then, the above identity holds good, the intersection of u and v must be wholly situated upon r; therefore

$$r = au+bv,$$

and, similarly,

$$q = cu-bw,$$

$$p = -cv-aw;$$

a, b, c are perfectly arbitrary forms. The diminution to be effected would therefore appear to be $\phi\,(n-\alpha-\beta)+\phi\,(n-\alpha-\gamma)+\phi(n-\beta-\gamma)$ were it not that some of these identities are counted several times, since

a might be changed into $a+A\,.\,v$,

simultaneously b ,, ,, ,, $b-A\,.\,u$,

c ,, ,, ,, $c-A\,.\,w$,

without adding to the number of identities, A denoting any form of order $n-\alpha-\beta-\gamma$. So then the diminution is only

$$\phi\,(n-\alpha-\beta)+\phi\,(n-\beta-\gamma)+\phi\,(n-\gamma-\alpha)-\phi\,(n-\alpha-\beta-\gamma)\,;$$

and therefore the order of the condition that a surface of order n should contain the intersection of u, v, w is $\Delta_\alpha\Delta_\beta\Delta_\gamma\phi\,(n)$. Repeating this process, we obtain the general theorem: If $u_1 \dots u_p$ are any h forms of orders $\lambda_1, \lambda_2, \dots \lambda_h$ whose resultant does not identically vanish, then the order of the condition that a surface of order n should contain their intersection is

$$N = \Delta_{\lambda_1}, \Delta_{\lambda_2}, \dots \Delta_{\lambda_h}\,\phi\,(n),$$

or, what is the same thing, the n^{th} powers of any $N+1$ points upon the intersection of the U are linearly dependent.

Now, it will be noticed, k being the manifoldness of the space in which these surfaces are situated, that $\phi\,(n)$ is a function of n of order k; and that N is some integer function of n of order $k-h$. It appears, therefore, at least when $\mathfrak{C}$ is a geometrical formation generated by the intersection of surfaces, that the order of the condition for a surface of order n to contain $\mathfrak{C}$ is an integer function N of n, whose degree is equal to the degree of manifoldness of $\mathfrak{C}$; a linear function, for instance, for curves. We shall discuss this result later, and show that it is valid without any restriction on the nature of the generation of $\mathfrak{C}$; and, that, moreover the coefficients of the function N of n are numbers in intimate relation to $\mathfrak{C}$.

If $h = k$, N is a constant whose value is found, according to elementary theorems of the calculus of differences,

$$N = \lambda_1 \dots \lambda_k.$$

Hence it follows that this is the number of points of intersection common to k surfaces of orders $\lambda_1 \dots \lambda_k$ in space S_k—another demonstration of this famous proposition. But the order of the condition that a surface of order n should contain these points will not be equal to their number whenever n is so small that some of the identities counted above do not exist; for instance, when n is smaller than

$$n' = \lambda_1 + \lambda_2 + \dots + \lambda_k.$$

When $n = n'-1$, $\phi(n-n')$ will be 0; also, when

$$n = n'-2 \dots n = n'-k.$$

But, when $n = n'-k-1,$

then $\phi(n-n')$ will be ∓ 1;

and therefore the order of the condition that a surface of order

$$\lambda_1 + \lambda_2 + \dots + \lambda_k - k - 1$$

should contain the $\lambda_1 . \lambda_2 . \dots \lambda_k$ points of intersection is not $\lambda_1 . \lambda_2 . \dots \lambda_k$, but, since one of the identities which were counted above will cease to exist, only $\lambda_1 \dots \lambda_k - 1$. So, then, any surface ν of order

$$\lambda_1 + \dots + \lambda_k - k - 1,$$

containing $\lambda_1 \dots \lambda_k - 1$ of the points of intersection of k surfaces of orders $\lambda_1 + \dots + \lambda_k$ in space S_k, will also contain the last one. In other words, the $(\lambda_1 + \dots + \lambda_k - k - 1)$th powers of these points are linearly dependent.

If, finally, $h = k+1$, then $N = 0$. Hence the *Theorem:* If the resultant of any $k+1$ surfaces $u_1 \dots u_{k+1}$ in space S_k does not vanish, any surface V of order n, n being assumed large enough, is expressible in the form

$$V = p_1 u_1 + \dots + p_{k+1} u_{k+1}.$$

The orders of $u_2 \dots u_{k+1}$ being denoted by $\lambda_1 \dots \lambda_{k+1}$, this theorem will hold good if n is at least $= \Sigma\lambda_i - k$. If, however, $n = \Sigma\lambda_i - k - 1$, it is seen, as above, that one condition has to be satisfied for the above identity to exist. For instance, if u_1, u_2, u_3 are three conics in a plane, V any cubic in the same plane, V will not be expressible in the form

$$V = p_1 u_1 + p_2 u_2 + p_3 u_3$$

(the p denoting lines), unless the six points of intersection of V and u_1 and the four points common to u_2 and u_3 are situated upon one cubic. If $V = S^3$, the cube of a line, A, B, C, D the four points common to u_2 and u_3, and

$$aA + bB + cC + dD = 0,$$

the linear equation connecting them, then $S^3 - p_1 . u_1$ will contain A, B, C, D; hence

$$\frac{(SA)^3}{u_1 A} = p_1 A,$$

$$\frac{(SB)^3}{u_1 B} = p_1 B,$$

$$\dots \quad \dots \quad \dots,$$

and $$a\frac{(SA)^3}{u_1 A} + b\frac{(SB)^3}{u_1 B} + c\frac{(SC)^3}{u_1 C} + d\frac{(SD)^3}{u_1 D} = 0;$$

S must therefore belong to the manifoldness

$$\theta = \frac{a}{u_1 A} . A^3 + \frac{b}{u_1 B} . B^3 + \frac{c}{u_1 C} . C^3 + \frac{d}{u_1 D} . D^3.$$

Hence generally the condition for V to be expressible modulo u_1, u_2, u_3 is to be conjugate to θ.

Generally to any set of $k+1$ surfaces of orders $\lambda_1 \dots \lambda_{k+1}$ in the space S_k must belong a point-form θ of order $\lambda_1 + \dots + \lambda_{k+1} - k - 1$ in intimate relation to it (which is, in fact, that $u_1 \dots u_{k+1}$ are apolar to θ) to be found in the manner given above. If the resultant of the forms vanish, θ will reduce to the $(\lambda_1 + \dots + \lambda_{k+1} - k - 1)^{\text{th}}$ power of the point common to the surfaces. θ will vanish identically when $u_1 \dots u_{k+1}$ have more than one point in common.

Any form V of order $\lambda_1 + \dots + \lambda_{k+1} - k$ will, as we have seen, be expressible by $p_1 u_1 + \dots + p_{k+1} u_{k+1}$. If the coefficients of the u_i undergo continuous changes until the resultant R of the u_i vanishes, this will cease to be true, for V must then obey the one condition to pass through the point common to the u to be thus expressible. This can only be explained by the supposition that, in consequence of the vanishing of R, a *new* relation such as

$$p_1 u_1 + \dots + p_{k+1} u_{k+1} = 0$$

is created. Hence, if the form of order $\lambda_1 + \dots + \lambda_{k+1} - k$,

$$p_1 u_1 + \dots + p_{k+1} u_{k+1},$$

vanishes, unless p_i belongs to the system of moduli $u_1 \dots u_{k+1}$ (u_i excepted), one condition must be fulfilled, viz., that R should vanish, and *vice versâ*.

III. If we have any group of points $A_1, \dots A_h$ in the space S_k, any $k+2$ of them will be connected by a linear equation, and the h points therefore by $h-k-1$ such equations. This system of equations will have the following form:—

$$\begin{gathered} a_1 A_1 + a_2 A_2 + \ldots + a_h A_h = 0, \\ b_1 A_1 + b_2 A_2 + \ldots + b_h A_h = 0, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ l_1 A_1 + l_2 A_2 + \ldots + l_h A_h = 0. \end{gathered}$$

Multiplying the first by α_1, the second by α_2, ..., the last by α_l, where $l = h-k-1$, we can write the whole system of equations in one line

$$\mathfrak{A}_1 \,.\, A_1 + \mathfrak{A}_2 \,.\, A_2 + \ldots + \mathfrak{A}_h \,.\, A_h = 0,$$

where

$$\begin{gathered} \mathfrak{A}_1 = a_1 \,.\, \alpha_1 + b_1 \,.\, \alpha_2 + \ldots + l_1 \,.\, \alpha_l, \\ \mathfrak{A}_2 = a_2 \,.\, \alpha_1 + b_2 \,.\, \alpha_2 + \ldots + l_2 \,.\, \alpha_l, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{gathered}$$

The $\alpha_1, \ldots \alpha_l$ will be perfectly arbitrary and independent of each other. We may therefore use the geometrical calculus, interpreting the α_i as corners of a pyramid in an auxiliary space. The $\mathfrak{A}_i$ will consequently be $\equiv$ points in some space $\mathfrak{S}_{i-1}$ of perfectly arbitrary situation. It will also be noticed that, on account of the perfect freedom in the choice of the α, the group $\mathfrak{A}$ may be subjected to any linear transformation without ceasing to make the equation connecting the A_i and $\mathfrak{A}_i$ true.

From (θ) $\qquad A_1 \,.\, \mathfrak{A}_1 + A_2 \,.\, \mathfrak{A}_2 + \ldots + A_h \,.\, \mathfrak{A}_h = 0,$

we may deduce any relation connecting the A_i, for instance, the one connecting $A_1, A_2, \ldots A_{k+2}$ by composing the $\mathfrak{A}$ in (θ) with $\mathfrak{A}_{k+3} \mathfrak{A}_{k+4} \ldots \mathfrak{A}_h$. The A and $\mathfrak{A}$ being situated in totally different spaces, any operation may be performed on the one group, while the symbols of the other group are treated as constants.

If A, B, C, D are four points on a line, they are connected by two relations, and we shall have to introduce four points $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ on some other line, so that

$$a \,.\, A \,.\, \mathfrak{A} + b \,.\, B \,.\, \mathfrak{B} + c \,.\, C \,.\, \mathfrak{C} + d \,.\, D \,.\, \mathfrak{D} = 0,$$

the a, b, c, d denoting constants. Composing with C and $\mathfrak{D}$, we obtain

$$a \,.\, AC \,.\, \mathfrak{A}\mathfrak{D} + b \,.\, BC \,.\, \mathfrak{B}\mathfrak{D} = 0;$$

similarly, composing with $\mathfrak{C}$ and D,

$$a \,.\, AD \,.\, \mathfrak{A}\mathfrak{C} + b \,.\, BD \,.\, \mathfrak{B}\mathfrak{C} = 0;$$

hence $$\frac{AC}{AD}:\frac{BC}{BD}=\frac{\mathfrak{AC}}{\mathfrak{AD}}:\frac{\mathfrak{BC}}{\mathfrak{BD}};$$

or the necessary and sufficient condition for the two groups to be corresponding in the above-mentioned manner is that their cross-ratios should be equal.

(If the group A is given, the group $\mathfrak{A}$ defined by the relation

$$\Sigma a_i \,.\, A_i \,.\, \mathfrak{A}_i = 0$$

may be linearly constructed, $l+1$ of them being arbitrary. To k points A_i on a straight line correspond k points $\mathfrak{A}_i$ of a S_{k-3}. If, of the k points A, $k-1$ remain fixed, the last one describing its straight line, while also, of the k points $\mathfrak{A}$, the $k-1$ arbitrarily to be assumed remain fixed, the last point $\mathfrak{A}$ will describe a certain rational curve passing through the fixed points $\mathfrak{A}$. *Vide Nature*, 17th October, 1895.)

There is no reason why no more than two different space-symbols should be used. Our definitions will apply also in this case, the general rule being that any operation may be performed upon the symbols of any one space, while the symbols of the other spaces are treated as constants.

A form like the following,

$$a_1 \,.\, A_1 \,.\, A_1' \,.\, A_1'' \ldots A_1^{(k)} + \ldots + a_h \,.\, A_h \,.\, A_h' \,.\, A_h'' \ldots . A_h^{(k)},$$

where the A_i belong to one space, the A_i' to another, &c., is called a $(k+1)$-linear form. The point symbols $A_i^{(s)}$ and their compositions are called "different space-symbols."

An equation between point-forms, such as, for instance,

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_h u_h = 0,$$

imposes a condition upon the coefficients of u_1, u_2, ... u_h. In fact, if the order of the $u_1 \ldots u_h$ is n, the α denoting unknown constants, the order of the condition expressed by this equation is $\phi\,(n)+1-h$ (since a form of order n in S_k vanishes, the form containing h homogeneous parameters). If, for instance, it is known that three point-pairs $A \,.\, A'$, $B \,.\, B'$, $C \,.\, C'$ in a plane are linearly dependent, the six points must satisfy a four-fold condition.

If, in an equation $\alpha_1 u_1 + \ldots + \alpha_h u_h = 0$,

the point-symbols implicitly contained in the $u_1 \ldots u_h$ are subjected to one and the same linear transformation, and they are projected into some other space, only the constants $\alpha_1 \ldots \alpha_h$ will be affected, but the projections $u_1' \ldots u_h'$ of $u_1 \ldots u_h$ will again be linearly dependent. This follows immediately if we compose the equation with some point P

outside the space S of the u, and then cut it by a space Σ contained in SP, and of the same manifoldness as S.

If we polarize the equation

$$a_1 u_1 + \ldots + a_h u_h = 0$$

with $P_1 . P_2 . \ldots P_n$, where $P_1 \ldots P_n$ are *not* contained in S, and $SP_1 \ldots P_n$ is different from 0, the point-symbols $A_1 \ldots A_m$ implicitly contained in the $u_1 \ldots u_h$ will combine with the $P_1 \ldots P_n$, and we shall obtain a relation between

$$\begin{matrix} A_1 P_1, & A_2 P_1, & \ldots & A_m P_1, \\ A_1 P_2, & A_2 P_2, & \ldots & A_m P_2, \\ \ldots & \ldots & \ldots & \ldots \\ A_1 P_n, & A_2 P_n, & \ldots & A_m P_n, \end{matrix}$$

which evidently represent in the geometry of lines through $P_1, P_2, \ldots P_n$ *different space-symbols.* Cutting these by spaces $\Sigma_1, \Sigma_2, \ldots \Sigma_n$ of the same manifoldness as S in $SP_1, SP_2, \ldots SP_n$, respectively, we obtain a relation between

$$\begin{matrix} A_1', & A_2', & \ldots & A_m', \\ A_1'', & A_2'', & \ldots & A_m'', \\ \ldots & \ldots & \ldots & \ldots, \\ A_1^{(n)}, & A_2^{(n)}, & \ldots & A_m^{(n)}, \end{matrix}$$

where the $A', A'', \ldots$ are projections of the $A_1, A_2, \ldots A_m$, but situated in different spaces.

Thus from the original equation we obtain another which is evidently an n linear form of different space-symbols denoting point-groups in each separate space which are projectively identical with the original point-group. Therefore it follows that we may perform any operation with any one of the point-groups $A', A'', \ldots$, treating the others as constants, and afterwards identify again the symbols of that point-group with the original point-group A (which is but a specialization).

The polarized form of the power of a point P is especially simple. That of P^n is evidently $P' . P'' \ldots P^{(n)}$.

To give a few instances: an equation

$$(\theta) \quad aA^2 + bB^2 + cC^2 + dD^2 + eE^2 + fF^2 = 0,$$

where the $a, \ldots f$ are unknown constants, expresses one condition, which is evidently that the six points $A, \ldots F$ are situated upon one conic. Writing (θ),

$$aA^2 + bB^2 + cC^2 = -dD^2 - eE^2 - fF^2,$$

it appears that A, B, C and D, E, F form self-conjugate triangles to some conic. Polarizing each side of the last equation with itself, we obtain

$$abAB^2+bcBC^2+caCA^2 = deDE^2+\dots;$$

hence AB, BC, CA, DE, EF, FD touch one conic. Polarizing (θ) in regard to the cubic u formed by $DE.EF.FD$, which obviously contains D, E, F as double points, so that

$$D^2\times u = 0, \quad E^2\times u = 0, \quad F^2\times u = 0,$$

we have

$$aA^2\times u+bB^2\times u+cC^2\times u = 0,$$

or the three polar lines of A, B, C to the triangle D, E, F (which can be linearly constructed) are concurrent. Polarizing (θ),

$$aA.\mathfrak{A}+bB.\mathfrak{B}+\dots+fF.\mathfrak{F} = 0,$$

further composing with E and $\mathfrak{F}$,

$$aAE.\mathfrak{A}\mathfrak{F}+bBE.\mathfrak{B}\mathfrak{F}+c.CE.\mathfrak{C}\mathfrak{F}+d.DE.\mathfrak{D}\mathfrak{F} = 0,$$

or the cross-ratio of the lines AE, BE, CE, DE is equal to that of AF, BF, CF, DF.

If we put

$$aA^2+bB^2 = h.A'.B',$$

$$cC^2+dD^2 = k.C'.D',$$

$$eE^2+fF^2 = l.E'.F',$$

then

$$h.A'.B'+k.C'.D'+l.E'.F' = 0.$$

It is obvious that the line joining C' and E', for instance, must contain either A' or B'. The figure must therefore be that of a triangle $A'C'\mathfrak{F}'$, cut by a straight line $B'D'F'$, B' to be on $C'E'$, &c. Its reciprocal may be derived from the identity which connects four points upon a straight line

$$AD.BC+AB.CD+AC.DB = 0,$$

which must also exist for four points in a plane, since this form would contain all points of the plane, and must therefore identically vanish. The equation connecting $A'.B'$, &c., may be interpreted as meaning that if, $A'.B'$ and $C'.D'$ are conjugate to any conic, $E'.F'$ will be so also. If L, M, N are the three corners of the triangle formed by AB, CD, EF, it is at once seen that

$$A'.B' \equiv aL^2-bM^2,$$

$$C'.D' \equiv bM^2-cN^2,$$

$$E'.F' \equiv cN^2-aL^2,$$

a, b, c denoting parameters. $A'B'$ are therefore the double points of the involution formed by AB and LM, &c.

Let P, Q, R be any three points, and p, q, r their polars in respect to some conic u. If, then, $p'p''$, $q'q''$, $r'r''$ are points upon p, q, r, respectively, $P.p'$, $P.p''$, $Q.q'$, $Q.q''$, $R.r'$, $R.r''$ will be conjugate to the same conic; therefore linearly dependent. It follows that points P', Q', R' must exist upon p, q, r, so that $P.P'$, $Q.Q'$, $R.R'$ are linearly dependent. Now P' is collinear with QR; hence it is the cut of QR and p. The relation is then expressed thus, QR/p, RP/q, PQ/r are collinear, and, the configuration being its own reciprocal, $P\,q/r$, $Q\,r/p$, $R\,p/q$ are concurrent. This may also be proved in a different manner thus:—Let u be any conic, A, B, C the corners of a triangle. Then $(AB*u)\times C+(BC*u)\times A+(CA*u)\times B$ is a line-form $\times$. Composed with C it is

$$(BC*u)*AC+(CA*u)*BC=BC\,.\,AC*u+CA\,.\,BC*u=0.$$

Hence $\times A=\times B=\times C=0$. $\times$ will therefore vanish identically. This may also be generalized. If $u\equiv\mathfrak{J}$, this shows that the four heights of a tetrahedron belong to one quadric.

The curve whose points have the property that the cross-ratios of the twice four points

$$\begin{array}{cccc} A, & B, & C, & D, \\ A', & B', & C', & D', \end{array}$$

stand in a certain proportion $\beta:\alpha$ is the numerator of

$$\alpha\left(\frac{AC}{BC}:\frac{AD}{BD}\right)-\beta\left(\frac{A'C'}{B'C'}:\frac{A'D'}{B'D'}\right)$$

(the line-symbols written in their normal form), a curve of the fourth order. If $A\ldots D$, $A'\ldots D'$ are situated upon a conic u, $\beta:\alpha$ being the proportion of their cross-ratios with respect to the points of u, then the curve above defined must degenerate into two conics u, u'; u' contains the eight points $AC/A'C'$, $AC/B'D'$, $BD/A'C'$, ..., and is thus defined.

If u, v, w are three conics having two points in common, the form $a\,.\,u+b\,.\,v+c\,.\,w$, the a, b, c denoting lines, can only denote an involution of manifoldness 7; hence some identity such as

$$a\,.\,u+b\,.\,v+c\,.\,w=0$$

must exist, and consequently the lines a, b, c joining the two variable points of intersection of v, w; w, u; u, v respectively are concurrent. If now u is a conic, A, B, C, D, E, F six points upon it, then u, $AC\,.\,BD$ and $AE\,.\,BF$ have the points A and B in common; therefore the cut of CD and EF must be situated on the line joining the cuts of AC/BF and BD/AE, which is Pascal's theorem.

Let $A_1 \ldots A_9$ be nine points common to two cubics. Their configuration is expressed by the fact that their cubes are linearly dependent. Eight points being given, the last one is obviously determined. A construction may, for instance, be arrived at thus: We may have

$$a_1 A_1^3 + a_2 A_2^3 + \ldots + a_8 A_8^3 + a_9 A_9^3 = 0.$$

Polarizing, we obtain as a consequence

$$a_1 A_1^2 \,.\, \mathfrak{A}_1 + a_2 A_2^2 \,.\, \mathfrak{A}_2 + \ldots + a_8 A_8^2 \,.\, \mathfrak{A}_8 + a_9 A_9^2 \,.\, \mathfrak{A}_9 = 0.$$

Polarizing with the A_i^2 in regard to the cubic u represented by the sides of the triangle $A_6 A_7 A_8$, and denoting the lines $A_1^2 \times u$ by α_1, $A_2^2 \times u$ by α_2, ...,

$$a_1 \,.\, \alpha_1 \mathfrak{A}_1 + \ldots + a_5 \,.\, \alpha_5 \,.\, \mathfrak{A}_5 + a_9 \,.\, \alpha_9 \,.\, \mathfrak{A}_9 = 0.$$

Composing with $\mathfrak{A}_9$,

$$a_1 \alpha_1 \,.\, \mathfrak{A}_1 \mathfrak{A}_9 + a_2 \alpha_2 \,.\, \mathfrak{A}_2 \mathfrak{A}_9 + \ldots + a_5 \alpha_5 \,.\, \mathfrak{A}_5 \mathfrak{A}_9 = 0.$$

Now $\alpha_1, \ldots \alpha_5$ are known. The cross-ratio in which α_1, α_2, α_3, α_4 is cut by α_5 is equal to that of $\mathfrak{A}_1\mathfrak{A}_9$, $\mathfrak{A}_2\mathfrak{A}_9$, $\mathfrak{A}_3\mathfrak{A}_9$, $\mathfrak{A}_4\mathfrak{A}_9$, and that of α_1/α_4, α_2/α_4, α_3/α_4, α_5/α_4, equal to that of $\mathfrak{A}_1\mathfrak{A}_9$, $\mathfrak{A}_2\mathfrak{A}_9$, $\mathfrak{A}_3\mathfrak{A}_9$, $\mathfrak{A}_5\mathfrak{A}_9$. Hence $\mathfrak{A}_9$ can be constructed as the fourth point common to two conics through $\mathfrak{A}_1 \mathfrak{A}_2 \mathfrak{A}_3 \mathfrak{A}_4$ and $\mathfrak{A}_1 \mathfrak{A}_2 \mathfrak{A}_3 \mathfrak{A}_5$.

If seven points are given, $A_1, \ldots A_7$, and the other two points A_8, A_9 are restricted to a given line l, only one solution is possible. Through the seven points an involution of three cubics is possible. They cut l in an involution whose reciprocal will be a certain point-group $P \,.\, Q \,.\, R$ conjugate to any cubic containing the seven points. Therefore $P \,.\, Q \,.\, R$ is linearly dependent upon the cubes of the seven points. Representing $P \,.\, Q \,.\, R$ in its canonical form as the sum of two cubes, we find A_8 and A_9.

If two of the points P, Q, R coincide, there is no solution, but, if $Q \equiv P$, then $P^2 R$ will be linearly dependent on the cubes of the seven points, and this will express that any cubic passing through $A_1 \ldots A_7$, and P will touch l; that, in other words, A_8 and A_9 are consecutive in P upon l. It may also express that a cubic is possible through $A_1 \ldots A_7$ having P as a double point, the linear dependence between $A_1 \ldots A_7$ and $P^2 R$ reducing the number of conditions this implies to nine.

We may add, though without demonstration, that, if $A_1 \ldots A_8$ are fixed and A_8 moves upon any curve of order λ, A_9 will generally move upon a curve of order 8λ. If, however, the curve A_8 contains the fixed points $A_1 \dots A_7$ as $(\alpha_1 \dots \alpha_7)$-fold points, denoting $\alpha_1 + \dots + \alpha_7$ by β, the order of the curve A_9 will be $8\lambda - 3\beta$, and it will contain A_1 as $3\lambda - \beta - \alpha_1$, A_2 as $(3\lambda - \beta - \alpha_2)$-fold points, &c.

Eight points common to three quadrics satisfy a relation

$$a_1 A_1^2 + \dots + a_8 A_8^2 = 0.$$

By reading it $\qquad a_1 A_1^2 + \dots + a_4 A_4^2 = -a_5 A_5 \dots,$

it is evident that $A_1 A_2 A_3 A_4$ and $A_5 A_6 A_7 A_8$ are two self-conjugate pyramids to one and the same quadric. Polarizing

$$a_1 A_1 \,.\, \mathfrak{A}_i + \dots + a_8 A_8 \mathfrak{A}_8 = 0,$$

and composing with $A_8 A_7$ and $\mathfrak{A}_6 \mathfrak{A}_5$ we see that $A_1 A_2 A_3 A_4$ have the same cross-ratio, whether projected from $A_5 A_6$ or from $A_7 A_8$. We also see that any quadric through six of the points $A_1 \dots A_6$ will cut $l \equiv A_7 A_8$ in an involution whose double points are B and C, so that

$$B \,.\, C = a_7 A_7^2 + a_8 A_8^2.$$

Any point-pair on l harmonic with $B \,.\, C$ will therefore, if joined to $A_1 \dots A_6$, complete the configuration. The two points A_7, A_8, though moving upon l, will coincide in B and C, and l will be a tangent to any quadric containing $A_1 \dots A_6$, B or $A_1 \dots A_6$, C.

Now a twisted cubic having seven points in common with a quadric must be wholly contained in it. So any eight points upon a twisted cubic form the configuration. Seven points $A_1, \dots A_7$ being given, $A_7 A_8$ is the straight line through A_7 cutting the twisted cubic through $A_1 \dots A_6$ twice. So, then, l can always be determined, except in the following cases:—

(1) That A_7 is collinear with any two of the fixed points $A_1 \dots A_6$. Then A_8 is any point on that line, the squares of four points on a line being always linearly dependent.

(2) That any five of the fixed points are coplanar. Then A_8 is any point upon their conic.

(3) That any six of the fixed points are coplanar. Then A_8 is any point of their plane; or, if the six given points are also upon one conic, A_8 is any point whatever.

(4) That the seven given points are upon the same twisted cubic. Then A_8 is any point of that twisted cubic.

In no other case can the configuration degenerate. If, for instance,

$A_1 \dots A_4$ are coplanar in S, the line common to S and $A_5A_6A_7$ will be cut in involution by the conics through $A_1 \dots A_4$; hence a point-pair $B.C$ will exist upon it which is linearly dependent upon $A_1^2 \dots A_4^2$. But the conics passing through A_5, A_6, A_7 and containing $B.C$ as conjugate point-pair have a fourth point A_8 in common, so that $B.C$ is also linearly dependent upon the A_5^2, A_6^2, A_7^2, A_8^2.

If six of the points $A_1 \dots A_6$ are given, while A_7 is restricted to a plane S, and A_8 to a line l, l to be within S, we shall find three point-pairs A_7, A_8 to complete the configuration. For $A_1, \dots A_6$ determine an involution of four quadrics which cut S in an involution of conics, the reciprocal of which is formed by the two point-forms u and v. Their involution contains three point-pairs $L.L'$, $M.M'$, $N.N'$. LL' cuts l in A_8, while A_7 is $\equiv A_8 \times L.L'$. Thus the three point-pairs $A_7.A_8$ are constructed. The lines LL', MM', NN' are obviously the sides of the triangle, in which the twisted cubic through $A_1 \dots A_6$ cuts S.

Proceeding similarly when five points $A_1, \dots A_5$ are given, and a plane S as locus for the other points, we obtain in S one point-form u of the second order linearly dependent upon the squares $A_1, \dots A_5$; A_6, A_7, A_8, will therefore be the corners of any triangle self-conjugate to u_7. And we shall have two solutions, if A_6, A_7, A_8 are restricted to lie on given lines in S.

A configuration of some importance, at least in the theory of surfaces of the second order in any space, is that of $2n$ points characterized by (θ),

$$a_1A_1.B_1 + a_2A_2.B_2 + \dots a_nA_n.B_n = 0,$$

where it is understood that $A_1, \dots A_n$ are the corner-points of a (non-vanishing) pyramid in space S_{n-1}. The order of the condition imposed by (θ) is $\frac{n.n+1}{2} - n + 1$. Since $[A_1 \dots A_n]$ is different from 0, it follows that

$$[A_1 \dots A_{n-1}B_n] = 0.$$

Hence $[A_1 \dots A_{n-2}B_{n-1}B_n]$ is generally different from 0, and

$$[A_1 \dots A_{n-3}B_{n-2}B_{n-1}B_n] \text{ again} = 0.$$

Any space composed of an odd number of the B has a point in common with the space composed by the residual A. If, then, B_n and B_{n-1} are assumed anywhere in $A_1 \dots A_{n-1}$, and $A_1 \dots A_{n-2}A_n$, respectively, B_{n-2} will be in the cut of

$$A_1, \dots A_{n-3}A_{n-1}A_n \text{ and } A_1, \dots A_{n-3}B_{n-1}B_n;$$

B_{n-3} will be subject to three conditions to lie in the cut of

$$A_1 \dots A_{n-4} A_{n-2} A_{n-1} A_n ; \quad A_1 \dots A_{n-4} B_{n-2} B_{n-1} A_n ;$$
$$A_1 \dots A_{n-4} A_{n-2} B_{n-1} B_n ;$$

B_{n-4} will similarly be subject to 4 conditions;

… … … … … … …

B_1 will similarly be subject to $n-1$ conditions;

raising the number of conditions to $\dfrac{n.n-1}{2}+1$.

In space the two pyramids $A_1, \dots A_4$ and $B_1, \dots B_4$ are so related that each has its corner-points upon the faces of the other.

Whenever n is an even number, the relation of the pyramids is reciprocal. When n is odd, the pyramid of the B points must vanish.

Let $A_1, \dots A_8$ be eight points common to three quadrics. Join A_1A_2, A_3A_4, A_5A_6, A_7A_8. There are two lines cutting these four (the two lines of the involution reciprocal to the one formed by the four) say a and b, cutting A_1A_2 in L_1, L_2, A_3A_4 in L_3, L_4 … . Now, L_1, L_3, L_5, L_7 being collinear, their squares are linearly dependent; also those of L_2, L_4, L_6, L_8. Consequently, for any values of λ, μ, a relation will exist, viz.,

$$c_1\lambda L_1^2 + c_2\mu L_2^2 + c_3\lambda L_3^2 + c_4\mu L_4^2 + \dots = 0.$$

Putting

$$c_1\lambda L_1^2 + c_2\mu L_2^2 = M_1 . M_2,$$
$$c_3\lambda L_3^2 + c_4\mu L_4^2 = M_3 . M_4,$$
$$\dots \quad \dots \quad \dots \quad \dots .$$

$M_1.M_2$, $M_3.M_4$, $M_5.M_6$, and $M_7.M_8$ will be linearly dependent. The order of the condition implied by this statement is 7. Consequently, only a singly infinite series of such quadruples of point-pairs *can* exist on the four given lines. Hence that series is exactly represented by

$$(c_1\lambda L_1^2 + c_2\mu L_2^2), \quad (c_3\lambda L_3^2 + c_4\mu L_4^2) \dots .$$

This gives rise to the theorem: The double-points of the involutions $(L_1.L_2,\ A_1.A_2)$, $(L_3.L_4,\ A_3.A_4)$, &c., form the configuration of two pyramids, of which each has its corner-points upon the faces of the other.

To multiply these theorems to any extent would only require some imagination. To write what may be considered a complete theory of such configurations requires however more resources than are developed in this essay. We shall leave, therefore, the further discussion of configurations for a future occasion.

Metrical relations can also be derived with ease. If u is a surface, A and B any two points, the line AB cutting the surface in the n points $w_1, w_2, \dots w_n$, then the points w are found by the solution of

$$u \times (\alpha A + \beta B)^n = 0,$$

the n roots $\alpha : \beta$ corresponding to the n points w,

$$\alpha : \beta = -w_i B : w_i A.$$

Since

$$u \times (\alpha A + \beta B)^n = \alpha^n . uA + n . \alpha^{n-1} \beta . u \times A^{n-1} . B + \dots + \beta^n . uB,$$

it follows that the product of the roots

$$w_1 B . w_2 B \dots w_n B : w_1 A . w_2 A \dots w_n A = uB : uA.$$

If A, B are on u—say, and λ points w coincide in A, μ points in B—only $n-\lambda-\mu$ points w will remain corresponding to the equation

$$(n)_\lambda \alpha^{n-\lambda-\mu} . u \times A^{n-\lambda} . B^\lambda + \dots + (n)_\mu \beta^{n-\lambda-\mu} . u \times A^\mu B^{n-\mu} = 0;$$

hence

$$\Pi w_i B : \Pi w_i A = (n)_\mu . u \times A^\mu . B^{n-\mu} : (n)_\lambda . u \times A^{n-\lambda} . B^\lambda.$$

Similar theorems exist for any geometrical formations.

If a surface of order $2n$ contains $\mathfrak{I}^n$, points of its space will have a "power" in regard to it. If u is such a surface, P any point, l any line through it, $w_1, \dots w_{2n}$, its cuts with u, then $\Pi w_i P$ is constant, and independent of the situation of l. We have, denoting l/I by D,

$$\Pi P w_i : \Pi D w_i = uP : uD.$$

But, D being the I of l, all Dw are 1; and, $\mathfrak{I}^n$ being the cut of I and u, uD is a constant multiple of $(ID)^n$, which is 1. Putting

$$uD = k,$$

we have

$$\Pi P w_i = k . uP.$$

To give a few examples: If any surface u cuts the sect AB in n points $w_1, \dots w_n$, then the ratio $\Pi A_w : \Pi B_w$ may be called the ratio in which u cuts AB. This ratio, which is $= uA : uB$, being given, $= c$, it follows that $A^n - c . B^n$ is conjugate to u; it is, therefore, equivalent to a linear condition for u.

A conic which cuts five given sects in given proportions is therefore generally uniquely determined. A conic cuts any six given sects in six ratios $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ between which a six-linear relation exists; for, $\alpha, \beta, \gamma, \delta, \epsilon$ being given, ζ is uniquely determined. This may evidently be generalized. The equation illustrating the situation of six points on a conic

$$aA^2+bB^2+cC^2+dD^2+eE^2+fF^2=0$$

may be interpreted as meaning that any conic cutting the sect AB in a given ratio $\left(-\frac{b}{a}\right)$ and CD in another $\left(-\frac{d}{c}\right)$ will cut EF in a certain ratio $\left(-\frac{f}{e}\right)$. Similarly, any curve of the nth order cutting n of the $n+1$ given sects AA', ... LL' in certain ratios will also cut the last one in a known ratio if A, A', ... L, L' are upon one conic. The identity $(aA^n-bB^n)+(bB^n-cC^n)+(cC^n-aA^n)=0$ gives Carnot's theorem. All this might be much generalized and varied.

Finally, as regards the calculus with $\mathfrak{J}$, it follows, from our previous results, that all plane spaces ξ in contact with $\mathfrak{J}$ are to be regarded as isotropic spaces. If the $\mathfrak{J}$ of a S_k is defined as a point-form, a space S_a is isotropic, when

$$S_a^2\times\mathfrak{J}\times\mathfrak{J}\ldots(k-a\text{ times})=0,$$

and in its normal form, when the magnitude on the left-hand side is $=1$. $S_a\times\mathfrak{J}\times\mathfrak{J}\ldots$ is any S_{k-a} perpendicular to S_a, since any two points at I perpendicular to each other are conjugate to $\mathfrak{J}$. If S_a is in its normal form, so also

$$S_{k-a}=S_a\times\mathfrak{J}\times\mathfrak{J}\ldots.$$

For $[S_{k-a}\,S_a]$ is the same as

$$S_{k-a}\times S_a=S_a.S_a\times\mathfrak{J}\times\mathfrak{J}\ldots=1.$$

But the magnitude formed by the two spaces S_{k-a} and S_a perpendicular to each other is 1, it being a product of the sines of angles, all of which are right angles. S_a is in its normal form. Hence $[S_{k-a}\,S_a]$ cannot be equal to 1 unless S_{k-a} is also in its normal form. These statements may be regarded to express all cosine theorems, &c., in fact, all metrical relations based upon the measurement of angles in their simplest form. To give only one instance, D_1, D_2, D_3, D_4, denoting points at the I of our space in their normal form (points of a sphere), from the identity

$$D_1D_2.D_3D_4+D_1D_3.D_4D_2+D_1D_4.D_2D_3=0,$$

we conclude [considering that D_1D_2 is not in its normal form, but multiplied by $\sin(D_1, D_2)$, and similarly for the other line-symbols], by polarizing with $\mathfrak{J}$,

$$\begin{aligned}&\sin(D_1, D_2).\sin(D_3, D_4).\cos(D_1D_2, D_3D_4)\\+&\sin(D_1, D_3).\sin(D_4, D_2).\cos(D_1D_3, D_4D_2)\\+&\sin(D_1, D_4).\sin(D_2, D_3).\cos(D_1D_4, D_2D_3)=0.\end{aligned}$$

Thursday, June 10th, 1897.

Prof. E. B. ELLIOTT, F.R.S., President, in the Chair.

Seven members present.

Mr. W. W. Taylor exhibited numerous models of the regular convex and star solids.

Major MacMahon, Vice-President, having taken the Chair, communicated papers by Mr. H. MacColl, "The Calculus of Equivalent Statements" (Sixth Paper); by Dr. G. A. Miller, "On the Primitive Substitution Groups of Degree Fifteen."

Mr. Love, Hon. Sec., read "A Generalized Form of the Binomial Theorem," which had been sent by the Rev. F. H. Jackson. The Chairman (Major MacMahon) stated that the form was a known one.

The following present was made to the Society's Album :—

Cabinet likeness of Mr. W. Esson, F.R.S. (now Savilian Professor of Geometry, Oxford).

The following presents were made to the Library :—

Carruthers, G. T.—"The Origin of the Celestial Laws and Motions," 8vo; London, 1897.

Parasada Ganesh.—"On the Potential of a Solid Ellipsoid of Revolution at an External Point" (4 copies), 8vo; Allahabad, 1897.

"Proceedings of the Royal Society," Vol. LXI., Nos. 371–373.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 4, 5; Leipzig, 1897.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLI., Pt. 3, 1896–97.

"Jahresbericht der Deutschen Mathematiker Vereinigung," Bd. V., Heft 1, 1896; Leipzig, 1897.

"Proceedings of the Physical Society," Vol. XV., Pt. 5, No. 80; May, 1897.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich," 1897, Heft 1.

"Wiskundige Opgaven," Amsterdam, Deel VII., St. 3; 1897.

"Nieuw Archiev voor Wiskunde," Deel III., St. 2; Amsterdam, 1897.

"Bulletin of the American Mathematical Society," Vol. III., No. 8; May, 1897.

"Adresse présentée à M. Mittag-Leffler le 16 Mars 1896, 1846–1896," 8vo; Paris, 1896.

"Proceedings of the Canadian Institute," Vol. I., No. 1, Pt. 1, February, 1897; Toronto.

"Bulletin des Sciences Mathématiques," 1897, Mai, Tome XXI.

"In Memoriam N. J. Lobatschevskii," 8vo; Kasan, 1897.

"Bulletin de la Société Physico-Mathématique de Kasan," Série 2, Tome VI., No. 2; 1896.

"Prace Matematyczno-Fizyczne," Warsaw, Tome VIII.; 1897.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," 1897, 1–25.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VI., Fasc. 8–10; Roma, 1897.

"Annales de la Faculté des Sciences de Marseille," Tome VI., Fasc. 4–6, Tome VIII., Fasc. 1–4.

"Journal für die reine und angewandte Mathematik," Bd. CXVIII., Heft 1, 2.

"Educational Times," June, 1897.

"Annales de la Faculté des Sciences de Toulouse," Tome XI., Fasc. 2; Paris, 1897.

"Annali di Matematica," Tomo XXV., Fasc. 3; Milano, 1897.

"Annals of Mathematics," Vol. XI., Nos. 3, 4; Virginia, 1897.

"Indian Engineering," Vol. XXI., Nos. 17–20, April 24–May 15, 1897.

"Collected Mathematical Papers of Arthur Cayley," Vol. XII., 4to; Cambridge, 1897.

On the Primitive Substitution Groups of Degree Fifteen. By G. A. MILLER, Ph.D.

Received June 2nd, 1897. Read June 10th, 1897.

If any group (G) of order g contains a non-self-conjugate sub-group (G_1) of order g_1 that does not include any self-conjugate sub-group of G, with the exception of identity, then is G simply isomorphic to a transitive substitution group (G') of degree $g \div g_1$. When G_1 is a maximal sub-group of G, *i.e.*, when it is not contained in a larger sub-group of G, G' is a primitive group. When this condition is not satisfied, G' is non-primitive.*

It is a singular fact that we can find all the primitive groups of degree 15 which do not contain the alternating group of this degree by means of these well-known principles. The four groups

$$(+abcdef)_{24}, \quad (abcdef)_{48}, \quad (abcdefg)_{168}, \quad (abcdefgh)_{1344}\dagger$$

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* Dyck, *Mathematische Annalen*, Vol. XXII., p. 94.

† Noether, *Mathematische Annalen*, Vol. XV., p. 90. We follow the notation employed by Professor Cayley in his lists of substitution groups published in the *Quarterly Journal of Mathematics*, Vol. XXV.

are, respectively, maximal sub-groups of the following:—

(*abcdef*) pos., (*abcdef*) all, (*abcdefg*) pos., (*abcdefgh*) pos.

Hence there are four primitive groups of degree 15 which are, respectively, simply isomorphic to the last four. Three of them are simple groups. In what follows we shall prove that there can be no more than four primitive groups of this degree whose order is less than $15! \div 2$.*

§ 1. *Simply Transitive Groups.—General Theorems.*

We shall suppose that G represents any primitive group of degree n, and that G_1 represents its maximal sub-group of degree $n-1$, g and g_1 having the same meaning as before. G_1 is intransitive. We proceed to prove several general theorems with respect to G_1, which will be frequently employed in the following investigation.

THEOREM I.—*When* G_1 *contains a self-conjugate sub-group* (H) *of degree* $n-\alpha$, H *must be intransitive and it must be the transform with respect to substitutions of* G *of any one of* $\alpha-1$ *other sub-groups of* G_1 ($H'_1, H'_2, \ldots, H'_{\alpha-1}$). *The substitutions of* G *which transform* H'_β ($\beta = 1, 2, \ldots, \alpha-1$) *into* H *transform also all the substitutions of* G_1 *that are commutative to* H'_β *into substitutions of* G_1.

That H is intransitive follows from the theorem that a simply transitive primitive group cannot contain a transitive sub-group whose degree is lower than the degree of the group. Since G_1 is a maximal sub-group of G, it must contain all the substitutions that are commutative to H. As H occurs in $\alpha-1$ other conjugates of G_1, it must be the transform of just $\alpha-1$ different sub-groups of this group. The substitutions that are commutative to H'_β are transformed into substitutions that are commutative to H by every substitution that transforms H'_β into H.

COROLLARY I.—*When* G_1 *is an Abelian or a Hamilton*† *group, all its substitutions except identity are of degree* $n-1$, *and the order of* G *cannot exceed* $\frac{n(n-1)}{2}$.

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* In his enumeration of these groups, *Comptes Rendus*, Vol. LXXV., pp. 175–7, Jordan gives only three of them.

† A Hamilton group is non-commutative, and all its sub-groups are self-conjugate. Dedekind, *Mathematische Annalen*, Vol. XLVIII., p. 549.

Corollary II.—*When G_1 contains a self-conjugate sub-group whose degree is less than $n-1$, it must also contain non-self-conjugate similar sub-groups such that the degree of the largest sub-group of G_1 that transforms one of them into itself is less than $n-1$.*

Theorem II.—*G_1 transforms $H'_1, H'_2, \ldots, H'_{a-1}$ in the same manner as the elements of one of its constituent groups of degree $a-1$, and no two of the sub-groups $H, H'_1, H'_2, \ldots, H'_{a-1}$ have all their elements in common.*

Each substitution of G transforms the n sub-groups that are conjugate to G_1 in exactly the same manner as it transforms its own elements. Hence the substitutions of G_1 must transform the $n-1$ conjugates of H which differ from H in the same manner as they transform their own elements. As G_1 does not include any of these conjugates besides $H'_1, H'_2, \ldots, H'_{a-1}$, it must transform the latter among themselves. If two of the given sub-groups had all their elements in common, all the substitutions that are commutative to one or the other of them could not generate G.

Theorem III.—*The group generated by $H'_1, H'_2, \ldots, H'_{a-1}$ is of degree $n-1$.*

If this group were of a lower degree, it would occur in a conjugate of G_1 differing from G_1. If it would include H, the group generated by $H, H'_1, H'_2, \ldots, H'_{a-1}$ would be self-conjugate in G_1, and this conjugate. This is impossible, since its degree is less than n. If it would not include H, it would be transformed into itself by the other transform of H which the given conjugate would have to contain. As this transform would have to contain at least one substitution not contained in G_1, this is again impossible.

Corollary.—*If G_1 transforms $H'_1, H'_2, \ldots, H'_{a-1}$ according to the elements of a transitive constituent, each of them must contain at least one cycle from each of the systems of intransitivity of G_1, and the number of systems of intransitivity of H'_β $(\beta = 1, 2, \ldots, a-1)$ cannot be less than that of G_1.*

Theorem IV.—*All the prime numbers which divide the order of one of the transitive constituents of G_1 divide also the order of each of its other transitive constituents.*[*]

If the prime number p divides the order of a transitive constituent of G_1 without dividing that of another, the group formed by all the transitive constituents of G whose orders are divisible by p must have an ap, β (β being not divisible by p) isomorphism to that formed by all the other transitive constituents of G_1. Hence all the substitutions of order p contained in G_1 must generate a self-conjugate sub-group of G_1 whose degree is less than $n-1$. As G_1 could not contain any other similar sub-group, this is impossible.

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[*] Jordan, *Traité des Substitutions*, p. 284.

Corollary I.—*If one of the transitive constituents of G_1 is of a prime degree, each of its other transitive constituents is of the same or of a larger degree, and the order of G_1 is the same as the order of the group formed by these other transitive constituents.*

Corollary II.—*If the order of G_1 is not divisible by the square of a prime number, all its transitive constituents are of the same order, and G_1 is formed by establishing a simple isomorphism between them.*

Theorem V.—*If a transitive constituent of G_1 is of a prime order, the order of G_1 is the same prime number, and G is of class $n-1$.*

It follows from the preceding theorem that the order of G_1 is p^a, p being the given prime number. Since all its sub-groups of order p^{a-1} are self-conjugate,* the sub-group that corresponds to identity in the constituent of a prime order must be self-conjugate in more than one of the n groups that are conjugate to G_1, *i.e.*, it must be identity.

Corollary.—*If G_1 contains a constituent of degree 2, its order is 2, and the degree of G is a prime number.*

Special Considerations for Degree 15.

It follows directly from these theorems that G_1 cannot contain a constituent of degree 2, and that the largest possible degree of one of its transitive constituents is 8. We shall first consider all the possible groups when the maximum degree of a transitive constituent of G_1 is 8, then those when this maximum degree is 7, &c. When G_1 contains a transitive constituent of degree 8, the constituent of degree 6 is either transitive or it contains two equal systems of intransitivity. We proceed to prove that the latter condition cannot be satisfied.

Since the order of the constituent of degree 6 could not be divisible by 8, G_1 would have to contain an intransitive self-conjugate sub-group of the transitive constituent of degree 8. The class of this sub-group would have to exceed 4,* and the degree of one of its equal systems of intransitivity could not exceed the degree of one of the other transitive constituents of G_1. Hence it would have to be

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* Frobenius, *Crelle's Journal*, Vol. ci., p. 283.

$$(ab\,.\,cd\,.\,ef\,.\,gh).$$

This is impossible, because there is no transitive group of order 72 and degree 8.

Systems of Intransitivity of G_1, 8, 6.

Neither of the constituent groups can be primitive, since such a group cannot contain an intransitive self-conjugate sub-group, and a primitive group in one system cannot have a 1, α isomorphism to a transitive group in the other. Hence each one of the two given constituent groups is non-primitive, and the order of G_1 is $2^{\alpha}\,.\,3^{\beta}\,.\,24$.

If the order of G_1 would exceed that of the transitive constituent of degree 8, it would contain a self-conjugate sub-group of one of the following two types:—

$$(ab\,.\,cd\,.\,ef), \quad (abc\,.\,def) \text{ cyc.}$$

Each of the two transitive constituents of G_1 would contain $8 \div \alpha$ conjugate sub-groups of order 2 or 3, respectively, and of a degree that could not exceed 4. Each of the substitutions of G_1 would transform these sub-groups in exactly the same manner as its own elements belonging to the constituent of degree 8. Since the substitutions of the form *ab* or *ab . cd* in a non-primitive group of degree 6 are conjugate in sets of multiples of 3, the given self-conjugate sub-group of order 2 cannot occur. If that of order 3 would occur, the order of the quotient group of the constituent of degree 6 would be a divisor of 12. Since the corresponding constituent of degree 8 could not contain a substitution of degree 3, this is also impossible.

If the order of G_1 exceeds that of the transitive constituent of degree 6, it must contain one of the following two intransitive self-conjugate sub-groups:—

$$(ab\,.\,cd\,.\,ef\,.\,gh), \quad (abcd\,.\,efgh)_4,$$

because the class of such a sub-group must exceed 4, its systems of intransitivity must be systems of non-primitivity of the constituent of degree 8, the order of this constituent must be divisible by 24, and its quotient group, with respect to the intransitive sub-group that corresponds to identity in the constituent of degree 6, must be of order 6α. The latter of the two given sub-groups cannot occur, since one of its constituents would have to be contained in each of the six similar conjugate sub-groups of G_1. Hence the order of G_1 is either equal to that of the transitive constituent of degree 8, or it is the double of this order.

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* Netto, *Theory of Substitutions* (Cole's edition), p. 138.

If the transitive constituent of degree 8 is of order 24, that of degree 6 must be of the same order, since none of the three transitive groups of degree 8 and order 24 contains a substitution of degree 4. Each of the 4 sub-groups of order 3 and degree 12 contained in such a G_1 is transformed into itself by 18 substitutions of G, which permute its systems of intransitivity, according to the elements of a group which contains substitutions of degree 3. Hence the 6 substitutions of G_1 which transform the same sub-group into itself cannot interchange more than two of its systems, *i.e.*, G_1 can only be constructed by establishing a simple isomorphism between the following two groups :—

$$(abcd\,.\,efgh)\ \text{pos.}\ (ae\,.\,bg\,.\,cf\,.\,dh),^{*} \qquad (+abcdef)_{24}.$$

The substitutions of this G_1 may be written as follows :—

1

$ab\,.\,cd\,.\,ef\,.\,gh\,.\,ik\,.\,jl$,

$ac\,.\,bd\,.\,eg\,.\,fh\,.\,ik\,.\,mn$,

$ad\,.\,bc\,.\,eh\,.\,fg\,.\,jl\,.\,mn$,

$abc\,.\,efg\,.\,ijm\,.\,kln$,

$abd\,.\,efh\,.\,inj\,.\,kml$,

$acd\,.\,egh\,.\,ilm\,.\,jnk$,

$bcd\,.\,fgh\,.\,inl\,.\,jkm$,

$bdc\,.\,fhg\,.\,iln\,.\,jmk$,

$acb\,.\,egf\,.\,imj\,.\,knl$,

$adb\,.\,ehf\,.\,ijn\,.\,klm$,

$adc\,.\,ehg\,.\,iml\,.\,jkn$,

$af\,.\,be\,.\,cg\,.\,dh\,.\,ij\,.\,kl$,

$ag\,.\,bf\,.\,ce\,.\,dh\,.\,im\,.\,kn$,

$ah\,.\,bf\,.\,cg\,.\,de\,.\,jn\,.\,lm$,

$ae\,.\,bg\,.\,cf\,.\,dh\,.\,jm\,.\,ln$,

$ae\,.\,bh\,.\,cg\,.\,df\,.\,in\,.\,km$,

$ae\,.\,bf\,.\,ch\,.\,dg\,.\,il\,.\,jk$,

$afch\,.\,bgde\,.\,inkm\,.\,jl$,

$ahcf\,.\,bedg\,.\,imkn\,.\,jl$,

$afdg\,.\,bhce\,.\,ik\,.\,jmln$,

$agdf\,.\,bech\,.\,ik\,.\,jnlm$,

$agbh\,.\,cfde\,.\,ilkj\,.\,mn$,

$ahbg\,.\,cedf\,.\,ijkl\,.\,mn$.

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* Miller, *Quarterly Journal of Mathematics*, Vol. XXVII., p. 102.

The 9 substitutions of G that transform $abc \,.\, efg \,.\, ijm \,.\, kln$ into itself must clearly include one of the following 18 substitutions:

$$(1,\ abc \,.\, cgf,\ acb \,.\, efg)(aei \,.\, bfj \,.\, cgm)(dho,\ doh)(1,\ kln,\ knl).$$

Since the group of order 18 that transforms the group

$$(abc \,.\, efg \,.\, ijm \,.\, kln)\ \text{cyc.}$$

into itself must permute its first three systems according to (abc) all, this must contain 9 substitutions of order 2, and 8 of order 3, and is therefore fully determined. The required substitution of order 3 must be transformed into its square by $af \,.\, be \,.\, cg \,.\, dh \,.\, ij \,.\, kl$, and hence must be one of the following six:—

$$(aei \,.\, bfj \,.\, cgm)(dho,\ doh)(1,\ kln,\ knl).$$

From the following products,

$afch \,.\, bgde \,.\, inkm \,.\, jl \times aei \,.\, bfj \,.\, cgm \,.\, doh = ajlbm \,.\, cdink \,.\, efgoh$,
,, ,, ,, ,, $\times aei \,.\, bfj \,.\, cgm \,.\, dho = ajlbm \,.\, codink \,.\, efgh$,
,, ,, ,, ,, $\times aei \,.\, bfj \,.\, cgm \,.\, doh \,.\, kln = ajnlbm \,.\, cdik \,.\, efgoh$,
,, ,, ,, ,, $\times aei \,.\, bfj \,.\, cgm \,.\, dho \,.\, kln = ajnlbm \,.\, codik \,.\, efgh$,
,, ,, ,, ,, $\times aei \,.\, bfj \,.\, cgm \,.\, doh \,.\, knl = ajkcdilbm \,.\, efgoh$,
,, ,, ,, ,, $\times aei \,.\, bfj \,.\, cgm \,.\, dho \,.\, knl = ajkcodilbm \,.\, efgh$,

it follows that only $aei \,.\, bfj \,.\, cgm \,.\, doh$ can occur in G, and hence there can be no more than one primitive group that contains the given G_1. Its order is $6! \div 2$, and its class is 12.

If G_1 is formed by a simple isomorphism between two groups of order 48, each one of its sub-groups of order 3 is transformed into itself by 36 substitutions of G. This group has to permute its four systems of intransitivity according to a transitive group of degree 4. This is impossible, since the group of degree 6 and order 48 contains 6 substitutions that do not permute the systems. The only 2, 1 isomorphism between a transitive group of degree 8 and order 24 and a transitive group of degree 6, which contains a multiple of 7 similar substitutions of degree 8, is that which may be obtained by multiplying the preceding G_1 by

$$(ae \,.\, bf \,.\, cg \,.\, dh).$$

The G that contains this G_1 contains 36 substitutions that transform the group $(abc.efg.ijm.kln)$ cyc. into itself, and 18 that are commutative to each of its substitutions. The group of order 36 contains therefore only one sub-group of order 9. Hence it follows directly from the proof with respect to the G_1 of order 24 that this G_1 of order 48 can also not occur in more than one primitive group. It would not be difficult to show that there is really one such primitive group, and that it is simply isomorphic to $(abcdef)$ all, while that which contains the preceding G_1 is simply isomorphic to $(abcdef)$ pos. This is, however, unnecessary, as we shall show that there can be no more than four such groups, and these have been given above.

The transitive constituent of degree 8 could not be of order 96, since the group of this order that contains a self-conjugate sub-group of order 2 is not isomorphic to $(abcdef)_{48}$ with respect to it, and there is no transitive group of degree 6 and order 96. Hence there cannot be more than two primitive groups of degree 15 whose G_1 contain a transitive constituent of degree 8. We shall find that no other simply transitive primitive groups of this degree are possible.

Systems of Intransitivity of G_1 7, 7.

From Theorem IV., Corollary I., it follows that G_1 contains two transitive constituents of degree 7 if it contains one of this degree, and that it must consist of a simple isomorphism between these two groups. The number of sub-groups of order 5 in G must be of the form

$$21+35x \text{ (x being an integer)},$$

since it must be divisible by 7 and also congruent to 1 with respect to modulus 5. Hence the order of G_1 cannot be either 7 or 14. If its order would be 21, G would contain

$15 \,.\, 6 = 90$ substitutions of order 7,
$5 \,.\, 14 = 70$,, ,, degree 12,
$21 \,.\, 4 = 84$,, ,, order 5.

Each of the 21 sub-groups of order 5 would be transformed into itself by a cyclical sub-group of order 15. This is impossible, as G could not contain $8 \,.\, 21$ substitutions of order 15. The order of G_1 could not be 42, since there would have to be 18 substitutions in G which would transform $(abcdef \,.\, hijklm)$ cyc. into itself and whose degree could not be less than 12.

G_1 could not be of a larger order, for it would have to contain substitutions of order 2 and degree 12. As all of these are conjugate in G_1, all of those of G must also be conjugate, and each of them has to be transformed into itself by $7a$ substitutions. As this is clearly impossible, there can be no primitive group of degree 15 in which G_1 contains a transitive constituent of degree 7.

Systems of Intransitivity of G_1 6, 4, 4.

When G_1 contains a transitive constituent of degree 6, but none of a larger degree, its systems of intransitivity must be 6, 4, 4. The groups of degree 4 must be alternating or symmetric groups of this degree, as the orders of the others are not divisible by 3. If one of these groups were alternating and the other symmetric, the order of the constituent group of degree 8 would be 288. This is impossible, since there is no transitive group of degree 6 and order 288, and the given group does not contain a self-conjugate sub-group whose class exceeds 4.

The only two self-conjugate sub-groups of the constituent group of degree 8, whose class exceeds 4, are

$$(abcd\,.\,efgh)\text{ pos.},\quad (abcd\,.\,efgh)_4.$$

According to Theorem III., Corollary, G_1 cannot contain either of them. Its order must therefore be the same as that of the transitive constituent of degree 6. It must also be divisible by 12, and the order of the constituent of degree 8 must be either 12 or 24. Hence the order of G_1 must be one of these two numbers.

If G_1 were of order 12, G would contain 6 conjugate sub-groups of order 5 which its substitutions would have to transform according to an isomorphic primitive group of degree 6, since it would contain substitutions of order 5. Hence G would contain a self-conjugate sub-group of order 3, and could therefore not be primitive. If G_1 were of order 24, one of its 6 sub-groups of degree 10 and order 2 would be transformed into itself by 20 substitutions of G, which would interchange its 5 systems of intransitivity according to a substitution of order 5 and also according to one of degree 2. As this is clearly impossible, there is no simply transitive primitive group of degree 15 whose G_1 contains a transitive constituent of degree 6 and an intransitive constituent of degree 8.

Systems of Intransitivity of G_1 4, 4, 3, 3.

If one transitive constituent of G_1 were of degree 5, the other would have to be transitive and of degree 9. We observed above that this is impossible. If G_1 contains a transitive constituent of degree 4 but none of a higher degree, its systems of intransitivity must be 4, 4, 3, 3. If the order of G_1 would exceed that of the constituent of degree 6, it would have to contain one of the following two groups as a self-conjugate sub-group :—

$$(abcd\,.\,efgh)_4,\quad (abcd\,.\,efgh)\text{ pos.}$$

This is impossible, since the degree of the systems of intransitivity of these sub-groups exceeds that of the other transitive constituents of G_1.

Hence the order of G_1 must be the same as that of the constituent of degree 6; *i.e.*, it must be 6, 18, or 36. Since it must also be a multiple of 12, and the group of order 36 does not contain a self-conjugate sub-group of order 3 and degree 6, this is impossible. As all the transitive constituents of G_1 could not be of degrees 2 and 3, this completes the proof that there are no more than 2 simply transitive primitive groups of degree 15.

§ 2. *Multiply Transitive Groups.*

If G_1 is non-primitive, its systems of non-primitivity cannot be 7, 7, for it would have to contain a self-conjugate sub-group whose systems of intransitivity are the same numbers. As its substitutions that interchange these systems would have to be of degree 14, we can use practically the same proof as was employed to show that there can be no simply transitive group of this degree whose G_1 contains a transitive constituent of degree 7. Hence G_1 must be either primitive or it must contain seven systems of non-primitivity without also containing two such systems.

When G_1 contains an intransitive self-conjugate sub-group with seven systems of intransitivity, this H cannot be of order 16, since the number of its substitutions of degree 6 is not divisible by 3. If it is of order 8, its 7 systems of intransitivity cannot be transformed by G_1 according to the metacyclic group or one of its sub-groups, since a substitution of degree 8 and order 2 would have to be transformed into itself by $7a$ substitutions of G. Hence these systems must be transformed according to $(abcdefg)_{168}$. There are two groups that satisfy this condition, but we can use only the one in which the substitutions that permute only four systems include substitutions of degree 8 and order 2.

For H we may use the following group:—

$$\begin{array}{ll} 1 & cd\,.\,ef\,.\,ij\,.\,mn, \\ ab\,.\,cd\,.\,ef\,.\,gh, & ab\,.\,gh\,.\,ij\,.\,mn, \\ ab\,.\,ef\,.\,ij\,.\,kl, & ab\,.\,cd\,.\,kl\,.\,mn, \\ cd\,.\,gh\,.\,ij\,.\,kl, & ef\,.\,gh\,.\,kl\,.\,mn. \end{array}$$

G_1 is generated by this H and the following two substitutions:—

$$aceikgm\,.\,bdfjlhn, \quad ac\,.\,bd\,.\,eg\,.\,fh.$$

Its substitutions corresponding to *acg . def* in the isomorphic group of degree 7 are:

aem . bfn . gik . hjl, *afm . ben . cd . gjlhik*,
afn . bem . cd . gikhjl, *aen . bfm . gjl . hik*,
afn . bem . gjk . hil, *aen . bfm . cd . gilhjk*,
aem . bfn . cd . gjkhil, *afm . ben . gil . hjk*.

Since the 6 substitutions of G_1 that transform *aem . bfn . gik . hjl* into itself interchange its last two systems, the 18 substitutions of G which transform it into itself must interchange its systems according to the symmetric group of degree 3. As *ab . cd . ef . gh . ij . kl . mn* transforms G_1 into itself, we may assume that G contains one of the following 6 substitutions:—

(*aimgek*, *agmkei*, *akmieg*) (*co*, *do*) *bfn . hjl*.

From the following products,

amgkiec . bnhljfd × *aimgek . bfn . co . hjl* = *ag . cikmeo . df . jn*,
aceikgm . bdfjlhn × *aimgek . bfn . do . hjl* = *ackemi . bodnfljh*,
,, ,, × *agmkei . bfn . co . hjl* = *aocie . bdnfljh . gkm*,
,, ,, × *agmkei . bfn . do . hjl* = *acie . bodnfljh . gkm*,
,, ,, × *akmieg . bfn . do . hjl* = *acgimk . bodnfljh*,
,, ,, × *akmieg . bfn : co . hjl* = *aocgimk . bdnfljh*,

it follows that we can use only *akmieg . bfn . co . hjl*, and hence there cannot be more than one primitive group that contains this G_1. It is simply isomorphic to the sy~~mmetric~~ alt group of degree 8.

G_1 cannot be the group of order 336 generated by

ab . cd . ef . gh . ij . kl . mn, *aem . bfn . gjk . hil*, *cek . dfl . gim . hjn*,

for its 6 substitutions that transform one of its substitutions of order 3 into itself permute its four systems of intransitivity. The 18 substitutions of G that transform the same substitution into itself could therefore not permute these systems according to a group of order 6. Only one non-primitive group remains to be considered, viz., the one which is generated by the last two substitutions given above. Since this G_1 is commutative to *ab . cd . ef . gh . ij . kl . mn*, we may assume that it contains one of the following 18 substitutions:—

(1, *aem . gkj*, *ame . gjk*) (*agh . eji . mkl*) (*cdo*, *cod*) (1, *bfn*, *bnf*).

As the required substitution is transformed into its square by

$$am \,.\, bn \,.\, cd \,.\, gi \,.\, hj \,.\, kl,$$

it must be one of the following 6 :—

$$akh \,.\, egi \,.\, jlm\,(cdo,\ cod)\,(1,\ bfn,\ bnf).$$

From the following products,

$$cek \,.\, dfl \,.\, gim \,.\, hjn \times akh \,.\, egi \,.\, jlm \,.\, cdo \,.\, bfn = akdn \,.\, bfmij \,.\, cgehlo,$$

,, ,, ,, ,, $\times\, akh \,.\, egi \,.\, jlm \,.\, cod \,.\, bfn = akodn \,.\, bfmij \,.\, cgehl,$

,, ,, ,, ,, $\times\, akh \,.\, egi \,.\, jlm \,.\, cdo \,.\, bnf = akdbn \,.\, cgehlo \,.\, fmij,$

,, ,, ,, ,, $\times\, akh \,.\, egi \,.\, jlm \,.\, cod \,.\, bnf = akodbn \,.\, cgehl \,.\, fmij,$

,, ,, ,, ,, $\times\, akh \,.\, egi \,.\, jlm \,.\, cdo = akdfmijn \,.\, cgehlo,$

,, ,, ,, ,, $\times\, akh \,.\, egi \,.\, jlm \,.\, cod = akodfmijn \,.\, cgehl,$

it follows that we can use only $akh \,.\, egi \,.\, jlm \,.\, cod \,.\, bfn$, and hence there can be no more than one primitive group that contains this G_1. This is simply isomorphic to the alternating group of degree 7.

There are only two primitive groups of degree 14 that do not include the alternating group of this degree, viz., the modular group of order 6 . 13 . 14 and the Matthieu group of twice this order. The former cannot be used for G_1, since all its sub-groups of order 6 and degree 12 are conjugate, and each of them would have to be transformed into itself by 36 substitutions of G, whose degree, excepting identity, could not be less than 12. It is equally evident that the latter cannot be used for G, since G would have to contain 71 substitutions whose degrees could not be less than 12, which would transform a cyclical group of order 12 into itself.

We have now examined all the possible cases, and found that there cannot be more than four primitive groups of degree 15 that do not include the alternating group of this degree. Two of these are simply transitive and the other two are doubly transitive. As we proved the existence of four such groups at the beginning, the investigation is completed. To obtain the total number of primitive groups of degree 15, we have to add the alternating and the symmetric groups. Hence there are six primitive groups of degree 15.

On the Degeneration of a Cubic Curve. *By* H. M. Taylor.

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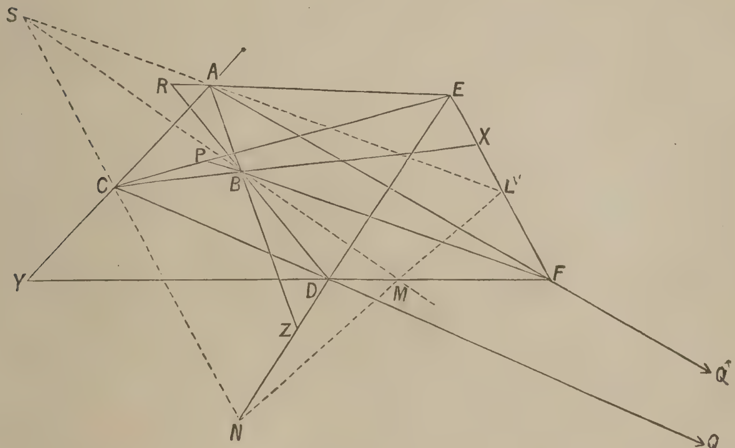
1. Grassmann seems to have been the first to give the following theorem relating to the generation of a cubic curve. The locus of a point, such that the straight lines joining it to the angular point of one triangle *ABC* cut the sides of a second triangle *DEF*, taken in order, in three collinear points, is a cubic curve passing through the angular points of both triangles.



2. We will first give a proof of the theorem, and then deduce other results from it.

Let *ABC* be taken as the triangle of reference, and let the coordinates of *D*, *E*, *F* be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$; then the condition that the straight lines joining $S(x, y, z)$ to *A*, *B*, *C* should cut *EF*, *FD*, *DE* in three collinear points *L*, *M*, *N* is

$$\begin{vmatrix} 0, & \beta_1 x - \alpha_1 y, & \gamma_1 x - \alpha_1 z \\ \alpha_2 y - \beta_2 x, & 0, & \gamma_2 y - \beta_2 z \\ \alpha_3 z - \gamma_3 x, & \beta_3 z - \gamma_3 y, & 0 \end{vmatrix} = 0,$$

or $$(\alpha_2 y - \beta_2 x)(\beta_3 z - \gamma_3 y)(\gamma_1 x - \alpha_1 z)$$
$$+ (\alpha_3 z - \gamma_3 x)(\beta_1 x - \alpha_1 y)(\gamma_2 y - \beta_2 z) = 0.$$

This is the equation of a cubic curve which passes through A, B, C: through D, E, F: through P, Q, R, the intersections of the pairs of lines BF, CE; CD, AF; AE, BD: and also through X, Y, Z, the intersections of the pairs of lines BC, EF; CA, FD; AB, DE.

3. If the triangles ABC, DEF be interchanged, the cubic locus will pass through the same twelve points, and is therefore the same cubic. The cubic will also be unaltered by the interchange of two corresponding vertices of the triangles ABC, DEF. For instance, the triangles ABF, DEC would give rise to the same cubic.

By considering the limiting case when S is close to A, we can obtain the tangent at the point A by the following geometrical construction. Let AB, DF intersect in M'; AC, DE in N'. Draw $M'N'$ cutting EF in L'; then AL' is the tangent to the cubic at A. A similar construction will give the tangents at any one of the other five points B, C, D, E, F.

4. The three sets of points A, E, R; A, Q, F; X, B, C lie on straight lines, and the points X, E, F lie on a straight line; therefore the remaining points A, A, B, C, Q, R lie on a conic; in other words, the conic $ABCQR$ touches the curve at the point A. Similarly, the conics $ABFRY$, $ACEQZ$, $AEFYZ$ touch the cubic at the point A. In the same way, it can be shown that at each of the points B, C, D, E, F four conics can be drawn to touch the curve, and to pass through four other points in the figure $ABCDEFPQRXYZ$.

5. Again, because each of the sets of points D, B, R; A, E, R; A, B, Z lie on straight lines, and the points D, E, Z lie on a straight line, therefore the six points A, A, B, B, R, R lie on a conic; in other words, a conic can be drawn having contact with the curve at the points A, B, R. There are twelve such conics having triple contact with the cubic.

Their points of contact can be found by writing down any three collinear points in the figure, *e.g.*, A, F, Q, and substituting for A or F its conjugate point D or C; thus conics will touch the cubic at D, F, Q and at A, C, Q.

Similarly, we can pick out A, B, Z; A, Y, C, two sets of three collinear points, which include a common point A but do not include any conjugate points. By writing D for A we obtain six points D, D, B, C, Y, Z, which lie on a conic. We thus determine the five points

D, *B*, *C*, *Y*, *Z*, which lie on one of the conics touching the cubic at *D*.

6. It can be seen from inspection of the equation of the cubic that in certain cases the cubic degenerates into a straight line and a conic, or into three straight lines.

For instance, if $\alpha_1 = 0$, the cubic reduces to *BC* and a conic through *A*, *D*, *E*, *F*.

If $\alpha_1 = 0$ and $\beta_2 = 0$, the cubic reduces to *BC*, *CA*, *FZ*.

If $\alpha_1 = 0$, $\beta_2 = 0$, and $\gamma_3 = 0$, the cubic reduces to the three sides of the triangle *ABC*, except in the case when

$$\gamma_1\alpha_2\beta_3 + \beta_1\gamma_2\alpha_3 = 0,$$

that is, when *D*, *E*, *F* are collinear: in which case the cubic becomes indeterminate.

7. It is at once seen that, if *D*, *E*, *F* be collinear, and *S* be any point in the straight line *DEF*, then *L*, *M*, *N* are collinear.

We see that in this case the straight line *DEF* is part of the locus, and therefore the remaining part must be a conic circumscribing the triangle *ABC*.

8. Let us inquire whether it is possible that the cubic should include the straight line

$$lx + my + nz = 0.$$

The equation of the cubic may be written

$$\begin{aligned}\gamma_3(\gamma_1\beta_2-\beta_1\gamma_2)\,x^2y+\gamma_3(\alpha_1\gamma_2-\alpha_2\gamma_1)\,xy^2+\alpha_1(\alpha_2\gamma_3-\alpha_3\gamma_2)\,y^2z \\ +\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2)\,yz^2+\beta_2(\alpha_1\beta_3-\alpha_3\beta_1)\,z^2x+\beta_2(\beta_1\gamma_3-\beta_3\gamma_1)\,zx^2 \\ +(\gamma_1\alpha_2\beta_3+\beta_1\gamma_2\alpha_3-2\alpha_1\beta_2\gamma_3)\,xyz = 0.\end{aligned}$$

If $lx+my+nz = 0$ is part of the cubic, the remaining part must be

$$xy\,\frac{\gamma_3(\gamma_1\beta_2-\beta_1\gamma_2)}{l}+yz\,\frac{\alpha_1(\alpha_2\gamma_3-\alpha_3\gamma_2)}{m}+zx\,\frac{\beta_2(\alpha_1\beta_3-\alpha_3\beta_1)}{n}=0,$$

and we must have the conditions

$$\frac{m}{l}\,\gamma_3(\gamma_1\beta_2-\beta_1\gamma_2) = \gamma_3(\alpha_1\gamma_2-\alpha_2\gamma_1),$$

$$\frac{n}{m}\,\alpha_1(\alpha_2\gamma_3-\alpha_3\gamma_2) = \alpha_1(\beta_2\alpha_3-\beta_3\alpha_2),$$

$$\frac{l}{n}\,\beta_2(\alpha_1\beta_3-\alpha_3\beta_1) = \beta_2(\beta_1\gamma_3-\beta_3\gamma_1),$$

and $$\frac{n}{l}\gamma_3(\gamma_1\beta_2-\gamma_2\beta_1)+\frac{l}{m}\alpha_1(\alpha_2\gamma_3-\alpha_3\gamma_2)+\frac{m}{n}\beta_2(\alpha_1\beta_3-\alpha_3\beta_1)$$

$$=\gamma_1\alpha_2\beta_3-2\alpha_1\beta_2\gamma_3+\beta_1\gamma_2\alpha_3.$$

The ratio $m : n$ becomes indeterminate when $\alpha_1=0$, and also when

$$\alpha_2\gamma_3-\alpha_3\gamma_2=\alpha_3\beta_2-\alpha_2\beta_3=0.$$

The first case has already been discussed.

In the second case E, F are coincident.

If none of the ratios $l : m : n$ be indeterminate, then

$$(\gamma_1\beta_2-\gamma_2\beta_1)(\alpha_2\gamma_3-\alpha_3\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$$
$$=(\alpha_1\gamma_2-\alpha_2\gamma_1)(\beta_2\alpha_3-\beta_3\alpha_2)(\beta_1\gamma_3-\beta_3\gamma_1).$$

This equation is equivalent to the condition that the points X, Y, Z should be collinear.

This will be the case when the points D, E, F are collinear, and also when the straight lines AD, BE, CF are concurrent: as is apparent also from the algebraical identity

$$(\gamma_1\beta_2-\beta_1\gamma_2)(\alpha_2\gamma_3-\alpha_3\gamma_2)(\alpha_1\beta_3-\alpha_3\beta_1)$$
$$-(\alpha_1\gamma_2-\alpha_2\gamma_1)(\beta_2\alpha_3-\beta_3\alpha_2)(\beta_1\gamma_3-\beta_3\gamma_1)$$
$$\equiv(\alpha_2\beta_3\gamma_1-\alpha_3\beta_1\gamma_2)\left\{\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)+\alpha_2(\beta_3\gamma_1-\beta_1\gamma_3)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)\right\}.$$

9. The coordinates of P, Q, R are given by the following equations:—

$$P;\quad \frac{x_1}{\alpha_2\alpha_3}=\frac{y_1}{\beta_2\alpha_3}=\frac{z_1}{\alpha_2\gamma_3},$$

$$Q;\quad \frac{x_2}{\alpha_1\beta_3}=\frac{y_2}{\beta_1\beta_3}=\frac{z_2}{\beta_1\gamma_3},$$

$$R;\quad \frac{x_3}{\alpha_1\gamma_2}=\frac{y_3}{\beta_2\gamma_1}=\frac{z_3}{\gamma_1\gamma_2}.$$

If we substitute the coordinates of P, Q, R for the coordinates of D, E, F in the equation of the cubic

$$(\alpha_2y-\beta_2x)(\beta_3z-\gamma_3y)(\gamma_1x-\alpha_1z)+(\alpha_3z-\gamma_3x)(\beta_1x-\alpha_1y)(\gamma_2y-\beta_2z)=0,$$

we obtain the equation

$$\beta_3(\alpha_1y-\beta_1x)\gamma_1(\beta_2z-\gamma_2y)\alpha_2(\gamma_3x-\alpha_3z)$$
$$+\gamma_2(\alpha_1z-\gamma_1x)\alpha_3(\beta_2x-\alpha_2y)\beta_1(\gamma_3y-\beta_3z)=0.$$

This proves that in general the substitution of P, Q, R for D, E, F

gives rise to a second cubic; but that the cubic is unaltered by this substitution in the case when $\alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2$, that is, when *AD*, *BE*, *CF* are concurrent.

10. In discussing the case when *AD*, *BE*, *CF* are concurrent it is more convenient to use a modified system of coordinates instead of the ordinary triangular coordinates.

We will take the coordinates of a point, multiples of the triangular coordinates of the point, such that the equations of *AD*, *BE*, *CF* will be $y = z$, $z = x$, $x = y$ respectively. Then we may take the coordinates of *D*, *E*, *F*, thus:

$$\alpha, 1, 1; \quad 1, \beta, 1; \quad 1, 1, \gamma.$$

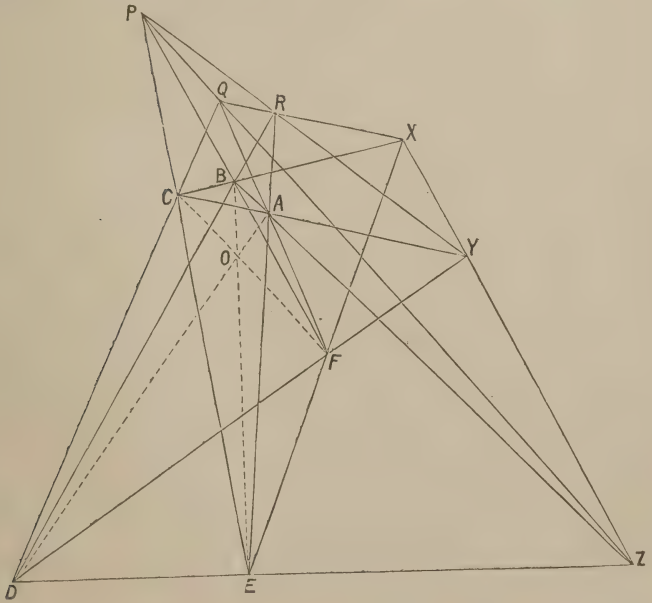
The equation of the cubic will then be

$$(x-\alpha y)(y-\beta z)(z-\gamma x)+(x-\alpha z)(y-\beta x)(z-\gamma y) = 0,$$

or $$\gamma(\beta-1)\,x^2y+\alpha(\gamma-1)\,y^2z+\beta(\alpha-1)\,z^2x+\beta(\gamma-1)\,x^2z$$
$$+\gamma(\alpha-1)\,y^2x+\alpha(\beta-1)\,z^2y+2(1-\alpha\beta\gamma)\,xyz = 0.$$

11. The equations of *EF*, *FD*, *DE* are now as follows:—

$$EF: (\beta\gamma-1)\,x+(1-\gamma)\,y+(1-\beta)\,z = 0,$$
$$FD: (1-\gamma)\,x+(\gamma\alpha-1)\,y+(1-\alpha)\,z = 0,$$
$$DE: (1-\beta)\,x+(1-\alpha)\,y+(\alpha\beta-1)\,z = 0.$$



In this case X, Y, Z are collinear, and the equation of XYZ is

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0.$$

12. The equations of the tangents to the cubic at the angular points of the triangle ABC are

$$\gamma(\beta-1)\,y+\beta(\gamma-1)\,z=0,$$
$$\alpha(\gamma-1)\,z+\gamma(\alpha-1)\,x=0,$$
$$\beta(\alpha-1)\,x+\alpha(\beta-1)\,y=0.$$

We see therefore that a conic can be described touching the cubic at A, B, C.

Its equation is

$$\frac{\alpha}{\alpha-1}\,yz+\frac{\beta}{\beta-1}\,zx+\frac{\gamma}{\gamma-1}\,xy=0.$$

As the cubic is unaltered by the interchange of any two of the three triangles ABC, DEF, PQR, we conclude that conics can be described to touch the cubic at the angular points of the triangles DEF, PQR.

13. Since the triangles ABC, DEF are compolar, the points X, Y, Z are collinear, also the triangles ABF, DEC are compolar; therefore P, Q, Z are collinear. Similarly, X, Q, R and P, Y, R are collinear. Therefore the triangle PQR is coaxial and also compolar with each of the triangles ABC, DEF. It follows, therefore, in this case that Grassmann's cubic is the same for each of the six possible permutations of the triangles ABC, DEF, PQR.

14. The triangle XYR is compolar with each of the compolar triangles ABF, DEC, and they have a common axis PQZ. Hence conics can be described to touch the curve at the angular points of each of the triangles ABF, DEC, XYR. In the same way, it can be shown that, if we take the 64 combinations of letters which it is possible to obtain by taking one letter out of each of the three sets $ADPX$; $BEQY$; $CFRZ$, each of 48 will give the angular points of a triangle through which a conic can be drawn having triple contact with the cubic, and the remaining 16 combinations will give sets of collinear points.

15. From the general theory of cubic curves, we see that the tangents to the curve at A, B, Z, three points on a straight line, meet

the curve again in A', B', Z', three points on a straight line, also the three tangents at A, B, C, the points of contact of a conic having triple contact with the curve, meet the curve again in A', B', C', three points on a straight line.

It follows therefore that the points Z', C' are coincident. Similarly, it can be shown that the tangents at F and R pass through the same point C'. Similarly, the tangents at A, D, P, X all pass through A', and the tangents at B, E, Q, Y all pass through B'.

16. It appears from what has gone before that, if $lx+my+nz=0$ be part of the cubic,

$$l(\alpha-1)=m(\beta-1)=n(\gamma-1),$$

and therefore the equation of the straight line may be written

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0,$$

which is the equation of XYZ. If this line be part of the cubic, it is seen, by comparing the coefficients of x^2y, y^2z, z^2x in the cubic, that the remaining part of the cubic must be the conic

$$\frac{\alpha}{\alpha-1}yz+\frac{\beta}{\beta-1}zx+\frac{\gamma}{\gamma-1}xy=0.$$

Multiplication of the left-hand sides of the last two equations gives at once the coefficients of the terms xy^2, yz^2, zx^2, the same as those of the cubic; but, in order that the cubic may admit of the two factors, α, β, γ must satisfy the condition

$$\frac{\alpha(\beta-1)(\gamma-1)}{\alpha-1}+\frac{\beta(\gamma-1)(\alpha-1)}{\beta-1}+\frac{\gamma(\alpha-1)(\beta-1)}{\gamma-1}=2(1-\alpha\beta\gamma),$$

or $$\alpha(\beta-1)^2(\gamma-1)^2+\beta(\gamma-1)^2(\alpha-1)^2+\gamma(\alpha-1)^2(\beta-1)^2$$
$$-2(1-\alpha\beta\gamma)(\alpha-1)(\beta-1)(\gamma-1)=0.$$

Let $p=\alpha\beta\gamma,\quad q=\beta\gamma+\gamma\alpha+\alpha\beta,\quad s=\alpha+\beta+\gamma.$

Now $$\Sigma\alpha(\beta-1)^2(\gamma-1)^2$$
$$=\Sigma\alpha(\beta^2-2\beta+1)(\gamma^2-2\gamma+1)$$
$$=\Sigma\alpha\left\{\beta\gamma(\beta\gamma-2\beta-2\gamma+4)+\beta^2-2\beta+\gamma^2-2\gamma+1\right\}$$
$$=pq-4ps+12p+sq-3p-4q+s$$
$$=pq-4ps+9p+sq-4q+s.$$

Therefore $\Sigma\alpha(\beta-1)^2(\gamma-1)^2-2(1-\alpha\beta\gamma)(\alpha-1)(\beta-1)(\gamma-1)$

$$= pq-4ps+9p+sq-4q+s+2p(p-q+s-1)-2(p-q+s-1)$$
$$= 2p^2+p(-q-2s+5)+sq-2q-s+2$$
$$= (p-s+2)(2p-q+1).$$

Therefore the cubic consists of the straight line

$$\frac{x}{\alpha-1}+\frac{y}{\beta-1}+\frac{z}{\gamma-1}=0$$

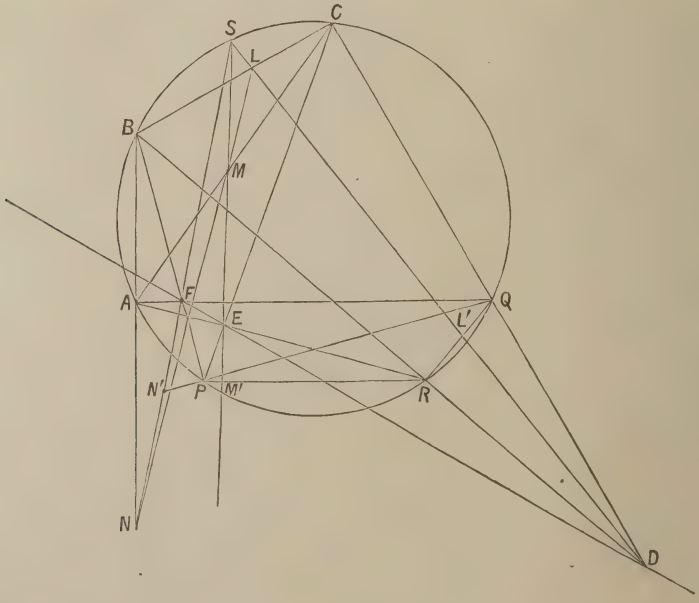
and the conic $$\frac{\alpha}{\alpha-1}yz+\frac{\beta}{\beta-1}zx+\frac{\gamma}{\gamma-1}xy=0,$$

if either $$\alpha\beta\gamma-\alpha-\beta-\gamma+2=0,$$

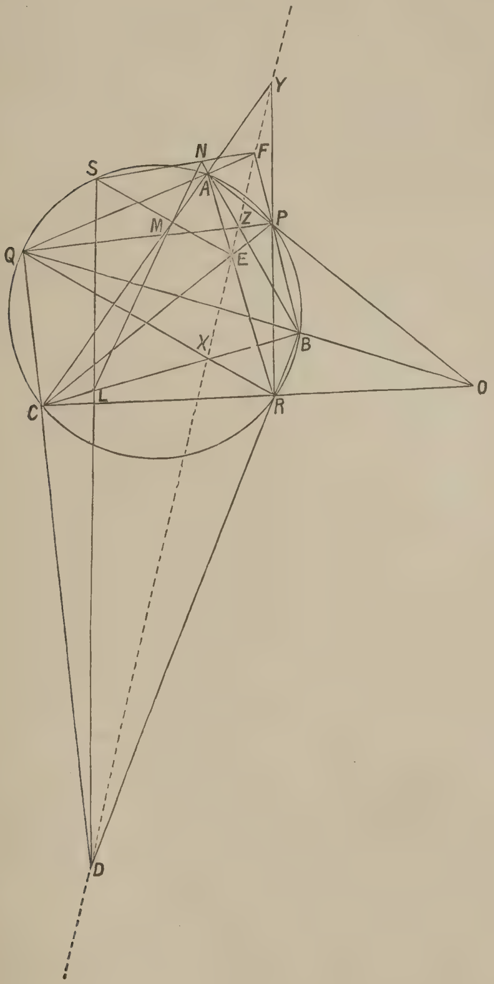
or $$\frac{1}{\alpha\beta\gamma}-\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}+2=0,$$

that is to say, either if the points D, E, F are collinear, or if the points A, B, C, D, E, F lie on a conic. In the first case, the line is DEF and the conic $ABCPQR$; in the second case, the line is PQR and the conic $ABCDEF$.

17. We have proved that Grassmann's cubic degenerates, in the case when D, E, F are collinear, into a straight line DEF and the conic $ABCPQR$.



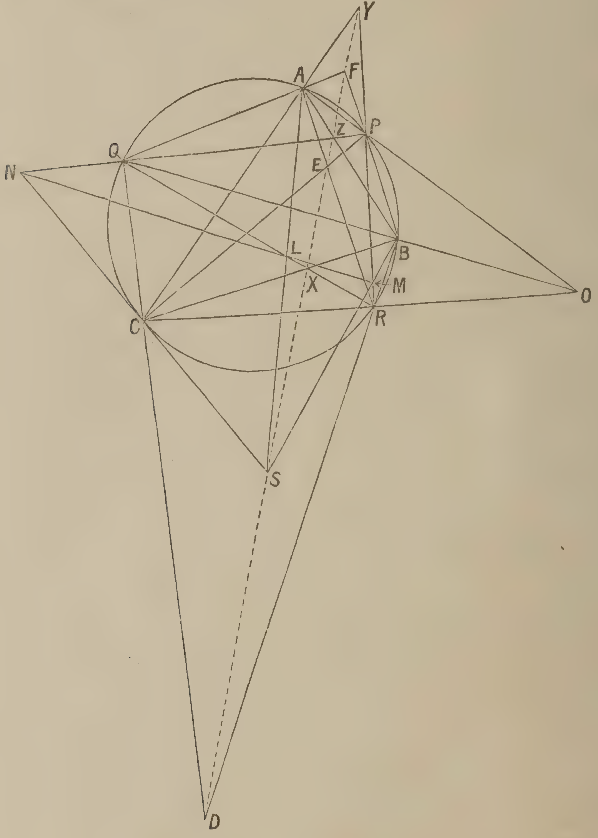
18. We have also proved that, in the case when AD, BE, CF are concurrent, the cubic is unchanged by the interchange of any two of the three triangles ABC, DEF, PQR.



19. The following theorem is a corollary from § 17 and Pascal's theorem :—

If BR, CQ ; CP, AR ; AQ, BP, pairs of opposite sides of a hexagon inscribed in a conic, intersect in D, E, F respectively, and if S be any point on the conic, then SD, SE, SF intersect the sides of each of the triangles ABC, PQR in three collinear points.

Professor Elliott has remarked that this theorem may also be obtained by projection from an extension of Simson's line theorem given in the *Pitt Press Euclid* (Book III., p. 272, Ex. 1).



20. The following theorems are corollaries from §§ 17, 18.

If ABC, PQR be a pair of compolar triangles, whose pole is O, inscribed in a conic, and S be any point on the conic, then the straight lines joining S to the angular points of either of the triangles ABC, PQR intersect the sides of the other triangle in three points lying on a straight line.

It can be proved by the help of Pascal's theorem that this straight line passes through O. (Townsend's *Modern Geometry*, Vol. II., p. 207).

Further: if ABC, PQR be a pair of compolar triangles inscribed in a conic, and D, E, F the points of intersection of BR, CQ; CP, AR; AQ, BP, and S any point on DEF, then the three straight lines joining S to the angular points of either of the triangles ABC, PQR intersect the sides of the other triangle in three collinear points.

Professor Elliott remarks that this result can be proved by comparing J. W. Russell, *An Elementary Treatise on Pure Geometry* (1893), p. 182, Ex. 6, and J. W. Russell, *l.c.*, p. 215, Ex. 6.

The Calculus of Equivalent Statements. (Sixth Paper.)[*] *By* HUGH MACCOLL, B.A.

Communicated June 10th, 1897, by Mr. TUCKER. Received, in revised form, September 4th, 1897.

It will be observed that the three-dimensional system of logic which I proposed in my last paper is closely connected with the theory of probability; so much so that the three classes of statements, ϵ, η, θ may be respectively defined as statements whose chances of being true are 1, 0, x, in which x is some proper fraction less than 1 and greater than 0. But it is clear that we might develop other analogous three-dimensional systems on similar, yet not identical, principles. For example, κ might denote all statements *known to be true*, or capable of being proved true; λ all statements *known to be false*, or capable of being proved false; and μ all statements which, through insufficiency of data, *can neither be proved true nor proved false*. Or, again, the symbol u might denote all *functional* statements which, like $\alpha\beta+\alpha'+\beta'$, are *true for all values of their constituents*; v all functional statements which, like $\alpha\beta\,(\alpha'+\beta')$, are *false for all values of their constituents*; and w all functional statements which, like $\alpha:\beta$ or $\alpha\beta+\gamma$, *may be true or false* according to the values we give to their elementary constituents. By combining the logical system (ϵ, η, θ) with the logical system (κ, λ, μ), we should have a logic of 3^2 dimensions; and by combining this again with the system (u, v, w) we should have a logic of 3^3 dimensions; and so on. But some of these systems—of which the *subjective* system (κ, λ, μ) is one—might open psychological or metaphysical questions whose discussion would be out of place in this paper; so I shall here confine myself mainly to the bearing of the (ϵ, η, θ) system upon mathematical probability.

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[*] Though I call this the *Sixth* Paper, and the one preceding it was called the *Fifth*, I may mention that my paper in the *Proceedings* on "The Limits of Multiple Integrals" (printed after my *Fourth*) belongs to the same subject, though it has a different title. It is a continuation on purely mathematical lines of my *First* Paper.

Probability Notation No. 2.

In my Fourth Paper I used the symbol α_β to denote *the chance that* α *is true on the assumption that* β *is true*, and the symbol α_ϵ to denote *the chance that* α *is true on the assumption* ϵ *only*, in which ϵ denotes the *data of the problem*; these data being assumed to be true *always*, *i.e.*, throughout the whole investigation. This notation answered very well for the questions there treated; but it becomes rather awkward when we have to express such a chance, for instance, as the chance that α^θ is true on the assumption that β^η is true; so in this paper I shall generally adopt a notation more suitable to the problems treated.

Def. 1.—When we have two *statements* A and B, the symbol $\frac{A}{B}$ denotes the chance that A is true on the assumption that B is true.

In this paper, therefore, when A and B denote *statements*, the symbol $\frac{A}{B}$ will not (as in my last paper) denote a *causal implication*, but a *chance;* the symbol $A:B$ will, as before, denote a *general implication*, and be synonymous with $(AB')^\eta$; while the symbol A_B will vary its meaning according to convenience and the nature of the problem treated. Another caution may be needed, namely, that $\frac{A}{B}$ and $\frac{B}{A}$ are not generally *reciprocals* when A and B are *statements*. For example,

the chance $$\frac{A}{AB} = 1;$$

but the chance $$\frac{AB}{A} = \frac{B}{A},$$

which is *not* generally $= 1$.

Def. 2.—The symbol $\frac{A}{\epsilon}$ denotes *the chance that* A *is true on the assumption that* ϵ *is true;* but, as ϵ is understood to be true throughout, the symbol $\frac{A}{\epsilon}$ simply denotes *the chance that A is true;* no assumption being made beyond the *data of the problem*, which are supposed to be always held in the recollection and understood when not stated.

DEF. 3.—The symbol $\frac{A}{B, C}$ means $\frac{A}{B}+\frac{A}{C}$, in which A and B are statements, and $\frac{A}{B}$ and $\frac{A}{C}$ denote *chances*, as explained in Def. 1.

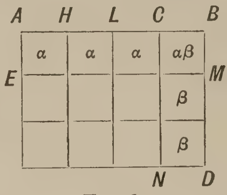

FIG. 1.

For example, in the accompanying diagram let α assert that a random point P will fall in the rectangle AM; let β assert that P will fall in the rectangle BN; and let ϵ assert that it will fall somewhere in the whole figure AD made up of 12 equal rectangles. Then we shall have

$$\frac{\alpha}{\beta, \beta'}=\frac{\alpha}{\beta}+\frac{\alpha}{\beta'}=\frac{CM}{CD}+\frac{CE}{AN}=\frac{1}{3}+\frac{3}{9}=\frac{2}{3}.$$

In this case, since $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta'}$ happen to be equal chances, α and β are independent; but α and β would not be independent if $\frac{CM}{CD}$ were not equal to $\frac{CE}{AN}$ (see my Fourth Paper). Thus, when two statements A and B are independent, we have

$$\frac{A}{B}=\frac{A}{B'}=\frac{A}{\epsilon}, \text{ and } \frac{A}{B, B'}=2\times\frac{A}{\epsilon}.$$

DEF. 4.—The symbol $\frac{A, B}{C}$ means $\frac{A}{C}+\frac{B}{C}$.

The following problem in probability will show the difference in meaning between $A'+B$ and $A : B$.

Problem 1.—Out of 25 mixed statements, consisting of 13 *certainties*, 7 *impossibilities*, and 5 *independent variables*, whose chances of being true are respectively $\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{2}{3}, \frac{3}{4}$, a statement A is taken at random; and out of a *duplicate* collection of 25 statements (mixed in the same proportions) a statement B is taken at random. What is the chance (1) that $A'+B$ is true? and (2) that $A : B$ is true?

We have $$A : A^{\epsilon}+A^{\theta};$$

therefore $$A=A\left(A^{\epsilon}+A^{\theta}\right)=A^{\epsilon}+AA^{\theta};$$

therefore $$\frac{A}{\epsilon} = \frac{A^{\epsilon}}{\epsilon} + \frac{AA^{\theta}}{\epsilon} = \frac{A^{\epsilon}}{\epsilon} + \frac{A^{\theta}}{\epsilon} \cdot \frac{A}{A^{\theta}} = \frac{13}{25} + \frac{5}{25} \cdot \frac{A}{A^{\theta}}.$$

Now let the symbol A^c be a *statement* and assert that A *will turn out to be the variable statement whose chance of being true is* c; in which, from the data, c has five possible values, $\frac{1}{4}$, $\frac{1}{3}$, &c. Then we get

$$\frac{A}{A^{\theta}} = \frac{AA^{\theta}}{A^{\theta}} = \frac{A\left(A^{\frac{1}{4}} + A^{\frac{1}{3}} + A^{\frac{2}{5}} + A^{\frac{2}{3}} + A^{\frac{3}{4}}\right)}{A^{\theta}} = \Sigma \frac{AA^c}{A^{\theta}}.$$

But, from the probability formula,

$$\frac{\alpha\beta}{u} = \frac{\alpha}{u} \cdot \frac{\beta}{\alpha u},$$

which may be written

$$\frac{\alpha\beta}{u} = \frac{\alpha}{u} \cdot \frac{\beta}{\alpha},$$

if the assumption u (like ϵ) as well as α be always understood in the second chance $\dfrac{\beta}{\alpha}$, we get

$$\frac{AA^c}{A^{\theta}} = \frac{A^c}{A^{\theta}} \cdot \frac{A}{A^c} = \frac{1}{5} \cdot \frac{A}{A^c} = \frac{1}{5}c;$$

therefore $$\frac{A}{A^{\theta}} = \Sigma \frac{1}{5}c = \frac{1}{5}\left(\frac{1}{4} + \frac{1}{3} + \frac{2}{5} + \frac{2}{3} + \frac{3}{4}\right) = \frac{12}{25},$$

a result which is almost evident *a priori*. Hence we get

$$\frac{A}{\epsilon} = \frac{13}{25} + \frac{5}{25} \cdot \frac{12}{25} = \frac{77}{125}.$$

Now, from the formula

$$\frac{\alpha + \beta}{u} = \frac{\alpha}{u} + \frac{\beta}{u} - \frac{\alpha\beta}{u},$$

we get

$$\frac{A' + B}{\epsilon} = \frac{A'}{\epsilon} + \frac{B}{\epsilon} - \frac{A'B}{\epsilon} = \left(1 - \frac{A}{\epsilon}\right) + \frac{B}{\epsilon} - \left(1 - \frac{A}{\epsilon}\right)\frac{B}{\epsilon};$$

for in this problem A and B are independent, so that

$$\frac{A'B}{\epsilon} = \frac{A'}{\epsilon} \cdot \frac{B}{\epsilon} = \left(1 - \frac{A}{\epsilon}\right)\frac{B}{\epsilon}.$$

Also $$\frac{A}{\epsilon} = \frac{B}{\epsilon}.$$

Hence $\dfrac{A'+B}{\epsilon} = 1 - \left(1 - \dfrac{A}{\epsilon}\right)\dfrac{A}{\epsilon} = 1 - \dfrac{48}{125}\cdot\dfrac{77}{125} = \cdot 763406.$

Next, to find $\dfrac{A:B}{\epsilon}$,

$$A:B = (AB')^\eta = A^\eta + B^\epsilon + A^\theta B^\theta (AB')^\eta\,;$$

and the two statements $(A^\eta + B^\epsilon)$ and $A^\theta B^\theta$ being mutually inconsistent, we get

$$\begin{aligned}\frac{A:B}{\epsilon} &= \frac{A^\eta + B^\epsilon}{\epsilon} + \frac{A^\theta B^\theta (AB')^\eta}{\epsilon} \\ &= \left(\frac{A^\eta}{\epsilon} + \frac{B^\epsilon}{\epsilon} - \frac{A^\eta B^\epsilon}{\epsilon}\right) + \frac{A^\theta B^\theta}{\epsilon}\cdot\frac{(AB')^\eta}{A^\theta B^\theta} \\ &= \left(\frac{A^\eta}{\epsilon} + \frac{B^\epsilon}{\epsilon} - \frac{A^\eta}{\epsilon}\cdot\frac{B^\epsilon}{\epsilon}\right) + \frac{A^\theta}{\epsilon}\cdot\frac{B^\theta}{\epsilon}\cdot\frac{(AB')^\eta}{A^\theta B^\theta} \\ &= \left(\frac{7}{25} + \frac{13}{25} - \frac{7}{25}\cdot\frac{13}{25}\right) + \frac{5}{25}\cdot\frac{5}{25}\cdot\frac{(AB')^\eta}{A^\theta B^\theta} \\ &= \frac{409}{625} + \frac{1}{25}\,\frac{(AB')^\eta}{A^\theta B^\theta}\,;\end{aligned}$$

for A^η and B^ϵ are independent, as are also A^θ and B^θ. It remains to find $\dfrac{(AB')^\eta}{A^\theta B^\theta}$. Now, since here the five *independent variables* from which B is taken at random are respectively *duplicates* of the five *independent variables* from which A is taken at random, the statement $(AB')^\eta$ asserts that B is a duplicate (or synonym) of A; for in no other way (since the five statements in each collection are independent of each other) could AB' be an *impossibility*, or *inconsistency*, though in other ways it might, in the long run, turn out to be *sometimes false.* The chance of A and B turning out duplicates of each other, when $A^\theta B^\theta$ is taken for granted, is evidently $\frac{1}{25} \times 5$ or $\frac{1}{5}$.

Hence we get

$$\frac{A:B}{\epsilon} = \frac{409}{625} + \frac{1}{25}\cdot\frac{1}{5} = \frac{414}{625} = \cdot 6624.$$

Thus we see that $\dfrac{A:B}{\epsilon}$ is less than $\dfrac{A'+B}{\epsilon}$, as it should be, since $A:B$ is a stronger statement than $A'+B$, and therefore less likely to be true.

In this problem the data were sufficient to determine the *exact* values of the chances $\frac{A'+B}{\epsilon}$ and $\frac{A:B}{\epsilon}$. In the following problem, the data will only allow us to determine the limits of these chances.

Problem 2.—Out of a collection of s mixed statements, consisting of m *certainties*, n *impossibilities*, and r *variables*, a statement A is taken at random; and out of another collection of s statements mixed in the same proportions a statement B is taken at random. Required the limits of the chances $\frac{A'+B}{\epsilon}$ and $\frac{A:B}{\epsilon}$.

As before, we get

$$\frac{A}{\epsilon}=\frac{m}{s}+\frac{r}{s}\cdot\frac{A}{A^{\theta}};$$

but this time we have no data for determining $\frac{A}{A^{\theta}}$, and can therefore only state the *limits* of the chance $\frac{A}{\epsilon}$. That is, we get

$$\frac{m}{s}<\frac{A}{\epsilon}<\frac{m}{s}+\frac{r}{s}.$$

Again, as before, we get

$$\frac{A'+B}{\epsilon}=\frac{A'}{\epsilon}+\frac{B}{\epsilon}-\frac{A'}{\epsilon}\cdot\frac{B}{\epsilon}=1-\left(1-\frac{A}{\epsilon}\right)\frac{A}{\epsilon}.$$

When $\frac{A}{\epsilon}=\frac{1}{2}$, this has a minimum value $\frac{3}{4}$; so that $\frac{A'+B}{\epsilon}$ lies between the *inferior* limit $\frac{3}{4}$ and the *superior* limit $1-\left(1-\frac{m}{s}\right)\frac{m}{s}$ or $1-\left(1-\frac{m+r}{s}\right)\frac{m+r}{s}$, whichever of these two proper fractions happens to be higher. This covers the widest case, $\frac{m}{s}<\frac{1}{2}<\frac{m+r}{s}$.

To find the limits of $\frac{A:B}{\epsilon}$, we have, as before,

$$\frac{A:B}{\epsilon}=\left(\frac{A^{\eta}}{\epsilon}+\frac{B^{\epsilon}}{\epsilon}-\frac{A^{\eta}}{\epsilon}\cdot\frac{B^{\epsilon}}{\epsilon}\right)+\frac{A^{\theta}}{\epsilon}\cdot\frac{B^{\theta}}{\epsilon}\cdot\frac{(AB')^{\eta}}{A^{\theta}B^{\theta}}$$

$$=\frac{n}{s}+\frac{m}{s}-\frac{mn}{s^2}+\frac{r^2}{s^2}\,\frac{(AB')^{\eta}}{A^{\theta}B^{\theta}}.$$

But $\frac{(AB')^{\eta}}{A^{\theta}B^{\theta}}$ may, without contradicting the data, have any value from

0 to 1, so that all we can legitimately infer from our premisses is

$$\frac{m+n}{s}-\frac{mn}{s^2}<\frac{A:B}{\epsilon}<\frac{m+n}{s}-\frac{mn}{s^2}+\frac{r^2}{s^2}.$$

In other words, we can assign limits between which the chance $\frac{A:B}{\epsilon}$ must lie, but we cannot determine its exact value.

If the data implied that the statements from which B was taken at random were respectively *duplicates* (or synonyms) of those from which A was taken at random, we could infer (as in the preceding problem) that $\frac{(AB')^\eta}{A^\theta B^\theta}$ was $\frac{1}{r}$, and, in that case, we should know the *exact* value of $\frac{A:B}{\epsilon}$ to be $\frac{m+n}{s}-\frac{mn}{s^2}+\frac{r}{s^2}$. Or, secondly, if our data implied that the statements from which B was taken at random were wholly unconnected with and independent of the statements from which A was taken at random (though mixed in the same proportions), so that AB' would, in the long run, be sometimes true and sometimes false, but never an *inconsistency or impossibility*, then $\frac{(AB')^\eta}{A^\theta B^\theta}$ would be *zero*, and the exact value of $\frac{A:B}{\epsilon}$ would be

$$\frac{m+n}{s}-\frac{mn}{s^2}.$$

Or, thirdly, every one of the variables from which A was taken at random might be known to imply or be a multiple of every one of the variables from which B was taken at random; in which case the value of $\frac{(AB')^\eta}{A^\theta B^\theta}$ would be *unity*, and the exact value of $\frac{A:B}{\epsilon}$ would be $\frac{m+n}{s}-\frac{mn}{s^2}+\frac{r^2}{s^2}$. But, as none of these assumptions is stated in or deducible from the data, we must be satisfied with determining the nearest obtainable *limits* of the chance $\frac{A:B}{\epsilon}$, till further data enable us to find its exact value or closer limits.

Note 1.—The following problem is Boole's "Challenge Problem" (*Laws of Thought*, p. 321). Boole professes to find the *exact* value of the chance required by means of what he calls a "General Method in Probabilities," and obtains a result which I proved to be wrong in my Fourth Paper in the *Proceedings*. But a wrong result may arise from a mere symbolic slip in the application of a general method, and does not necessarily imply a fundamental error in principle. Boole's

"General Method," however, *does* contain such a fundamental error, as it professes to obtain exact results from data which are demonstrably insufficient. Before touching upon what I believe to be the leading fallacy in his reasoning, I shall prove formally that the required chance in his "Challenge Problem" is unobtainable from its data.

Problem 3.—Boole's problem expressed in the notation of this paper and with other letters is as follows:—

Let A assert that a certain cause or circumstance A exists or occurs; let B assert that a certain cause or circumstance B exists or occurs; and let V assert that a certain event V takes place. Given

$$(1)\ \frac{A}{\epsilon}=a,\quad (2)\ \frac{B}{\epsilon}=b,\quad (3)\ \frac{V}{A}=p,\quad (4)\ \frac{V}{B}=q,$$

and (5) $V:A+B$;

from these five data required to determine $\frac{V}{\epsilon}$.

We have $V=V(A+B)=AV+BV$;

therefore

$$\frac{V}{\epsilon}=\frac{AV}{\epsilon}+\frac{BV}{\epsilon}-\frac{ABV}{\epsilon}$$

$$=\frac{A}{\epsilon}\cdot\frac{V}{A}+\frac{B}{\epsilon}\cdot\frac{V}{B}-\frac{AB}{\epsilon}\cdot\frac{V}{AB}$$

$$=ap+bq-\frac{AB}{\epsilon}\cdot\frac{V}{AB}.$$

Now, the negative term in this result may have numberless values, all consistent with the data. The causes A and B, though capable of being combined, might (like two strong poisons which neutralize each other) be powerless when so combined, so that $\frac{V}{AB}$ would be zero. In this case $\frac{V}{\epsilon}$ would $=ap+bq$. Again, A and B might be independent, while $\frac{V}{AB}$ might $=1$; that is to say, the independent causes A and B might be so powerful in combination, though not separately, that the event V would be *sure* to follow from their joint action. In this case we should have

$$\frac{V}{\epsilon}=ap+bq-ab.$$

Or, lastly (as seems likely in Boole's concrete illustration of the flooding of his field by overflow of the river Lee or tides from the ocean), the causes A and B might be independent of each other, and more powerful conjointly than separately, yet not *always* sufficiently powerful either separately or conjointly to produce the event V. In this case we should have $\frac{V}{AB}$ greater than $\frac{V}{A}$, and also greater than $\frac{V}{B}$, so that $\frac{V}{\epsilon}$ would have $ap+bq-ab$ as an *inferior* limit, and the smaller of the two $ap+bq-abp$ or $ap+bq-abq$ as the nearest *superior* limit. This is the hypothesis I took (as it was in conformity with Boole's own illustration) when, in my Fourth Paper, I showed that Boole's result was outside the above limits and therefore wrong.

Thus the required chance varies on different hypotheses, each of which is consistent with the data of the problem; and, as *the respective chances of the truth of these hypotheses are wholly unknown to us*, we cannot infer that the required chance has a fixed or constant value calculable from the data.

Note 2.—The fallacy that vitiates Boole's whole reasoning on probability is to be found in his definition of *independent events.* "Two events," he says, "are independent when the probability of the happening of either of them is unaffected by our expectation of the occurrence or failure of the other" (*Laws of Thought*, p. 255); and a little further on he says: "When the probabilities of events are given, but all information respecting their dependence withheld, the mind regards them as independent." The erroneous nature of this *subjective* view of the nature of probability will appear from the following geometrical examples:—

Diagrammatic Illustrations of Dependent and Independent Events.

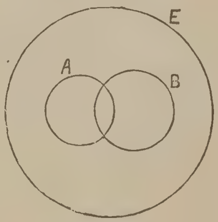

FIG. 2.

First let there be two intersecting circles A and B enclosed in a third circle E; and in the circle E let a point P be taken at random. Now consider the three possibilities: (1) the random point may fall in the circle A, (2) it may fall in B, and (3) it may fall in both A and B, that is, it may fall in the area common to A and B. Let the symbols α, β respectively assert that P will fall in A, and that P will fall in B. Then $\frac{\alpha}{\epsilon}$, $\frac{\beta}{\epsilon}$, $\frac{\alpha\beta}{\epsilon}$ will be the respective chances of the above three possibilities; and the third chance $\frac{\alpha\beta}{\epsilon}$ is equal to $\frac{\alpha}{\epsilon}.\frac{\beta}{\alpha}$. When $\frac{\beta}{\alpha}=\frac{\beta}{\epsilon}$, the ratio which the area common to A and B has to the area of A is the same as the ratio which the area of B has to the area of E; *in this case only* the events asserted by the statements α and β are independent. If the areal ratios above mentioned are *not* equal, the events are *not* independent, whether the mind chooses to regard them so or not.

But it may be objected that the *two possibilities* here asserted by the statements α and β constitute really but *one event*, as the statements have reference to the same random point. Let us therefore take another example. Referring to the same figure, suppose *two* points P and Q to be taken at random; first, the former in the circle E; and then the latter in the circle A *if* P *happens to fall in* A, but in the circle E if P happens *not* to fall in A. Let the symbol P_A assert that P will fall in A, P_B that P will fall in B, P_{AB} that P will fall both in A and B (*i.e.*, in the area common to both circles). Let Q_A, Q_B, Q_{AB} be similarly interpreted; and let P'_A, P'_{AB}, Q'_B, &c., be the *denials* of the corresponding unaccented statements. Are the two events asserted by the statements P_A and Q_B independent? The answer is *Yes* if the chances $\frac{Q_B}{P_A}$ and $\frac{Q_B}{P'_A}$ are equal, and *No* if they are *not* equal; and this equality or non-equality depends, as before, upon the sizes and positions of the circles, and in no way upon our *knowledge* (real or supposed) of those sizes and positions. (See Problem 4, further on.)

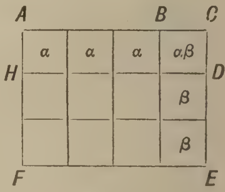

Fig. 3.

The accompanying diagram (Fig. 3), consisting of an assemblage of twelve equal rectangles, will exhibit to the eye the independence of statements or of the events which they assert. With reference to points P and Q taken at random in the whole rectangle AE, or in some defined portion of it, let the symbol P_α assert that P will fall somewhere in AD; let P'_α assert that P will *not* fall in AD; let P_β assert that P will fall somewhere in BE; let P'_β assert that P will *not* fall in BE; and let Q_α, Q'_α, Q_β, Q'_β be similarly interpreted. Now suppose P to be taken first at random in the whole rectangle AE, and that, if it happens to fall in AD, then Q is taken at random in AD; but that if P happens *not* to fall in AD, then Q (instead of being taken at random in AD) is taken at random in the whole rectangle AE.

From these data we evidently get

$$\frac{Q_\beta}{P_\alpha} = \frac{BD}{AD} = \frac{1}{4} \quad \text{and} \quad \frac{Q_\beta}{P'_\alpha} = \frac{BE}{AE} = \frac{1}{4};$$

so that here Q_β and P_α are independent.

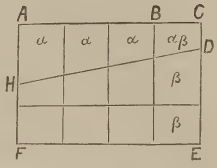

FIG. 4.

But suppose now we alter one of the lines in Fig. 3, moving HD so as to make it inclined to AC as in Fig. 4, AH being now greater than CD. As before, we shall have

$$\frac{Q_\beta}{P'_\alpha} = \frac{BE}{AE} = \frac{1}{4};$$

but now we shall have

$$\frac{Q_\beta}{P_\alpha} = \frac{BD}{AD} = \text{a fraction } \textit{less than } \tfrac{1}{4};$$

so that, in this case, Q_β and P_α are *not* independent.

Note 3.—I showed in my Fourth Paper in the *Proceedings* (but using different notation) that any one of the following four equations implies the other three :—

$$\left(\frac{\alpha}{\beta} = \frac{\alpha}{\beta'}\right), \quad \left(\frac{\alpha}{\beta} = \frac{\alpha}{\epsilon}\right), \quad \left(\frac{\beta}{\alpha} = \frac{\beta}{\alpha'}\right), \quad \left(\frac{\beta}{\alpha} = \frac{\beta}{\epsilon}\right);$$

so that any one of the four may be taken as a definition of the independence of the statements α and β, or of the independence of the events whose occurrence the statements assert. If any one of the four equations be true, all four are true, and the statements (or events) α and β are independent. If any one of the equations be false, all four are false, and the statements (or events) α and β are *not* independent.

Problem 4.—Two intersecting circles A and B of areas a and b respectively, and with an area c common to both, are enclosed in a third circle E of area *unity*.

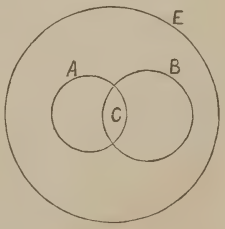

FIG. 5.

Let a point P be taken at random in E. If P happens to fall in A, let a second point Q be taken at random in A; but, if P does *not* happen to fall in A, let Q be taken at random in E. Assuming (1) that Q falls in B, what is the chance that P had fallen in A? And assuming (2) that Q does *not* fall in B, what is the chance that P had fallen in A?

This is a question in inverse probability. Let the symbols P_A, P_B, P_{AB} respectively assert that P falls in A, that it falls in B, that it falls in both A and B (*i.e.*, in the area c common to both); and let P'_A, P'_B, P'_{AB} be the denials of those three statements. Let Q_B, Q'_B, &c., be similarly interpreted. Assuming the formula

$$\frac{\alpha}{\beta}=\frac{\frac{\alpha}{\epsilon}\cdot\frac{\beta}{\alpha}}{\frac{\alpha}{\epsilon}\cdot\frac{\beta}{\alpha}+\frac{\alpha'}{\epsilon}\cdot\frac{\beta}{\alpha'}},$$

which in the following problem will be proved true for any statements α and β, whether or not these statements have reference to causes and consequences, we get for the first question (remembering that the area of E is *unity*)

$$\frac{P_A}{Q_B}=\frac{\frac{P_A}{\epsilon}\cdot\frac{Q_B}{P_A}}{\frac{P_A}{\epsilon}\cdot\frac{Q_B}{P_A}+\frac{P'_A}{\epsilon}\cdot\frac{Q_B}{P'_A}}=\frac{a\cdot\frac{c}{a}}{a\cdot\frac{c}{a}+(1-a)\,b}=\frac{c}{c+b-ab},$$

and for the second question we get, putting Q'_B for Q_B throughout,

$$\frac{P_A}{Q'_B}=\frac{\frac{P_A}{\epsilon}\cdot\frac{Q'_B}{P_A}}{\frac{P_A}{\epsilon}\cdot\frac{Q'_B}{P_A}+\frac{P'_A}{\epsilon}\cdot\frac{Q'_B}{P'_A}}=\frac{a\cdot\frac{a-c}{a}}{a\cdot\frac{a-c}{a}+(1-a)(1-b)}$$

$$=\frac{a-c}{(a-c)+(1-a)(1-b)}.$$

When the events P_A and Q_B are independent, these two results will be equal, and *vice versa*, and then (but not otherwise) we get $c=ab$. Substituting this value of c in either result, we get

$$\frac{P_A}{Q_B}=\frac{P_A}{Q'_B}=a=\frac{P_A}{\epsilon},$$

as we ought to get from our definitions of independent statements or events.

Problem 5.—To prove the probability formula assumed in the preceding problem.

Let α and β be any statements; then, whatever their nature, and whatever the *time* order of the events to which they refer (α happening before or after β), we always have

$$\frac{\alpha\beta}{\epsilon}=\frac{\alpha}{\epsilon}\cdot\frac{\beta}{\alpha}=\frac{\beta}{\epsilon}\cdot\frac{\alpha}{\beta},$$

and therefore
$$\frac{\alpha}{\beta}=\frac{\dfrac{\alpha}{\epsilon}\cdot\dfrac{\beta}{\alpha}}{\dfrac{\beta}{\epsilon}}.$$

But
$$\frac{\beta}{\epsilon}=\frac{(\alpha+\alpha')\,\beta}{\epsilon}=\frac{\alpha\beta+\alpha'\beta}{\epsilon}=\frac{\alpha}{\epsilon}\cdot\frac{\beta}{\alpha}+\frac{\alpha'}{\epsilon}\cdot\frac{\beta}{\alpha'}.$$

Hence
$$\frac{\alpha}{\beta}=\frac{\dfrac{\alpha}{\epsilon}\cdot\dfrac{\beta}{\alpha}}{\dfrac{\alpha}{\epsilon}\cdot\dfrac{\beta}{\alpha}+\dfrac{\alpha'}{\epsilon}\cdot\dfrac{\beta}{\alpha'}}.$$

Note 4.—From this follows the formula for inverse probability,

$$\frac{C}{V}=\frac{\dfrac{C_r}{\epsilon}\cdot\dfrac{V}{C_r}}{\Sigma\dfrac{C_r}{\epsilon}\cdot\dfrac{V}{C_r}},$$

in which r takes the values 1, 2, 3, ..., n, and the symbol $\frac{C_r}{V}$ denotes the chance that the r^{th} cause (amongst the series of mutually exclusive and only possible causes C_1, C_2, C_3, ... C_n) is the real cause of the event V.

Note 5.—It should be remarked that in the fundamental formula

$$\frac{ABCD...}{\epsilon}=\frac{A}{\epsilon}\cdot\frac{B}{A}\cdot\frac{C}{AB}\cdot\frac{D}{ABC}\cdots,$$

on which the preceding formulæ are founded, it is not necessary to assume that the event asserted by the statement A precedes in *the order of time* the event asserted by the statement B; that the event asserted by B takes place before that asserted by C; and so on. In whatever *time* order the events may occur, and whether or not they are mutually independent, the formula always holds good; and it will still hold good if we interchange any two of the letters. This is a point which Mr. G. Chrystal appears to have overlooked in his excellent *Text Book of Algebra* (see Part II., pp. 548, 549).

Problem 6.—Out of a very large (say, infinite) collection of problems in probability, with the correct answers to the required chances ranging in arithmetical progression (see Note 6) between 0 and 1, a problem is taken at random. What is the *a priori* probability, before the problem is known, that the event whose chance is required in it will, upon trial, happen m times out of n?

Let P_x, for all values of x, assert that the correct answer to the random problem is x, and let V predict that the event whose chance is required in the random problem will, upon trial, happen exactly m times out of n. It is clear that

$$\epsilon = P_{dx} + P_{2dx} + P_{3dx} + \ldots + P_{1-dx} + P_1 ;$$

therefore $$V = (P_{dx} + P_{2dx} + \ldots + P_1)\, V ;$$

therefore $$\frac{V}{\epsilon} = \frac{P_{dx}}{\epsilon} \cdot \frac{V}{P_{dx}} + \frac{P_{2dx}}{\epsilon} \cdot \frac{V}{P_{2dx}} + \ldots + \frac{P_1}{\epsilon} \cdot \frac{V}{P_1}$$

$$= dx \frac{V}{P_{dx}} + dx \frac{V}{P_{2dx}} + \ldots + dx \frac{V}{P_1}$$

$$= \int_0^1 dx \frac{V}{P_x} = \int_0^1 dx \frac{n!}{m!\,(n-m)!} x^m (1-x)^{n-m}$$

$$= \frac{1}{n+1} ;$$

a result which is independent of m.

Note 6.—Though in Problem 5 the correct answers are (for the sake of simplicity) supposed to be arranged in arithmetical progression between 0 and 1, the reasoning and the result will be the same if we suppose (as would be more likely) the correct answers to be distributed irregularly and at random, but on an average evenly, between 0 and 1; so that, if any three fractions in arithmetical progression, u, v, w, be taken between 0 and 1, the number of correct answers between u and v will be nearly equal to the number between v and w, whatever be the fractions u, v, w, and however near each other.*

Problem 7.—A mathematician solved a question in probability and found the required chance to be c. To test this result he had recourse to experiment and found that the event in question happened

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* In this case the symbol P_x must be understood to mean that the correct answer lies between $x + dx$ and $x - dx$.

m times out of n trials. What is the chance of c being the correct answer, assuming (1) that the *a priori* chance of his being right, independently of the experiment, is a (that is, out of n problems he correctly solves na on an average); and (2) that the problem was taken at random out of a very large (say, infinite) number of problems of which the required chances ranged at random between 0 and 1—high, low, and medium values between those limits being all equally probable (see preceding problem, with note)?

Let the symbol P_c assert that the correct chance is c; let P'_c assert that the correct chance is *not* c; and let V assert that out of n trials the event in question will happen just m times. Then, by the formula assumed in Problem 4 and proved in Problem 5, we get

$$\frac{P_c}{V} = \frac{\dfrac{P_c}{\epsilon}\cdot\dfrac{V}{P_c}}{\dfrac{P_c}{\epsilon}\cdot\dfrac{V}{P_c}+\dfrac{P'_c}{\epsilon}\cdot\dfrac{V}{P'_c}}.$$

Now $\quad \dfrac{P_c}{\epsilon}=a, \quad \dfrac{V}{P_c}=\dfrac{n!}{m!\,(n-m)!}c^m(1-c)^{n-m}, \quad \dfrac{P'_c}{\epsilon}=1-a,$

and
$$\frac{V}{P'_c}=\int_0^1 dx\,\frac{V}{P_x}=\frac{1}{n+1}$$

(see Problem 6 and Note 6).

Substituting, we get

$$\frac{P_c}{V}=\frac{aM}{aM+(1-a)\dfrac{1}{n+1}},$$

in which
$$M=\frac{n!}{m!\,(n-m)!}c^m(1-c)^{n-m}.$$

The value of M can of course be found for high numbers by the aid of Stirling's theorem.

Note 7.—There are many problems in probability of which the result may be thus confirmed or disproved by experiment. For example, if we take the coefficients A, B, C in the quadratic equation

$$Ax^2-Bx+C=0,$$

each at random between 0 and a, the chance that the roots of the equation will be real is $\frac{5}{36}+\frac{1}{6}\log 2$, which $=\cdot 18906$, &c. (see my First Paper in the *Proceedings*). Choose any value of a say, 1, so that A, B, C are random fractions between 0 and 1. The roots will be real when B^2-4AC is positive. Open anywhere at random a table of logarithms calculated to seven places, and read *backwards* to three or four figures any logarithms taken at random to represent A, B, C as decimals. Thus, if we light upon ·8948641 (which is the logarithm of 7·8499), we read it backwards to four places as ·1468, and consider this to be the random value of B. We next, on another page, light, suppose, on ·8026096 (which is log 6·3476), and, reading it backwards to four places as ·6906, consider this as the random value of A. Lastly, we find ·6632108 (which is log 4·6048), and, reading backwards, take ·8012 as the random value of C. It is evident at a glance, without the trouble of multiplying out, that

$$(\cdot 1468)^2-(4\times\cdot 6906\times\cdot 8012)$$

is negative; so that this first trial gives *imaginary* roots. The chance of real roots being only ·18906, &c., by previous calculation, we may expect this to happen (assuming our calculation correct) about 4 times out of 5, or a trifle oftener. Suppose the *a priori* chance of our making no error of reasoning or calculation to be $\frac{1}{2}$, and that, as the result of 1,000 trials, we find 180 quadratics with real roots and 820 with imaginary ones, and that we are in doubt as to the correctness of the calculated chance ·18906, &c. Then (by Problem 7) the chance of our result ·18906, &c., being correct would be $\frac{M}{M+\frac{1}{1001}}$, in which (putting c for ·18906, &c.)

$$M=\frac{(1000)!}{(180)!\,(820)!}c^{180}(1-c)^{820}=\cdot 06336 \text{ nearly},$$

by Stirling's theorem; so that

$$\frac{P_c}{V}=\cdot 985 \text{ nearly}.$$

Had we found 150 (instead of 180) real roots, and therefore 850 (instead of 820) imaginary ones, the chance $\frac{P_c}{V}$ would be no higher than ·325—a lowering of the chance which might have been expected from the considerable deviation of 150 from the most probable number 189 in the large number of trials 1,000. Had we, as the result of only 20 trials, found 3 real roots and 17 imaginary ones (*which is the same proportion as on the last supposition*), the chance $\frac{P_c}{V}$ would be ·821. This result, though at the first blush paradoxical, might also have been anticipated from the comparatively small number of trials (20 instead of 1,000), and from the nearness of 3 to the most likely number of real roots out of 20 random equations, namely 4, or rather 3·78.

Problem 8.—From the same data as in Problem 7, with the experiment V added as an *a priori*, what is the chance that the event, whose probability the mathematician had concluded to be c, will happen on the $(n+1)^{\text{th}}$ trial?

Let P_c, as before, assert that the correct value of the chance calculated by the mathematician is really c; and let Q assert that the event which happened m times out of n trials will happen on the $(m+1)^{\text{th}}$ trial. We have

$$Q = (P_c + P'_c)\,Q = P_cQ + P'_cQ;$$

therefore
$$\frac{Q}{\epsilon} = \frac{P_c}{\epsilon}\cdot\frac{Q}{P_c} + \frac{P'_c}{\epsilon}\cdot\frac{Q}{P'_c}.$$

Now, since this time $V=\epsilon$, the chance $\dfrac{P_c}{\epsilon}$ = the chance $\dfrac{P_c}{V}$, whose value we found in Problem 7. Call it A.

Therefore
$$\frac{Q}{\epsilon} = A\,\frac{Q}{P_c} + (1-A)\,\frac{Q}{P'_c}$$
$$= Ac + (1-A)\int_0^1 x\,dx$$
$$= Ac + \tfrac{1}{2}(1-A) = \tfrac{1}{2} + A\,(c-\tfrac{1}{2}).$$

Problem 9.—Let n quadratic equations, each of the form

$$Ax^2 + Bx + C = 0,$$

be written down at random, the coefficients A, B, C being each taken at random between a and $-a$ (a being real and positive); and out of the $2n$ roots of these random quadratics let two roots, x and y, be taken at random. What is the chance that $x+y$ will be real and positive?

It is evident that the chance required will be the same if we assume A to be *always positive* and taken at random between 0 and a, while B and C may have any value between a and $-a$. We shall, therefore, assume this to be the case throughout.

Let the symbols x^u, x^v, x^w respectively assert that x is *real and positive*, that x is *real and negative*, that x is *imaginary*; and let similar meanings be given to y^u, y^v, y^w, $(x+y)^u$, B^u, C^v, &c. Let the symbol α assert that x and y are roots of the *same quadratic*, while β (or α') asserts that x and y belong to *different* quadratics. Let

$$\phi = (x+y)^u.$$

We are required to find $\dfrac{\phi}{\epsilon}$.

$$\frac{\phi}{\epsilon} = \frac{(\alpha+\alpha')\,\phi}{\epsilon} = \frac{\alpha\phi+\beta\phi}{\epsilon} = \frac{\alpha}{\epsilon}\cdot\frac{\phi}{\alpha} + \frac{\beta}{\epsilon}\cdot\frac{\phi}{\beta}$$

$$= \frac{1}{2n-1}\cdot\frac{\phi}{\alpha} + \frac{2n-2}{2n-1}\cdot\frac{\phi}{\beta} \quad \text{(see Note 8 further on)}$$

$$= \frac{1}{2n-1}\cdot\frac{1}{2} + \frac{2n-2}{2n-1}\cdot\frac{\phi}{\beta};$$

for

$$\frac{\phi}{\alpha} = \frac{B^v}{\alpha} = \frac{1}{2}.$$

To find $\dfrac{\phi}{\beta}$, we have

$$\phi = x^u y^u + (x^u y^v + x^v y^u + x^w y^w)\,\phi,$$

as shown in the appendix to my last paper.

Therefore

$$\frac{\phi}{\beta} = \frac{x^u y^u}{\beta} + \left(\frac{x^u y^v \phi}{\beta} + \frac{x^v y^u \phi}{\beta}\right) + \frac{x^w y^w \phi}{\beta}$$

$$= \frac{x^u y^u}{\beta} + 2\,.\,\frac{x^u y^v \phi}{\beta} + 0;$$

for the chance $\dfrac{x^w y^w \phi}{\beta}$ $\left(\text{which} = \dfrac{x^w y^w}{\beta}\cdot\dfrac{\phi}{\beta\,.\,x^w y^w}\right)$ is infinitesimal, since $\dfrac{\phi}{\beta\,.\,x^w y^w}$ is infinitesimal.

Therefore

$$\frac{\phi}{\beta} = \frac{x^u}{\beta}\cdot\frac{y^u}{\beta\,.\,x^u} + 2\,.\,\frac{x^u}{\beta}\cdot\frac{y^v}{\beta\,.\,x^u}\cdot\frac{\phi}{\beta\,.\,x^u y^v}$$

$$= \frac{x^u}{\beta}\cdot\frac{y^u}{\beta} + 2\,.\,\frac{x^u}{\beta}\cdot\frac{y^v}{\beta}\cdot\frac{1}{2} \quad \text{(see Note 9 further on)}$$

$$= \frac{x^u}{\beta}\cdot\frac{y^u}{\beta} + \frac{x^u}{\beta}\cdot\frac{y^v}{\beta}$$

$$= \left(\frac{x^u}{\beta}\right)^2 + \left(\frac{x^u}{\beta}\right)^2 = 2\left(\frac{x^u}{\beta}\right)^2 = 2\left(\frac{x^u}{\epsilon}\right)^2.$$

To find $\dfrac{x^u}{\epsilon}$. Remembering that A, B, C are the coefficients in the quadratic of which x is a root, and that A is understood to be always positive, we have (A^u being understood throughout)

$$x^u = x^u\,(B^uC^v + B^vC^v + B^vC^u)\,;$$

therefore
$$\frac{x^u}{\epsilon} = \frac{B^uC^v}{\epsilon}\cdot\frac{x^u}{B^uC^v} + \frac{B^vC^v}{\epsilon}\cdot\frac{x^u}{B^vC^v} + \frac{B^vC^u}{\epsilon}\cdot\frac{x^u}{B^vC^u}$$
$$= \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{x^u}{B^vC^u}$$
$$= \frac{1}{4} + \frac{1}{4}\cdot\frac{x^u}{B^vC^u}.$$

But, in a quadratic equation

$$Ax^2 + Bx + C = 0,$$

the statement $A^uB^vC^u$ implies that both roots are real and positive, or else both imaginary. Hence $\dfrac{x^u}{B^vC^u}$ is the chance that the two roots of the equation

$$Ax^2 - Bx + C = 0$$

are real and positive when A, B, and C are each taken at random between 0 and a; and this chance, as shown in my First Paper, is $\frac{5}{36} + \frac{1}{6}\log 2$, which $=$ ·18906, &c. Putting k for $\frac{5}{36} + \frac{1}{6}\log 2$ or ·18906, &c., we get

$$\frac{x^u}{\epsilon} = \tfrac{1}{4}\,(1+k),$$

and therefore
$$\frac{\phi}{\beta} = \tfrac{1}{8}\,(1+k)^2\,;$$

so that, finally

$$\frac{\phi}{\epsilon} = \frac{1}{2\,(2n-1)} + \frac{n-1}{4\,(2n-1)}\,(1+k)^2,$$

in which
$$k = \tfrac{5}{36} + \tfrac{1}{6}\log 2.$$

Note 8.—To find $\dfrac{\alpha}{\epsilon}$ and $\dfrac{\beta}{\epsilon}$ in this problem, let $Q_r(x)$, $Q_r(xy)$, $Q_r(xy')$, &c., respectively assert that the r^{th} quadratic of the series $Q_1, Q_2, Q_3, \ldots, Q_n$ contains the random root x, that it contains both x and y, that it contains x but not y, and so on. Then we get

$$\frac{Q_r(xy)}{\epsilon} = \frac{Q_r(x)}{\epsilon}\cdot\frac{Q_r(y)}{Q_r(x)} = \frac{1}{n}\cdot\frac{1}{2n-1}\,;$$

for y may be any one of the total $2n$ roots *except* x, when the random root x has been taken. Giving r successively the values 1, 2, 3, ... , n

we get $$\frac{\alpha}{\epsilon} = \Sigma\,\frac{\phi_r\,(xy)}{\epsilon} = n\left(\frac{1}{n}\cdot\frac{1}{2n-1}\right) = \frac{1}{2n-1}.$$

and $$\frac{\beta}{\epsilon} = \frac{\alpha'}{\epsilon} = 1 - \frac{\alpha}{\epsilon} = \frac{2n-2}{2n-1}.$$

Note 9.—In Problem 9 it is clear that, on the assumption β (*i.e.*, that x and y are roots of *different equations*), x^u and y^u are independent, as are also x^u and y^v; so that

$$\frac{x^u y^u}{\beta} = \frac{x^u}{\beta}\cdot\frac{y^u}{\beta},$$

and $$\frac{x^u y^v}{\beta} = \frac{x^u}{\beta}\cdot\frac{y^v}{\beta}.$$

It is also evident that $\dfrac{\phi}{\beta x^u y^v}$ is $\dfrac{1}{2}$,

and that $$\frac{y^v}{\beta} = \frac{y^u}{\beta} = \frac{x^u}{\beta} = \frac{x^u}{\epsilon};$$

so that $$\frac{\phi}{\beta} = 2\left(\frac{x^u}{\epsilon}\right)^2.$$

The Symbol $\partial\alpha$.

In the following examples the symbols $\phi^\epsilon(\alpha, \beta, \gamma)$, $\phi^\eta(\alpha, \beta, \gamma)$, $\phi^\theta(\alpha, \beta, \gamma)$ assert respectively that the function $\phi(\alpha, \beta, \gamma)$ is true for *all* values of α, β, γ, for *no* values of α, β, γ, for *some* value or values of α or β or γ, but not for all. In other words, $\phi^\theta(\alpha, \beta, \gamma)$ denies both $\phi^\epsilon(\alpha, \beta, \gamma)$ and $\phi^\eta(\alpha, \beta, \gamma)$, so that we have the identity

$$\phi^\theta(\alpha, \beta, \gamma) = \phi^{\epsilon'}(\alpha, \beta, \gamma)\,\phi^{\eta'}(\alpha, \beta, \gamma).$$

DEF. 5.—The symbol $\partial\phi$ bears the same relation to ϕ that ϕ bears to ϕ^ϵ; so that $(\partial\alpha = \beta)$ means $(\alpha = \beta^\epsilon)$. To avoid brackets $\partial\alpha_\beta$ will mean $\partial(\alpha_\beta)$; the symbol α_β being now (as in my last paper) synonymous with $\alpha : \beta$. Hence we get

$$\partial\alpha^\epsilon = \alpha, \quad \partial\alpha^\eta = \alpha';$$

$$\partial\alpha_\beta = \partial(\alpha : \beta) = \alpha' + \beta.$$

Should the necessity arise we can, of course, have $\partial^2\alpha$ for $\partial(\partial\alpha)$, $\partial^3\alpha$ for $\partial(\partial^2\alpha)$, and so on. The symbol $\partial'\alpha$ will mean $(\partial\alpha)'$.

Problem 10.—To show that the statements $v : (u : \alpha)$ and $vu : \alpha$ are not always equivalent. (See my preceding paper, p. 176.)

$$\partial(v : u_\alpha) = v' + u_\alpha,$$

$$\partial(vu : \alpha) = (vu)' + \alpha = v' + (u' + \alpha) = v' + \partial(u_\alpha).$$

But u_α is generally stronger, and never weaker, than $\partial(u_\alpha)$; so that $v' + u_\alpha$ is generally stronger, and never weaker, than $v' + \partial(u_\alpha)$. Hence $\partial(v : u_\alpha)$ always implies, but is not always implied in, $\partial(vu : \alpha)$; and therefore $v : u_\alpha$ always implies, but is not always implied in, $vu : \alpha$. This follows from the formula

$$(\alpha : \beta) : (\alpha^\epsilon : \beta^\epsilon),$$

or its equivalent $$(\partial\alpha : \partial\beta) : (\alpha : \beta).$$

Note 10.—In Problem 10 let ϕ denote $v : u_\alpha$, and let ψ denote $vu : \alpha$. The cases of non-equivalence are

$$(v^\epsilon u^\theta + v^\theta u + v^\theta u^\theta)\,\alpha^\theta + v^\theta u^\theta \alpha^\eta.$$

To verify this we have

$$v^\epsilon u^\theta \alpha : (\phi = \epsilon : u_\alpha) \quad \text{and} \quad v^\epsilon u^\theta \alpha^\theta : (\psi = \epsilon u : \alpha),$$

that is, $v^\epsilon u^\theta \alpha^\theta$ reduces ϕ to $\epsilon : u_\alpha$ and reduces ψ to u_α; so that ϕ implies ψ, but is not always implied in ψ, just as α^ϵ implies α, but is not always implied in α, and α implies, but is not always implied in, $\partial\alpha$.

For the next term we have

$$v^\theta u^\epsilon \alpha^\theta : (\phi = v : \epsilon_\theta = v : \eta = \theta : \eta = \eta),$$

$$v^\theta u^\epsilon \alpha^\theta : (\psi = v\epsilon : \alpha = v : \alpha).$$

Now η always implies $v : \alpha$, but $v : \alpha$ does not (when, as here, v and α are variables) necessarily imply η. So here again we have $\phi : \psi$, but not necessarily $\psi : \phi$.

For the next term, we have

$$v^\theta u^\theta \alpha^\theta : (\phi = v : u_\alpha),$$

$$v^\theta u^\theta \alpha^\theta : (\psi = vu : \alpha = v : u' + \alpha = v : \partial u_\alpha);$$

in which v, u, α are all variables. Now u_α is generally stronger, and is never weaker, than $\partial(u_\alpha)$. Hence $v : u_\alpha$ is generally stronger, and is never weaker, than $v : \partial(u_\alpha)$; so that here, as before, ϕ implies, but is not necessarily implied in, ψ. (See proof and figure, p. 176 of Fifth Paper.)

For the last term we have

$$v^{\theta}u^{\theta}\alpha^{\eta} : (\phi = v : u_{\eta} = v : \theta_{\eta} = v : \eta = \theta : \eta = \eta),$$

$$v^{\theta}u^{\theta}\alpha^{\eta} : (\psi = vu : \eta);$$

and, since η always implies $(vu : \eta)$, while $vu : \eta$ may possibly be true (as when the variable v is the *denial* of the variable u), it follows, as before, that ϕ implies ψ, but is not necessarily implied in ψ.

For the 23 other terms in the product $v^{\epsilon, \eta, \theta} u^{\epsilon, \eta, \theta} \alpha^{\epsilon, \eta, \theta}$ the two expressions $v : u_{\alpha}$ and $vu : \alpha$ are equivalent, each implying the other.

Problem 11.—The implication $(\alpha : \beta) : (\beta^{\eta} : \alpha^{\eta})$ is true for all values of α and β; for what values of α and β is the converse implication $(\beta^{\eta} : \alpha^{\eta}) : (\alpha : \beta)$ false?

Let ϕ denote $\quad (\beta^{\eta} : \alpha^{\eta}) : (\alpha : \beta),$

$$\phi^{\eta} ! \ (\beta^{\eta} : \alpha^{\eta})^{\epsilon} (\alpha : \beta)^{\eta}$$

$$! \ (\beta^{\eta\iota} + \alpha^{\eta})(\alpha^{\epsilon}\beta^{\epsilon\iota} + \alpha^{\theta}\beta^{\eta})$$

$$! \ \alpha^{\epsilon}\beta^{\theta}.$$

Hence, the converse implication $(\beta^{\eta} : \alpha^{\eta}) : (\alpha : \beta)$ is false when α is a certainty and β a variable.

Note 11.—Putting $\alpha = zw'$ and $\beta = xy'$ in this problem, we get

$$x_y = (xy')^{\eta} = \beta^{\eta}, \quad \text{and} \quad z_w = \alpha^{\eta};$$

so that the implication $\quad (zw' : xy') : (x_y : z_w),$

being synonymous with $\quad (\alpha : \beta) : (\beta^{\eta} : \alpha^{\eta}),$

is always true. Let ϕ denote the converse implication

$$(x_y : z_w) : (zw' : xy').$$

We have $\quad \phi^{\eta} ! \ \alpha^{\epsilon}\beta^{\theta} ! \ (zw')^{\epsilon} (xy')^{\theta}$

$$! \ W(zw')^{\epsilon} W(xy')^{\theta} ! \ z^{\epsilon} w^{\eta} (x^{\theta}y^{\eta} + x^{\epsilon}y^{\theta}).$$

(See next problem and Postscript to Appendix of last paper.)

Problem 12.—The formula $\alpha_u + \beta_u : (\alpha\beta)_u$ is always true. When is the converse implication $(\alpha\beta)_u : \alpha_u + \beta_u$ false?

Let ϕ denote this converse implication. We then get

$$\phi = (\alpha\beta u')^{\eta} : (\alpha u')^{\eta} + (\beta u')^{\eta}$$

$$= (\alpha u' . \beta u')^{\eta} : (\alpha u')^{\eta} + (\beta u')^{\eta}$$

$$= (AB)^{\eta} : A^{\eta} + B^{\eta},$$

putting A for $\alpha u'$ and B for $\beta u'$. But evidently this is not true for all values of A and B, and therefore it is not true for all values of α, β, and u; for

$$W(AB)^\eta = A^\eta + B^\eta,$$

and

$$\phi^\eta !\ A^\theta B^\theta A_{B'}.$$

For example, let α be some variable statement x; let β be the denial of this, namely, x'; and let $u' = \epsilon$. Then we get

$$\phi = (xx')^\eta : x^\eta + (x')^\eta$$
$$= \eta^\eta : \eta + \eta = \epsilon : \eta + \eta = \eta.$$

Note 12.—In my last paper the cases of $W\phi(\alpha, \beta)$ and $S\phi(\alpha, \beta)$ considered were cases in which α and β had *fixed* values; *i.e.*, they were *singulars*, each letter being understood to denote *one statement only, and always the same statement.* On this supposition, statements like α', $\alpha\beta'$, $\alpha+\beta$, &c., may be of the class ϵ or η or θ; but statements with indices ϵ, η, θ, of the form α^θ, $(\alpha\beta')^\eta$, $(\alpha+\beta)^\epsilon$, $\alpha : \beta$, &c. (the fourth of which is synonymous with the second), cannot in this case be *variables;* they must be *constants* of the class ϵ or of the class η. But when α and β may change their meanings, so that any letter α may denote sometimes one statement, sometimes another, then statements of the form α^ϵ, α^η, $(\alpha\beta)^\theta$, $\phi^\eta(\alpha, \beta)$, &c., may (like their constituents α, β, &c.) be either constants or variables. A case of this is given in the first problem of this paper, in which the chance of $\alpha : \beta$ being true is found to be ·6624; so that in that problem $\alpha : \beta$ was a *variable.* The following is another case, and from a different point of view.

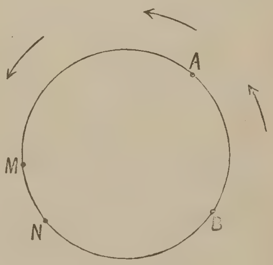

FIG. 6.

Let A, B, M, N be four *fixed* points in the circumference of a circle, as in the accompanying figure; and let a fifth point P be taken *at random* in the circumference. Understanding all arcs to be measured in the *positive* direction, so that any arc AM or BN may be greater or less than a semicircle, let α assert that the random point P will fall in the arc AM, and let β assert that P will fall in the arc BN. What is the chance of $\alpha : \beta$ being true? Here, since the points A, B, M, N are by hypothesis *fixed*, the implication $\alpha : \beta$ is either a *certainty* or an *impossibility;* it cannot be a *variable.* The answer is therefore 1 or 0; *unity* if (measuring in the positive direction) we have the arc AM wholly inside the arc BN, as in the figure; *zero* if the arc AM is *not* wholly inside the arc BN. But if the points A, B, M, N, as well as P, be *random* points which vary their positions in the circumference, all positions being equally probable, then evidently $\alpha : \beta$ is a *variable*, whose chance of being true, it is easy to see, is $\frac{1}{3\,.\,2\,.\,1}$ or $\frac{1}{6}$, the denominator expressing the number of possible permutations of the letters A, M, N; for the chance is clearly the same if we suppose the point B to be *fixed* in the circumference while the other points vary.

Note 13.—Also, when any statement α is a *singular*, so that it *never changes its class*, be it ϵ or η or θ, we have (as stated in my last paper) $\alpha^{\eta\epsilon}$ synonymous with α^{η}, $\alpha^{\eta\eta}$ synonymous with $\alpha^{\eta\iota}$, &c.; but when α is a *plural* or *functional* statement—the symbol α denoting a statement sometimes of one class (ϵ or η or θ), sometimes of another—then, though (as before) we shall always have the implications $\alpha^{\eta\epsilon} : \alpha^{\eta}$, $\alpha^{\eta\eta} : \alpha^{\eta\iota}$, &c., we shall *not* always have the converse implications $\alpha^{\eta} : \alpha^{\eta\epsilon}$, &c.

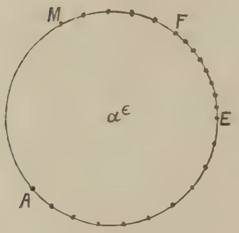

Fig. 7.

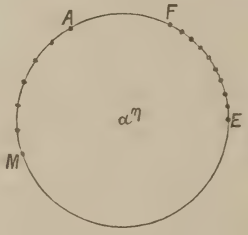

Fig. 8.

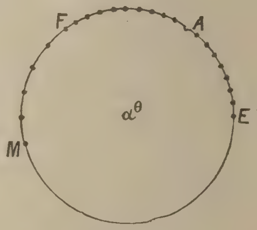

Fig. 9.

For example, in the accompanying three figures, let the points E, F, A, M be *fixed* in the circumference; let the points in the arc EF (measured in the positive direction) be our *universe of discourse;* in this universe let a point P be taken at random; and let the symbol α assert that P will be in the arc AM. Then, evidently, in Fig. 7, we shall have α^{ϵ} a *certainty*,* and α^{η} and α^{θ} both *impossibilities;* in Fig. 8 we shall have α^{η} a *certainty*, and α^{ϵ} and α^{θ} *impossi-*

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* For in Fig. 7 the arc AM, measured *positively*, is greater than a semicircle and *includes the universe of discourse* EF.

bilities; and in Fig. 9 we shall have α^{θ} a *certainty*, and α^{ϵ} and α^{η} *impossibilities.* But if (while E remains fixed) the points F, M, A be all three taken at random *in the whole circumference*, while P is taken at random in the *universe of discourse*, the arc EF, then, evidently, α^{ϵ}, α^{η}, α^{θ} will all three be *variables* (sometimes true, sometimes false), as will also their denials $\alpha^{\epsilon\iota}$, $\alpha^{\eta\iota}$, $\alpha^{\theta\iota}$; while $\alpha^{\epsilon\epsilon}$, $\alpha^{\eta\epsilon}$, $\alpha^{\theta\epsilon}$, $\alpha^{\epsilon\eta}$, $\alpha^{\eta\eta}$, $\alpha^{\theta\eta}$ will all be *impossibilities*, since each of the six either asserts that a *variable* is a *certainty* or else asserts that a *variable* is an *impossibility.*

[*Postscript.*—In my preceding paper there are one or two obscurities which I ask permission here to explain. On pp. 170, 171, 179 I treated $(\alpha : \beta)^{\epsilon}$ and $(\alpha : \beta)^{\eta}$ as respectively equivalent to $(\alpha : \beta)$ and $(\alpha : \beta)'$. This equivalence, however, assumes that α and β represent each only *one* statement, which (throughout the argument) *never changes its class* (be it ϵ or η or θ) in the three-divisional classification which constitutes our *Universe of Statements.* When this is understood, the implication $\alpha : \beta$ always $=$ either ϵ or η, and never θ. When the statement α or β may belong sometimes to one and sometimes to another of the three classes ϵ, η, θ, the formulæ $(\alpha : \beta)^{\epsilon} : (\alpha : \beta)$ and $(\alpha : \beta)^{\eta} : (\alpha : \beta)'$ will of course still be valid, but *not* always the converse formulæ $(\alpha : \beta) : (\alpha : \beta)^{\epsilon}$ and $(\alpha : \beta)' : (\alpha : \beta)^{\eta}$. Similarly, we may still accept $\alpha^{\epsilon\epsilon} : \alpha^{\epsilon}$, $\alpha^{\epsilon\eta} : \alpha^{\epsilon\iota}$, $\alpha^{\eta\eta} : \alpha^{\eta\iota}$, &c., as valid, but *not* their converses $\alpha^{\epsilon} : \alpha^{\epsilon\epsilon}$, $\alpha^{\epsilon\iota} : \alpha^{\epsilon\eta}$, $\alpha^{\eta\iota} : \alpha^{\eta\eta}$, &c.]

The following presents to the Library were received during the Recess :—

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"Smithsonian Report" for 1893, 1894, 1895, Washington.

APPENDIX.

Mr. Blythe's paper "On the Geometrical Construction of Models of Cubic Surfaces" (p. 4) is published in the *Quarterly Journal of Mathematics* (No. 115, November, 1897, pp. 206–223). In the same number will be found a paper by Mr. Brill, "On certain Analogues of Anharmonic Ratio" (*cf.* p. 447).

An interesting article "On the Four-Colour Map Theorem," by Mr. P. J. Heawood, occupies pp. 270–285 of the above-cited number (see *Proc.*, Vol. XXI., p. 456).

In the "Mathematical Notes" of No. 12 of the *Mathematical Gazette* (October, 1897, pp. 127–129) is given "A Property of Two Triangles and its extension to Quadrilaterals," of which Mr. Macaulay's paper (pp. 442–446) is an extension to polygons of any number of sides.

For the following sketch of the late Professor Sylvester, the Council are indebted to Mr. J. J. Walker.

By the death of Professor James Joseph Sylvester, on the "Ides

[15th] of March," 1897, the Society lost one of its earliest and most valued members, whose reawakened interest in his mathematical work had recently renewed his presence at the Society's meetings. He appeared among us for the last time on 14th January, taking an active part in the proceedings, which those present little thought was to be the final flickering of a flame which had so often illumined our gatherings.

Born of Jewish parentage in London, of a family which is said to have been originally resident in Liverpool, on 3rd September, 1814, Sylvester proceeded to St. John's College, Cambridge, from the Royal Institution, Liverpool, coming out Second Wrangler in the Mathematical Tripos List of 1837—the year of Green and Gregory. His creed being an obstacle to taking a degree at Cambridge, he obtained those of B.A. and M.A. as *ad eundem*, at Dublin, by accumulation in 1841; but later both the same degrees were conferred upon him *honoris causâ* by his own University, and he was enrolled among its honorary Fellows by St. John's College.

He first contributed an analytical proof of Fresnel's wave-surface and properties thereof, to Vols. XI. and XII. of the *Philosophical Magazine*, New Series, at the conclusion of which he acknowledges having been anticipated by Sir W. R. Hamilton—"our living Lagrange," as he terms that mathematician—in his "System of Rays," published in *Trans. R. I. A.* for 1838. Early in 1838, Sylvester—perhaps owing to the supposed bent indicated in this paper—was appointed Professor of Natural Philosophy in University College, London, in succession to Dr. Lardner, who had become immersed in numerous literary undertakings.

In various subsequent papers, of an algebraic character, which he contributed to the *Philosophical Magazine* down to April, 1841, he gives as his address "University College or 22 Dogherty Street." About the above date he was appointed Professor of Mathematics in the University of Virginia, which had been founded in 1824, at Charlottesville, Albemarle Co., and where his colleagues Key and Long, of University College, London, had previously, for a few years, first occupied the Chairs of Mathematics and Classics respectively.

Of Sylvester's tenure* of this Chair no record seems to exist, but it is believed he looked back on his first American experience, when he returned to England in 1844, with little satisfaction; but, on his return, the series of papers on algebraical subjects soon began to reappear in the *Philosophical Magazine*, dated from 4 Park Street, Grosvenor Square, until in August, 1847, he dates from 26 Lincoln's Inn Fields, having entered the Inner Temple as law student in the previous year, and being called to the Bar in November, 1850. During the ten years following his return from the United States he was engaged in actuarial business, and is believed never to have practised at the Bar.*

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* An obituary notice in the *American Journal of Mathematics* limits it to two years.

The last number of the *Cambridge and Dublin Mathematical Journal*, which had been edited first by W. Thomson (now Lord Kelvin), M.A., alone, subsequently by "W. Thomson, M.A., and N. M. Ferrers, M.A.," jointly, having been issued in March, 1855, that periodical was succeeded, in the following month, by the *Quarterly Journal of Pure and Applied Mathematics*, Nos. 1 and 2 being edited by "J. J. Sylvester, M.A., F.R.S., late Professor of Natural Philosophy in University College, London, and N. M. Ferrers, M.A. . . ."; but on the cover of No. 3 (December, 1855) Sylvester is described as "Professor of Mathematics in the Royal Military Academy, Woolwich," to which post he had been appointed in the interval. Assisted by Stokes, Cayley, and Hermite, corresponding editor in Paris, this joint editorship continued unchanged until on the cover of No. 57, issued June, 1877, Sylvester is designated "Professor of Mathematics in the Johns Hopkins University, Baltimore" (Maryland, U.S.A.), "and late† . . . in the Royal Military Academy, Woolwich." On the cover of No. 61 (December, 1878) his name finally disappeared as editor.

In January, 1865, The London Mathematical Society had been founded under the Presidency of De Morgan, Professor of Mathematics in University College, London, and on June 19th of that year Sylvester was elected into the Society, of which he became the second President (November 8th, 1866, to November 12th, 1868), while he served on the Council until his departure from England, and, as Vice-President, both before and after (1884–85) that change. In 1887 he was awarded the De Morgan Medal by the Society.

Sylvester had been elected a Fellow of the Royal Society as far back as April 5th, 1839—he and the late Mr. James Heywood being twin-seniors among the Fellows at the time of his death. This election in his twenty-fifth year he, no doubt, owed in great part to the post he held in University College at the time. In 1861 a Royal Medal was awarded to him, and in 1880 the Copley Medal; Cayley, who had preceded him by two years in the award of a Royal, following in 1882 as recipient of the Copley Medal.

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* In the second paper on the "Calculus of Forms" (*Cambridge and Dublin Mathematical Journal*, VII., 1852, p. 179) he describes himself as "Barrister-at-law," and in *Nouvelles Annales* (below) as "Avocat"; but in Vol. VIII. of the former *Journal* the affix is dropped.

† The Professorship at Woolwich he had, in fact, vacated some years back, having reached the limit of age at which it could be held.

The next event which marked an epoch in Sylvester's career was his second appointment in the United States of America; this time to the newly-founded Johns Hopkins University, which he held for more than seven years, usually revisiting* England between the successive sessions. "Wherever Sylvester was, there was a centre of mathematical activity," has been said of him—a saying exemplified by, if not originating from, the activity shown by the superannuated (!) ex-Professor of the Royal Military Academy in England, in this new post. An eager following of learners and sympathizing helpers soon gathered round him, and in 1878 he, with their cooperation, founded the *American Journal of Mathematics*, which, enriched by European, especially English, contributions, has maintained a high position in its class. Sylvester continued to edit, as well as contribute to, this *Journal* for six years; but, having been elected Savilian Professor of Geometry at Oxford, in December, 1883, he was destined to pass the last period of a long and honourable career in his native land.

A list of Sylvester's academic distinctions would occupy half a page in their bare enumeration. Some have been alluded to, or will be mentioned below; but of all it may be said that greater honour was in every case conferred on the body in which he was enrolled than on the recipient.

Sylvester himself valued most his association with the Royal Society of London and the Academy of Sciences at Paris, if one may judge by the greater frequency with which he designated himself as a member of these august associations.

Sylvester's published papers amount to some two hundred and fifty in number, of which are that on "Fresnel's Wave-Surface" above mentioned; a short paper on "Motion and Rest of Fluids," *Phil. Mag.*, XIII., and Note, *ib.* XIV.; one on the "Equation of Secular Inequalities in the Planetary Theory," communicated to the *Nouvelles Annales de Mathématiques* (XI., 1852)—which is virtually an essay of pure mathematics; a kinematic paper "On the Rotation of a Rigid Body about a Fixed Point," *Phil. Mag.*, XXXVII., 1850; a short paper "On Projectiles," *ib.* XI., 1856; an incomplete* essay "On the Pressure of Earth on Revetment Walls," *ib.* XX., 1860; and one in the *Comptes Rendus*, I., II., 1861, "Sur l'Involution des Lignes droites dans l'espace, considérées des axes de rotation," which was followed by some "Observations de M. Chasles," to whom "Les beaux resultats dans la communication de M. Sylvester" suggested a similar property of the directions of six forces in equilibrium; also an "Addition to Poinsot's Ellipsoidal Mode of representing the Motion of a Rigid Body about a Fixed Point," *Lond. Math. Soc. Proc.* These half-dozen are believed to form the total of the papers published with Sylvester's name attached the subjects of which were in applied mathematics. Of these the last but one is remarkable as being treated by the method of synthetic geometry,† and having, perhaps, insured Sylvester being chosen correspondent in mathematics by the French Academy of Sciences (7th December, 1863), in place of the great geometer Steiner, who had died on the previous 1st April.

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* On the voyages home and out he has described himself as carrying on mathematical investigations. So the solution of the ternary cubic was worked at during "a hasty tour on the Continent" in 1847 (*Phil. Mag.*, XXXI.).

But Sylvester's mathematical writings, in Professor Cayley's words,‡ "relate chiefly to finite analysis, and cover by their subjects a large part of it: algebraical determinants, elimination, the theory of equations, partitions, tactic, the theory of forms, matrices, reciprocants. A leading feature is the power which is shown of originating a theory, or of developing it from a small beginning; there is a breadth of treatment and determination to make the most of a subject, an appreciation of its capabilities and real enjoyment of it. There is not unfrequently an adornment of enthusiasm of language. . . . There is always a generous and cordial recognition of the merits of others, his fellow-workers in the science."

Only twice, as far as known to the present writer, was any asperity in his allusions to other contemporaries manifested: the earlier occasion being called forth in a reply to the late Professor Boole, who had claimed to have anticipated him in one of his results in the theory of linear transformations (*Cambridge and Dublin Mathematical Journal*, VI.), which elicited from Boole an ample *amende;* the other occasion during a lecture delivered at King's College, an account of which—quite accurate in its details according to the recollection of the present writer, who was one of the audience on the occasion—is given in the very able obituary notice which appeared in *Nature* soon after the lamented death of Professor Sylvester.

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* Another paper, never completed, is "On the Integral of the General Equation in Differences," *Phil. Mag.*, XXIV., 1862.

† In his opening address as President of Section A of the meeting of the British Association at Exeter, 1869, he went so far as to express himself as having contracted a "hatred of geometry" from the study of Euclid.

‡ *Nature series* of "Scientific Worthies." *Nature*, Vol. XXXIX.

[Professor Sylvester's communications to our *Proceedings* were:—

"An Elementary Proof and Generalization of Sir I. Newton's hitherto undemonstrated Rule for the discovery of Imaginary Roots," Vol. I. (*cf. supra*).*

"On an Addition to Poinsot's Ellipsoidal Mode of representing the Motion of a Rigid Body turning freely round a Fixed Point, whereby the Time may be made to Register itself Mechanically," Vol. I.

"A Problem of Arrangement connected with a New Method of Tesselation," Vol. II.

"Outline Trace of the Theory of Reducible Cyclodes," Vol. II.

"On the Partition of an Even Number into Two Primes," Vol. IV.

"On the Theorem that an Arithmetical Progression which contains more than one contains an Infinite Number of Prime Numbers," Vol. IV.

"On the Expression of the Curves generated by any given System whatever of Linkwork under the Form of an Irreducible Determinant," Vol. VI.

"An Orthogonal Web," Vol. VI.

"Note on certain Elementary Geometrical Notions and Determinations," Vol. XVI.

Some interesting particulars are appended to a notice of Professor Sylvester, by J. W., in the *Eagle*, Vol. XIX., No. 113, June, 1897.]

The Rev. Alexander Freeman was born January 28th, 1838, and was educated at the Merchant Taylors' School,† whence he proceeded to St. John's College, Cambridge, in 1857, having obtained a scholarship on the Foundation. He graduated as Fifth Wrangler in 1861, and in February, 1862, he gained the Chancellor's Gold Medal for Legal studies. He was elected Fellow of his College in May, 1862, and took his M.A. degree in 1864. He was examiner for the Mathematical Tripos in 1874 and 1875; he acted as deputy for the Plumian Professor of Astronomy at Cambridge from 1880 to 1882; he translated and compiled an English edition of Fourier's *Analytical Theory of Heat*, and added to it many valuable original notes; and recently brought out a new edition of Cheyne's *Planetary Theory*. To the *Monthly Notices* of the Royal Astronomical Society, of which he was elected a Fellow in 1864, he contributed about a dozen papers. He further acted as Director of the Saturn Section of the British Astronomical Association, from which office he retired, on account of failing health, in 1895. Mr. Freeman was presented in 1882 to the living of Murston, and in 1891 was made Rural Dean of Sittingbourne. His death took place somewhat suddenly on Saturday, June 12th, 1897. He leaves a widow and four young children. Mr. Freeman was elected a life-member of this Society December 14th, 1871. He never served on the Council, and attended but one of its meetings; he, however, took at one time a warm interest in its proceedings, and presented one or two rare books to its Library. He was also a Fellow of the Cambridge Philosophical Society and a member of the Physical Society.*

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* *Cf. Nature*, p. 493, March 25th, 1897.

† Entered April, 1846.

John Robert Campbell was the only child of Robert Campbell. This gentleman, who was born in 1798, was the eldest son of Major John H. Campbell, of the 23rd Royal Welsh Fusiliers, and grandson of Captain Robert Campbell, who served with distinction under Lord Clive in India. Mr. Campbell married early in life, his wife (who died in 1850) being a lady much esteemed and possessed of great musical talent. Before he came to Charing, he resided for many years in the Lake District. He was an intimate friend of Robert Southey, and at the Poet Laureate's house he occasionally met the leading literary men of the day—Wordsworth, Hartley Coleridge, Dr. Arnold, and others. Mr. Campbell died in his ninety-second year (March 15th, 1890).† Among such surroundings as these Lt.-Col. Campbell, who gained his rank in the Hampshire Artillery Militia, passed great part of his earlier life. In his younger days he travelled almost all over the world, being an accomplished linguist. He devoted much of his leisure to art and literature as a recreation. A startling novel from his pen appeared some few years ago in the *Kentish Express* as a serial. The Colonel took a keen interest in chess, and was a good average player. The village of Charing, in Kent, where he resided for some forty years, has lost by his decease a most liberal friend. Though he took little part in public business, he consented to be a manager of the school, and his interest in the education of the children, especially in subjects connected with drawing, music, and elementary science, was maintained to the end. In the spring of the present year (1897) he made a handsome gift of books, specially selected by himself, to the school library. A new parish hall was built on ground which Col. Campbell gave the promoters for a nominal sum of money, and he presented besides a donation of £30. Ringing cheers greeted the announcement of this act of liberality at the Jubilee rejoicings in June last. These rejoicings were hardly over before it became known that the kind donor, who had been suffering from an incurable internal malady for the two previous months, had passed away on June 23rd, in the seventieth year of his age. The day before his death he announced in a letter to the Vicar "a bequest in his will for the benefit of Charing Parish." It will thus be seen that he took a kindly and very practical interest in his neighbours' affairs, and many will miss his liberality.

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* We are indebted for our information to Professor Hudson, a contemporary of Mr. Freeman (Third Wrangler, 1861); to *The Taylorian*, July, 1897; and to the *Journal of the British Astronomical Association* for August 16th, 1897.

† The above particulars, and others telling how keen a sportsman his father was, are given in an obituary sketch by his son.

Before Col. Campbell became a member of the London Mathematical Society, he read a paper, at the meeting of January 14th, 1875, entitled "The Diagonal Scale Principle applied to Angular Measurement in the Circular Slide." An abstract of this paper is printed in the *Proceedings*, Vol. VI., pp. 73–78. Col. Campbell was the author also of the following:—*How to see Norway* (1871); *Range Finding for the Service of Artillery, with a description of Campbell's Range Indicator* (revised edition, Clowes, 1874). In the Preface (p. 4) he writes: "In working out my ideas, especially those respecting the construction of a range indicator, I have had to trust to my own general knowledge alone, rarely having met with any Artillery officer or instrument maker who took the slightest interest in the subject." From pp. 8 and 17 it appears that he also contributed papers on the same instrument to the *Journal of the Royal United Service Institution*, Vol. VIII., p. 431 (1864), and Vol. XVI. (1872).

Col. Campbell was a Fellow of the Geological Society. He was elected a member of this Society, December 9th, 1875, and served on the Council of the Society in the sessions 1893–94, 1894–95.

At the meeting of May 8th, 1890, a letter from Col. Campbell was read, in which he asked to be allowed to give a donation of £500 to the Society—the sum to be invested or otherwise made use of for the good of the Society, in any way the Council should judge best.

The following resolution was carried with acclamation, viz.: "That the cordial thanks of the London Mathematical Society be given to Lt.-Col. Campbell for his generous gift of £500 to the general fund of the Society"* (*Proc. Lond. Math. Soc.*, Vol. XXI., p. 273).

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* We are indebted for the personal details to the courtesy of Lt.-Col. Frederic Campbell, cousin of the deceased.

ERRATA AND ADDENDA.

VOL. XXVII.

Page 636, add to Lt.-Col. Cunningham "*Notes on High Primes, p. 328."
,, 637, prefix * to the second paper by Mr. Macdonald.

VOL. XXVIII.

Page 159, line 6, *for* "γ^{θ}" *read* "$\gamma^{\theta\iota}$."
,, 159, line 8, *for* "neither . . . nor" *read* "either . . . or."
,, 159, line 19, *for* "$\alpha_{\epsilon\iota}$" *read* "$\alpha^{\epsilon\iota}$."
,, 163, line 7 from bottom, *for* "η^{η}" *read* "η_2^{η}."
,, 164, line 5, *for* "0" *read* "η."
,, 167, line 7, *for* "which" *read* "from which."
,, 168, line 10 from bottom, *for* "$\beta^{\theta}\alpha^{,\epsilon}$" *read* "$\beta^{\theta}\alpha^{\theta,\epsilon}$."
,, 168, line 6 from bottom, *for* "$(\alpha, \beta)_{\epsilon}$" *read* "$(\alpha, \beta)^{\epsilon}$."
,, 170, line 6 from bottom, *for* "$\alpha\beta^{\theta}(\alpha : \beta)^{\eta}$" *read* "$\alpha^{\theta}\beta^{\theta}(\alpha : \beta)^{\eta}$."
,, 171, line 13, *for* "$W\phi_u(\alpha, \beta)$" *read* "$W\phi^u(\alpha, \beta)$."
,, 173, line 9 from bottom, *for* "$(\alpha\beta)^n$" *read* "$(\alpha\beta)^u$."
,, 176, line 7 from bottom, *for* "*identities*" *read* "*certainties*."

The above corrections are required, the Author believes, in *all* the copies; the following are only required in some, as the errors do not occur in all:—

Page 168, line 18, *for* "$= \alpha\beta^{\theta} + \&c.$" *read* "$\alpha^{\epsilon}\beta^{\theta} + \&c.$"
,, 169, line 14, *for* "$W = (\alpha', \beta')(\alpha', \beta')^{\theta}$" *read* "$W(\alpha', \beta')^{\epsilon}(\alpha', \beta')^{\theta}$."
,, 172. Formula 33 *should be*

$$\frac{u}{\alpha\beta} = u_{\alpha\beta}\left(\frac{u}{\alpha} + \frac{u}{\beta}\right).$$

,, 378, line of 9,367,291, *for* "$y = \bar{1}$" *read* "$y = 1$."
,, 378, line of 11,006,851, *for* "q" *read* "n."
,, 378, line of 18,837,001, *for* "n" *read* "q."
,, 378, line of 19,019,801, *for* "$y = \bar{1}$" *read* "$y = 1$."
,, 586, add the following reference:—*Mathem. Ann.*, Vol. L., Heft 1, pp. 133–156, "James Joseph Sylvester," von M. Noether, in Erlangen.

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† *Cf.* p. 32.

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